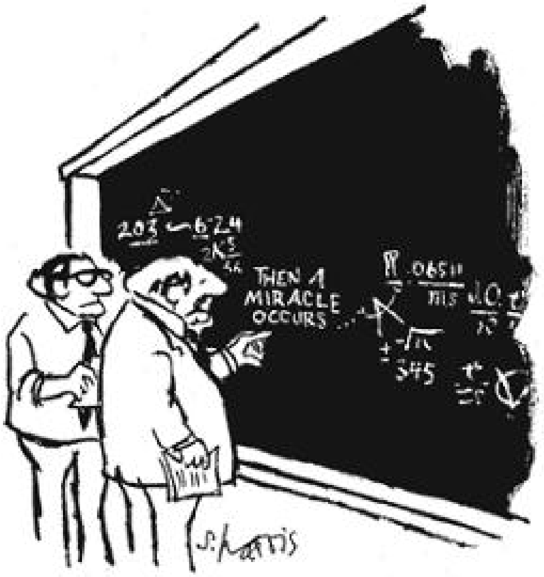
Final (Math 371):

Fall 2007



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Exams will be due in class on Thursday December 6. The exam is out of 100 points and the number of points a problem is worth is written next to the problem number. However, it will not be possible to get more than 100/100 points. You are allowed to use the text book Algebra by Michael Artin (1991) as well as the lecture notes from the course web page. However you are not allowed to refer to any other sources (including fellow students) for this exam. Don't forget to put your name on your exam and good luck!

Problem 1: An element of a ring S is called *idempotent* if $e^2 = e$. Note that in the product $R \times R'$ of rings R, R', $(0,1), (1,0)$ are idempotents. In this problem you will show the converse.

(2 pt) Part (a) Prove that if e is idempotent then $e' = 1 - e$ is also an idempotent.

(5 pt) Part (b) Let e be an idempotent of a ring S prove that the principle ideal (e) is a ring with identity element e.

(3 pt) Part (c) Give an example of a ring S and an idempotent e such that (e) is not a subring of S.

(5 pt) Part (d) Let $e \in S$ be an idempotent and let $e' = 1 - e$. Prove that S is isomorphic (as a ring) to the product ring $(e) \times (e')$.

Problem 2: R is a noncommutative ring if it satisfies all the axioms of a commutative ring except for $(\forall a, b) a \times b = b \times a$. If R is a non-commutative ring then an *ideal* of R is any set $I \subseteq R$ such that

- $(\forall r \in R)(\forall i \in I) r \times i \in I$ and $i \times r \in I$
- $(\forall a, b \in I) a + b \in I$

Let M_n be the non-commutative ring of $n \times n$ real matrixes.

(6 pt) Part (a) Show that if I is a non-zero ideal in M_n then I contains a symmetric matrix.

(5 pt) Part (b) Show that if I is a non-zero ideal then $I = M_n$ (so there are no proper ideals).

Problem 3: A subset of an integral domain R which is closed under multiplication and which does not contain 0 is called a multiplicative set.

Given a multiplicative set S we define S-fractions to be elements of the form a/b where $b \in S$ and we say $a/b \sim c/d$ if $ad = bc$. Let $R_S = \{a/b : a \in R, b \in S\}/\sim$.

Let R be a ring, F it's field of fractions and let S be a multiplicative set.

(4 pt) Part (a) Let P be a prime ideal. Show that $R - P$ is a multiplicative set.

(6 pt) Part (b) Determine all the maximal ideals of R_{R-P}.

Problem 4: Let R be a commutative ring. An element $x \in R$ is said to be nilpotent if for some $n \in \mathbb{Z}$, $x^n = 0$.

(5 pt) Part (a) Let $N = \{x \in R : x \text{ is nilpotent}\}$. Prove that N is an ideal and that R/N has no non-zero nilpotent elements.

(5 pt) Part (b) If R is a ring of characteristic p and x is nilpotent, prove that $1+x$ is unipotent. I.e. that for some $n \in \mathbb{Z}$ $(1+x)^n = 1$.

Problem 5: Let F be a field of characteristic 0. A primitive nth root of unity over F is an element ζ (of a field K) such that

– $\zeta^n - 1 = 0$.

– $(\forall m < n)\zeta^m \neq 1$

(3 pt) Part (a) Show that γ is a primitive nth root of unity (in K) if and only if $\gamma = \zeta^r$ for some $r > 0$ where r and n are relatively prime.

(4 pt) Part (b) Let $K = F(\zeta)$. Show that K is a Galois extension of F and that the Galois group $G(K/F)$ is commutative.

(2 pt) Part (c) If n is prime, what is $|G(K/F)|$ (Hint: Recall that if n is prime and $0 < a < n$ then there is a $0 < b < n$ such that $a \times b = 1 \mod n$)

For parts (d), (e) let n be a prime number let $a \in F$ be some number which is not a power of n and let $f(x) = x^n - a$.

(2 pt) Part (d) Show that the splitting field of $f(x)$ over F is $K = F(\zeta, \alpha)$ where α is any element of K such that $f(\alpha) = 0$.

(5 pt) Part (e) Show that $[K : F] = n(n-1)$

(8 pt) Problem 6: Let F be a field and let $f(x) \in F[x]$ be of degree n. If K is the splitting field of f then show $[K : F]$ divides $n!$.

(5 pt) Problem 7: Show that for any field extensions K/F of finite degree (of any characteristic) $|G(K/F)| \leq [K : F]$.

Problem 8: Let K/F be a field extension of characteristic $p \neq 0$. Let $f(x) = x^p - x + a$ be an irreducible polynomial over F and let $\alpha \in K$ be a root of $f(x)$.

(3 pt) Part (a) Prove that $\alpha + 1$ is also a root of $f(x)$.

(7 pt) Part (b) Let $F \subset L \subset K$ with L a splitting field of $f(x)$ over F. Prove that $G(L/F)$ is a cyclic group of order p.

Problem 9: Let K be a field of characteristic $p \neq 0$.

(3 pt) Part (a) Prove that the *Frobenius* map φ defined by $\varphi(x) = x^p$ is a homomorphism from K to itself.

(4 pt) Part (b) Prove that if K is a finite field then φ is an isomorphism.

(2 pt) Part (c) Give an example of an infinite field of characteristic p such that φ is not an isomorphism.

(5 pt) Part (d) Let $K = \boldsymbol{F}_q$ where $q = p^r$ and let $F = \boldsymbol{F}_p$. Prove $G(K/F)$ is a cyclic group of order r generated by φ.

(5 pt)Extra Credit 1: Write up a proof of the fundamental theorem of algebra

Theorem 0.0.0.1. *For all $f(x) \in \mathbb{C}[x]$ there is an $\alpha \in \mathbb{C}$ such that $f(\alpha) = 0$*

without using any references.

To clarify, there is a proof in the book (and in the lecture notes) and you are allowed to look at that proof and even memorize it. However when you go to write it up you should close the book and put aside all references while you are doing it.

(10 pt)Extra Credit 2: (Much Harder!)

Come up with a proof of the fundamental theorem of algebra which is fundamentally different from the one in the book and lecture notes (note you are NOT allowed to look in other books to do this)