Practice Midterm (Math 371):

Fall 2006

<u>Problem 1:</u> **Show that $c : G \times G \to G$ given by $c(g, x) = xg^{-1}$ is group action of G on itself.**

We know that

$$(\forall x \in G)c(e, x) = xe^{-1} = x$$

Similarly we have

$$(\forall g, g' \in G)c(gh, x) = x(gh)^{-1} = xh^{-1}g^{-1} = c(h, x)g^{-1} = c(g, c(h, x))$$

So we have c is a group action.

<u>Problem 2:</u> **Let $(A, +, 0)$ be a finitely generated Abelian group with no torsion elements. Show that the map $\varphi_d(x) : A \to A$ which is defined by $\varphi(x) = dx$ is an injective homomorphism for each $d \in \{1, 2, \dots\}$**

First note that

$$\varphi_d(0) = d0 = 0$$

$$\begin{aligned}
(\forall a, b \in A)\varphi_d(a + b) &= d(a + b) \\
&= (a + b) + \cdots + (a + b) \\
&= (a + \cdots + a) + (b + \cdots + b) \\
&= da + db \\
&= \varphi_d(a) + \varphi_d(b)
\end{aligned}$$

where each $x + \cdots + x$ represents x being added to itself d times and the middle equality is because our group is Abelian.
We then further have

$$(\forall a \in A)\varphi_d(-a) + \varphi_d(a) = d(-a) + d(a) = d(-a + a) = d0 = 0$$

and so $\varphi_d(-a) = -\varphi_d(a)$ and hence φ_d is a homomorphism.

Lets let $a, b \in A$ such that $\varphi_d(a) = \varphi_d(b)$. We then have $\varphi_d(a) - \varphi_d(b) = da - db = d(a - b)$. Hence, either $a - b$ is a torsion element of order dividing d or $a - b = 0$. And as we have no torsion elements we must have $a - b = 0$ or equivalently $a = b$. So, as a, b were arbitrary φ_d is injective.

Problem 3: **Let $\circ : G \times X \to X$ be a group action. Show that $H = \{g \in G : (\forall s \in X) g \circ s = s\}$ is a normal subgroup of G.**
First note that

$$(\forall s \in X) e \circ s = s$$

$$(\forall s \in X)(\forall g, h \in H) gh \circ s = g \circ (h \circ s) = g \circ s = s$$

and

$$(\forall s \in X)(\forall g \in H) s = e \circ s = g \circ g^{-1} \circ s = g^{-1} \circ s$$

where the last equality is because g fixes all elements of X and so in particular fixes $g^{-1} \circ s$. Hence we see that H is in fact a subgroup of G.

Now lets let $g \in G$ and $h \in H$. In order to show that H is normal we need to show that $ghg^{-1} \in H$. Or more specifically

$$(\forall s \in X)(ghg^{-1}) \circ s = s$$

Lets let $s \in X$ and let $t = (ghg^{-1}) \circ s$. We then have $g^{-1} \circ t = hg^{-1} \circ s$. But $h \in H$ so $hg^{-1} \circ s = h \circ (g^{-1} \circ s) = g^{-1} \circ s$. So we have

$$g^{-1} \circ t = g^{-1} \circ s$$

and hence $t = ghg^{-1} \circ s = s$. And, because $s \in X$ was arbitrary we have

$$(\forall s \in X)(ghg^{-1}) \circ s = s$$

and hence $ghg^{-1} \in H$. And because $g \in G$ was arbitrary this means that $gHg^{-1} = H$.

Problem 4: **Let N be a normal subgroup of G such that N is also a subset of a Sylow p-subgroup H of G. Show that N is a subset of every Sylow p-Group of G.**

Let H^* be a Sylow p-subgroup of G. Then $H = gHg^{-1}$ because by the second Sylow theorem all Sylow p-subgroups of G are conjugate. But then $N = gNg^{-1} \subseteq gHg^{-1} = H^*$ because N is normal. Hence N is contained in every Sylow p subgroup of G (because H was arbitrary).

Problem 5: **Find the orthonormal basis for $\mathbb{R}^3$ gotten by applying the Gram-Schmidt procedure to the basis vectors**

$$v_1 = [1, 2, 2]$$

$$v_2 = [1, 4, 0]$$

$$v_3 = [9, 0, 0]$$

First we need to normalize v_1. We see that $\langle v_1, v_1 \rangle = 1^2 + 2^2 + 2^2 = 1+4+4 = 9$. So in order to normalize v_1 we need to let $w_1 = 1/\sqrt{9}v_1 = 1/3v_1 = (1/3)[1, 2, 2]$.

In order to find a vector which is orthogonal to v_2 we use the Gram-Schmidt formula and see that

$$w = v_2 - \langle v_2, v_1 \rangle v_1$$

is orthogonal to v_1. In particular we have

$$\begin{aligned} w &= [1, 4, 0] - \langle v_2, w_1 \rangle [\frac{1}{3}, \frac{2}{3}, \frac{2}{3}] \\ &= [1, 4, 0] - (1/3 * 9)[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}] \\ &= [1, 4, 0] - [1, 2, 2] \\ &= [0, 2, -2] \end{aligned}$$

Then we must normalize w to get w_2. But we see $\langle w, w \rangle = 8$ and so $w_2 = 1/\sqrt{8}[0, 2, -2] = 1/\sqrt{2}[0, 1, -1] = [0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]$

