Math 104-006

Chapter 12.9: Representations of Functions as Power Series

Outline For Today

- Expressing Functions as Power Series
- Differentiating Power Series
- Integrating Power Series

Geometric Series

- Recall that the series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots$$

converges if and only if |r| < 1

If |r|< 1 $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Example

- Lets find a power series representative of $\dfrac{1}{x+2}$

$$\begin{aligned}
\frac{1}{x+2} &= \frac{1}{2\left(1-\left(-\frac{x}{2}\right)\right)} \\
&= \frac{1}{2}\left(1+\left(-\frac{x}{2}\right)+\left(-\frac{x}{2}\right)^2+\left(-\frac{x}{2}\right)^3+\ldots\right) \\
&= \frac{1}{2}-\frac{x}{4}+\left(\frac{x^2}{8}\right)-\left(\frac{x^3}{16}\right)+\ldots \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n
\end{aligned}$$

Example

- $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$ converges if |x/2|< 1 or |x|<2

- The series diverges if $|x/2| \geq 1$ or $|x| \geq 2$.

Try An Example

What is a power series representation of $\dfrac{x^2}{1+4x^2}$?

A) $\sum_{n=0}^{\infty} (-1)^n 2^{2n} x^{2n+2}$

B) $\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+2}}{2^{2n}}$

C) $\sum_{n=0}^{\infty} 2^{2n} x^{2n+2}$

D) $\sum_{n=0}^{\infty} \dfrac{x^{2n+2}}{2^{2n}}$

E) $\sum_{n=0}^{\infty} 2^n x^{n+2}$

F) None of the above

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D) $\sum_{n=0}^{\infty} \dfrac{x^{2n+2}}{2^{2n}}$

E) $\sum_{n=0}^{\infty} 2^n x^{n+2}$

F) None of the above

An Example Continued

What is the interval of convergence of

$$\sum_{n=0}^{\infty} (-1)^n 2^{2n} x^{2n+2} \text{ ?}$$

A) (-4, 4)

B) (-1/4,1/4)

C) (-1/2,1/2)

D) (-2,2)

E) [-1/2,1/2]

F) None of the above

An Example Continued

What is the interval of convergence of

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F) None of the above

Differentiate and Integrate

- If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence of R

- Then the function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

is differentiable (and therefore continuous) on

(a-R, a+R)

Differentiate and Integrate

- Further
- (i) $f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$
- (ii) $\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + \ldots$
- And the radius of convergence of both is R

Differentiate and Integrate

- We can write these two equations as
- (i) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n (x-a)^n\right] = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$
- (ii) $$\int\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty} \int\left[c_n (x-a)^n\right] dx = C + \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n}$$

Example

- Lets find a power series representative of $-\ln|1-x|$

$$
\begin{aligned}
-\ln|1-x| &= \int \frac{1}{1-x}\,dx \\
&= \int \sum_{n=0}^{\infty} x^n\,dx \\
&= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\
&= C + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots
\end{aligned}
$$

Example

- To determine the value of C notice that if
- x=0 then $-\ln|1-x|=0$ and so

$$0=-\ln|1-x|=C+0+\frac{0^2}{2}+\frac{0^3}{3}+\ldots=C$$

- Hence $-\ln|1-x| \quad = \quad x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots$
- When the series converges.

Example Continued

- The radius of convergence is 1 because the radius of convergence of $1 + x + x^2 + \ldots$ is 1.

Notice though that the interval of convergence is [-1,1)

Whereas the interval of convergence of $1 + x + x^2 + \ldots$ is (-1, 1).

Example

- Lets find a power series representative of

$\arctan(x)$

$$\begin{aligned}
\arctan(x) &= \int \frac{1}{1+x^2}\,dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\
&= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots
\end{aligned}$$

Example Continued

- But arctan(0) = 0 so C = 0 and

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

- We also have though that arctan(1) = π/4 so

$$\pi / 4 = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

Try An Example

What is a power series representation of $\dfrac{1}{(1-x)^2}$?

A) $\sum_{n=0}^{\infty} x^{2n}$

B) $\sum_{n=0}^{\infty} \dfrac{x^{2n}}{2^n}$

C) $\sum_{n=0}^{\infty} nx^n$

D) $\sum_{n=0}^{\infty} (n+1)x^n$

E) $\sum_{n=0}^{\infty} (n-1)x^n$

F) None of the above

Try An Example

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