# (Math 371) Midterm:

Due October 23, 2007

Problem 1: [21 points]

(a) [14 points] Show that a finite abelian group $G$ is cyclic if and only if for every prime $p$ dividing $|G|$ there is exactly one subgroup of $G$ of order $p$.

(b) [7 points] Let $G$ be a finite abelian group which is not cyclic. Show that there is a prime $p$ dividing $|G|$ and subgroups $C_1, C_2$ of $G$ such that $|C_1| = |C_2| = p$ and $G$ contains a subgroup $X$ isomorphic to $C_1 \times C_2$ with $C_1 \cup C_2 \subseteq X$.

Problem 2: [19 Points] Let $G$ be a finite group of order $pq$ where $p, q$ are primes, $p < q$ and $q \not\equiv 1 \mod p$. Prove that $G$ is abelian and cyclic.

Exercise 3: [20 Points] Let $R$ be a ring and let $f, g$ be polynomials in $R[x]$. Assume that the leading coefficient of $f$ is a unit in $R$. Then show that there are unique $q(x), r(x) \in R[x]$ such that

$$g(x) = f(x)q(x) + r(x)$$

and $deg(r) < deg(f)$ or $r = 0$.

Exercise 4: Let $T$ be a symmetric linear operator on a finite dimensional real vector space $V$ such that $V$ has a basis consisting of eigenvectors for $T$ none of whose eigenvalues are 0.

(a) [4 Points] Show there is a positive definite bilinear form $\langle , \rangle$ on $V$ and a basis $B$ of $V$ such that

* $B$ is orthonormal with respect to $\langle , \rangle$
* Every element of $B$ is an eigenvector of $T$.

(b) [4 Points] Define linear operators $T^0 = Id$ and $T^{n+1} = T \circ T^n$. Also define linear operators

* $E_0(T) = Id$
* $E_{n+1}(T) = E_n(T) + T^{n+1}/(n+1)!$

So $E_n(T) = \sum_{i=0}^{n} T^n/n!$

Show $E_n(T)$ is a symmetric linear operator for every $n$.

(c) [6 Points] Let $A_n$ be the matrix associated to $E_n(T)$ relative to some orthonormal basis. Show that there exists $n$ such that $(\forall m > n)A_m$ is positive definite.

(d) [6 Points] Show that $n$ is independent of the orthonormal basis.

In fact we can define $\lim_{n\to\infty} E_n(T)$ which we usually write as $e^T$.

Problem 5: Let $V$ be the space of continuous complex-valued functions on the unit circle in the complex plane and for $f, g \in V$ define

$$\langle f, g \rangle = \int_0^{2\pi} \overline{f(\theta)} g(\theta) d\theta$$

(a) [6 Points] Show that this is a hermitian and positive definite form.

(b) [4 Points]Show that $T = i\frac{d}{d\theta}$ is a hermitian operator on $V$

(c) [10 Points] Let $W$ be the subspace of $V$ of functions $f(e^{i\theta})$ where $f$ is a polynomial of degree $\leq n$. Find an orthonormal basis for $W$.

(Extra Credit 1) [5 Points] Consider the following paradox:

Suppose $A$ has a right inverse $B$. So $AB = Id$. We then have $A^tAB = A^t$ or equivalently $B = (A^tA)^{-1}A^t$. But this satisfies $BA = (A^tA)^{-1}(A^tA) = Id$ and hence $B$ is also a left inverse.

What is the mistake in the reasoning?