1

Prime factors of 210 are $\{2, 3, 5, 7\}$. None of these factors divide 1573. This means the greatest common divisor of 210 and 1573 must be 1.

2

We see that $d = gcd(210, 1573) = 1$. We would like to find $a, b \in \mathbf{Z}$ such that $210a + 1573b = 1$. Euclid's algorithm involves finding 1 in terms of 210 and 1573 by utilizing their remainders by the following manner:

$$\begin{aligned} 103 &= 1573 - 7 \times 210 && (1) \\ 4 &= 210 - 2 \times 103 && (2) \\ 3 &= 103 - 25 \times 4 && (3) \\ 1 &= 4 - 3 && (4) \end{aligned}$$

In the above equations (starting at the bottom row and going up), we know what 1 is in terms of 4 and 3. Then we know what 3 is in terms of 4 and 103. We know what 4 is in terms of 210 and 103. Finally we know 103 in terms of 1573 and 210. Because we know all this information, we can find 1 in terms of 210 and 1573.

$$\begin{aligned} 1 &= 4 - 3 && (5) \\ &= 4 - (103 - 25 \times 4) && (6) \\ &= 4 + 25 \times 4 - 103 && (7) \\ &= 26 \times 4 - 103 && (8) \end{aligned}$$

Above, we replace 3 with what we found 3 to equal in terms of 103 and 4; and then we simplify by rearranging the terms.

$$\begin{aligned} 1 &= 26 \times (210 - 2 \times 103) - 103 && (9) \\ &= 26 \times 210 - 52 \times 103 - 103 && (10) \\ &= 26 \times 210 - 53 \times 103 && (11) \end{aligned}$$

Above we replace 4 with what we found 4 to equal in terms of 210 and 103; and then we simplify by rearranging the terms.

$$\begin{aligned} 1 &= 26 \times 210 - 53 \times (1573 - 7 \times 210) && (12) \\ &= 26 \times 210 + 371 \times 210 - 53 \times 1573 && (13) \\ &= 397 \times 210 - 53 \times 1573 && (14) \end{aligned}$$

Finally we replace 103 with what we found 103 to equal in terms of 210 and 1573 and simplify to get the final answer of $a = 397$ and $b = -53$.

3

Remember that the function $\phi(n)$ is a function that tells us how many numbers k are relatively prime to n such that $k \in \{1, 2, 3, ..., n-1\}$.

Part a: $\phi(97) = 96$ because 97 is a prime number. Because 97 is a prime number, it is relatively prime to every number below it, i.e. the numbers $1, 2, 3, ..., 96$ are all relatively prime to 97.

Part b: $\phi(143) = 120$. Note that $143 = 11 \times 13$. Also remember that $\phi(k \times n) = \phi(k) \times \phi(n)$. Having taken this into account and, because 11 and 13 are both prime, $\phi(11) \times \phi(13) = 10 \times 12 = 120$

Part c: $\phi(504) = 226$ because $504 = 2 \times 227$. Since both 2 and 227 are prime, the above explanation applies to this case as well.

4

We have $n = 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^2 \cdot 11^{225}$. Therefore $k_0 = 3$ which means the function is a composition of functions, and it looks like $f(g(\bullet))$. Then $k_1 = 2$ implies the inner function takes 2 variables; and $k_2 = 1$ implies the outer function takes 1 variable.

Then $F(2)$ is the outer function f. The outer function is the successor function since, here, we're raising 2 to the first power; and as such, its k_0 value is 1. So this function is $f(x) = x + 1$.

Finally $F(225)$ is the first (and in this case, only) inner function. $225 = 2^0 \cdot 3^2 \cdot 5^2$. So the inner function has, as its k_0 value, 0 - which means the inner function is a constant. k_1 is 2 and k_2 is 2 so the inner function takes two variables and returns the constant 2; i.e. it looks like $g(x_1, x_2) = 2$

So the complete function looks like: $h(x_1, x_2) = f(g(x_1, x_2)) = f(2) = 3$

5

The following solution was written by Professor Ackerman:

We know $n = 2^{k_0} \times 3^{k_1} \times ...$

Now because the outermost operation is primitive recursion we know that $k_0 = 4$.

The next piece of information we need (which is encoded by k_1) is the number of variables of the base function. So $k_1 = 0$. This is because the base function always takes in 0 as its input and, as such, is not a variable.

The third piece of information we need (k_2) is a number encoding the base function. Looking at the definition of Pred we see that this is the number encoding the constant function 0 with 0 arguments. Looking at the table we see that the exponent of 2 in a constant function is 0. Similarly the exponent of 3 is the number of variables, which is this case is 0. And lastly the exponent of 5 is the value of the constant function, which is also 0. So a number encoding the function is $2^0 3^0 5^0 = 1 = k_2$

Notice however that there are lots of numbers which encode this function (e.g. 7 also encodes the function because the encoding for constant functions doesnt talk about the exponent of 7).

The last piece of information we need (which is encoded by k_3) is the iterating function. Looking at the definition we see that this function is the function which takes two variables and projects onto the second one.

Looking at the table we see that the exponent of 2 in a projection function is 2. We then also see that the exponent of 3 in the encoding is the number of variables. In this case 2. And similarly the exponent of 5 is the variable projected onto, which is also 2. So the number encoding the iterative function is $2^2 \times 3^2 \times 5^2 = 900 = k_3$

So the entire number is $2^4 \times 3^0 \times 5^1 \times 7^{900}$