Now our last step is to find a vector we need to find a vector orthogonal to both w_1 and w_2. So once again using Gram-Schmidt we see that the vector is

$$
\begin{aligned}
w' &= [9,0,0] - \langle v_3, w_1\rangle[\frac{1}{3},\frac{2}{3},\frac{2}{3}] - \langle v_3, w_2\rangle(1/\sqrt{2})[0,1,-1] \\
&= [9,0,0] - 3*[\frac{1}{3},\frac{2}{3},\frac{2}{3}] - 0*[0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}] \\
&= [9,0,0] - [1,2,2] \\
&= [8,-2,-2]
\end{aligned}
$$

And so all that is left is to normalize w' to get w_3. But we see that $\langle w', w'\rangle = 72$ and so $w_3 = (1/\sqrt{72})[8,-2,-2] = (1/6\sqrt{2})[8,-2,-2] = [\frac{2\sqrt{2}}{3}, -\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}]$ So the three orthonormal vectors gotten by the Gram-Schmidt method are

$$w_1 = [\frac{1}{3},\frac{2}{3},\frac{2}{3}]$$

$$w_2 = [0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}]$$

$$w_3 = [\frac{2\sqrt{2}}{3},-\frac{\sqrt{2}}{6},-\frac{\sqrt{2}}{6}]$$

Problem 6: **Let $\langle,\rangle$ be a Hermitian form on a complex vector space V. Let $\{v,w\}$ denote the real part of $\langle v,w\rangle$. Prove that if we consider V as a real vector space then $\{v,w\}$ is a symmetric bilinear form and if $\langle,\rangle$ is positive definite then $\{,\}$ is positive definite.**

Note that for all complex numbers a,b $Real(a+b) = Real(a) + Real(b)$.

Hence

$$\begin{aligned}(\forall v, w_1, w_2 \in V)\{v, w_1 + w_2\} &= Real(\langle v, w_1 + w_2\rangle) \\ &= Real(\langle v, w_1\rangle + \langle v, w_2\rangle) \\ &= \{v, w_1\} + \{v, w_2\}\end{aligned}$$

$$\begin{aligned}(\forall v_1, v_2, w \in V)\{v_1 + v_2, w\} &= Real(\langle v_1 + v_2, w\rangle) \\ &= Real(\langle v_1, w\rangle + \langle v_2, w\rangle) \\ &= \{v_1, w\} + \{v_2, w\}\end{aligned}$$

$$\begin{aligned}(\forall v, w \in V)(\forall c \in \mathbb{R})\{v, cw\} &= Real(\langle v, cw\rangle) \\ &= Real(c\langle v, w\rangle) \\ &= c\{v, w\} \\ &= Real(\overline{c}\langle v, w\rangle) \\ &= Real(\langle cv, w\rangle) \\ &= \{cv, w\}\end{aligned}$$

With the second to last equality because $\overline{c} = c$ if $c \in \mathbb{R}$. So in particular $\{,\}$ is a bilinear form.

To see that $\{,\}$ is symmetric observe that

$$\{v, w\} = Real(\langle v, w\rangle) = Real(\overline{\langle v, w\rangle}) = Real(\langle w, v\rangle = \{w, v\}$$

And to see that $\{,\}$ is positive definite if $\langle,\rangle$ is positive definite observe that $\langle,\rangle$ is positive definite if and only if $\langle v, v\rangle \in \mathbb{R}$ and $\langle v, v\rangle > 0$ for all $v \neq 0$. But as a real vector space $v \neq 0$ iff and only if $v \neq 0$ as a complex vector space. Therefore if $\langle , \rangle$ is positive definite $\{v, v\} > 0$ for all $v \neq 0$ and hence is also positive definite.

Problem 7: **Let V be a finite dimensional complex vector space with $\langle , \rangle$ a positive definite hermitian form on V. Let B be a basis for V such that $\langle , \rangle$ corresponds to the standard hermitian dot product (i.e. $\langle v, w\rangle = (B_v)^* B_w$). Let $P \in GL_n(\mathbb{C})$ be a change of basis matrix from B' to a basis B. Show that $\langle v, w\rangle = (B'_v)^* B'_w$ if and only if P is unitary.**

We know that $PB'_v = B_v$ and $PB'_w = B_w$. So in particular we have

$$\langle v, w\rangle = (B_v)^* B_w = (PB'_v)^*(PB'_w) = (B'_v)^* P^* P B'_w$$

Hence $\langle v, w\rangle = (B'_v)^* B'_w$ if and only if $P^*P = Id$ which is exactly what it means for P to be unitary.

Problem 8: **Show that if M is a real normal $n \times n$ matrix and O is any $n \times n$ orthogonal matrix then M conjugated by O is a real normal $n \times n$ matrix.**

We need to show that if $MM^* = M^*M$ (where M is real) and $OO^t = Id$ then $NN^* = N^*N$ where $N = OMO^{-1}$. Now the first thing to notice is that as we are dealing with real matrixes $\overline{M} = M$ and $\overline{O} = O$ so $M^* = M^t$ and similarly $N^* = N^t$. And in particular we know that M being normal is equivalent to $MM^t = M^tM$ and what we need to show is $NN^t = N^tN$. But we have

$$\begin{aligned} NN^t &= (OMO^{-1})(OMO^{-1})^t \\ &= OMO^{-1}(O^{-1})^t M^t O^t \\ &= OMM^tO^t \ (*) \\ &= OM^tMO^t \ (+) \\ &= (O^{-1})^t M^t O^t)(OMO^{-1}) \ (*) \\ &= (OMO^{-1})^t (OMO^{-1}) \\ &= N^tN \end{aligned}$$

where the steps marked by $(*)$ are due to the fact that $O^t = O^{-1}$ (by the definition of orthogonality) and steps marked by $(+)$ are because M is normal. Hence we have shown that N is normal.