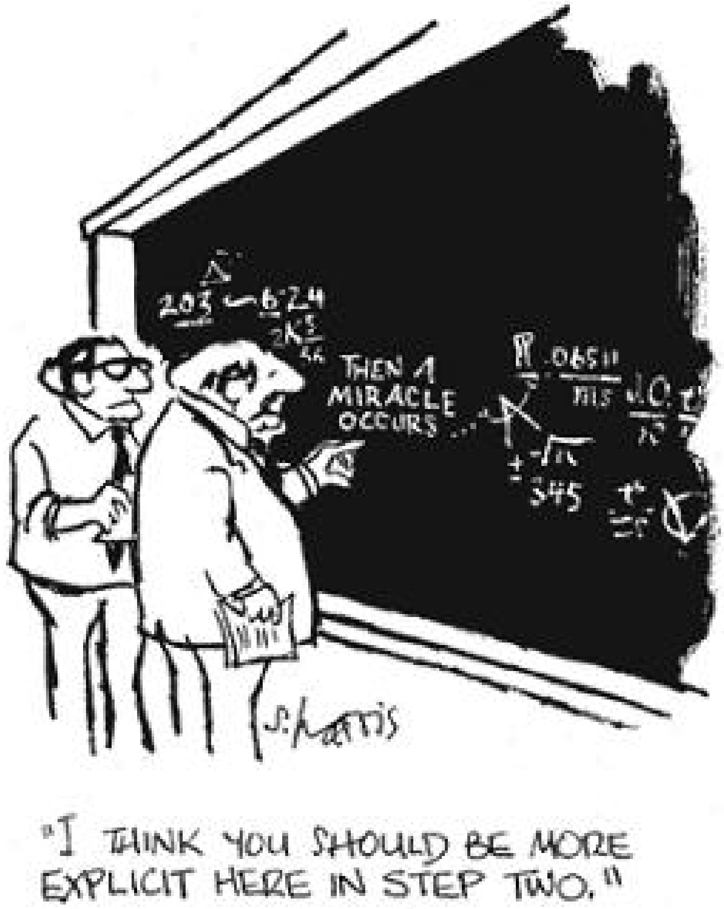
Midterm (Math 371):



You have 90 minutes to complete this exam. The exam is out of 100 points and the number of points a problem is worth is written next to the problem number. However, it will not be possible to get more than 100/100 points. Don't forget to put your name on your paper and good luck!

(8 pt) Problem 1: Let $\circ : G \times X \to X$ be a group action. Show that

$$s \sim s' \text{ if and only if } (\exists g \in G)\ s = g \circ s'$$

is an equivalence relation on X. In other words show

(Reflexivity) $(\forall s \in X)\ s \sim s$

(Symmetry) $(\forall s, t \in X)\ s \sim t \to t \sim s$

(Transitivity) $(\forall r, s, t \in X)\ [r \sim s \text{ and } s \sim t] \to r \sim t$

(12 pt) Problem 2: Let H be a Sylow p-subgroup of a finite group G. Show that gHg^{-1} is a Sylow p-subgroup for each $g \in G$.

(20 pt) Problem 3: Let A be an Abelian group of order $p_1 p_2 \dots p_n$ where p_i are distinct primes. Then prove $A \cong \mathbb{Z}/(p_1) \oplus \mathbb{Z}/(p_2) \oplus \mathbb{Z}/(p_n)$

(10 pt) Problem 4: Prove that if $(G, \cdot, e)$ is an Abelian group then for all $n \in \mathbb{Z}$

$$c_n : G \times G \to G \text{ with } c_n(g, x) = g^n x g^{-n}$$

is a group action of G on itself.

Extra Credit: Let $\varphi : G \to H$ be a group homomorphism and let $\circ : H \times X \to X$ be a group action of H on X.

(5pt) (a) Show that φ induces a group action of G on X.

(5pt) (b) Use part (a) to prove Problem 4.

(15 pt) Problem 5: Let

$$v_1 = [1, 1, 1, 0]$$

$$v_2 = [2, -4, 2, 1]$$

$$v_3 = [0, 3, 6, 25a]$$

and let $V = Span(v_1, v_2, v_3) \subseteq \mathbb{R}^4$. Find a vector $w \in \mathbb{R}^4$ (in terms of $a \in \mathbb{R}$) such that $\{v_1, v_2, w\}$ form an orthogonal basis for V under the standard dot product. You do not need to normalize w and you can assume $\{v_1, v_2, v_3\}$ are linearly independent. (Hint: Use Gram-Schmidt)

(10 pt) Problem 6: Let $(v \cdot w)$ be the standard Hermitian dot product on a finite dimensional complex vector space V relative to an orthonormal basis B. Prove that $(v \cdot Tw)$ is a Hermitian form if and only if T is a Hermitian operator.

Problem 7: Let $V, \langle , \rangle$ be a Hermitian Space and let T be a linear operator on V. Further let B be an orthonormal basis for V and let M be the matrix associated to T. We say that T^* is the adjoint operator to T if the matrix N for T^* relative to B is adjoint to the matrix for T (i.e. $N = M^*$)

(4 pt) (a) Prove that for all linear operators T on V TT^* is a Hermitian operator.

(7 pt) (b) Prove that for any linear operator $(\forall v \in V) TT^*(v) = T^*T(v)$ if and only if $(\forall v, w \in V)\langle Tv, Tw\rangle = \langle T^*v, T^*w\rangle$. We call such an operator normal.

(4 pt) (c) Prove that if T is a normal operator and v is an eigenvector of T with eigenvalue λ then $T^*(v)$ is also an eigenvector of T with eigenvalue λ.

(10 pt) Problem 8: Let $V, \langle , \rangle$ be a finite dimensional Hermitian space. Prove the collection of unitary operators on V form a group under composition.