Midterm 2

Solutions

1)

2

Problem 1: Find the greatest common divisor of 216 and 294.

$\frac{216}{6} = 36$ $\frac{294}{6} = 49$

$6 \cdot 6$ $7 \cdot 7$

$3 \cdot 2 \cdot 3 \cdot 2$

GCD(294, 216)

Apply Euclidean algorithm $G(a,d) = G(d,r)$

where $a = qd + r$

$294 = 1 \cdot 216 + r$

$-216 \quad -216$

$r = 78$

Apply again GCD(294, 216) = GCD(216, 78)

$216 = 2 \cdot 78 + r$

$216 = 156 + r$

$60 = r$

Again

$78 = 1 \cdot 60 + r$ = GCD(78, 60)

$18 = r$

Again $60 = 3 \cdot 18 + r$ = GCD(60, 18)

$6 = r$

Again = GCD(18, 6)

$18 = 3 \cdot 6 + r$

$0 = r$

= GCD(6, 0)

We know GCD(n, 0) = n

so $\boxed{6}$

2)

Problem 2: Suppose the following holds

- $x = p_1^a q_1^b$ where p_1, q_1 are prime and a, b are positive natural numbers.
- $y = p_2^c q_2^d$ where p_2, q_2 are prime and c, d are positive natural numbers.
- $x^2 = y^3$

Show that there is a natural number r such that $r^6 = x^2 = y^3$. Give a prime factorization of r.

$x^2 = p_1^{2a} q_1^{2b}$ $y^3 = p_2^{3c} q_2^{3c}$

$x^2 = y^3$ and we know that the prime factorization of a natural number is unique so $p_1^{2a} q_1^{2b} = p_2^{3c} q_2^{3c}$ where writing it out both sides have the same prime numbers. p_1 is equal either to p_2 or q_2, and q_1 is equal to the other. Because there must be a multiple of 2 p_1's and a multiple of 3 p_2's or q_2's in x^2 and those are equal, there are a multiple of LCM (3,2) p_1's and q_2 or p_2's in x^2.

$LCM(3,2) = \frac{3 \cdot 2}{GCD(3,2)}$ by homework $= \frac{6}{1} = 6$

So there are a multiple of 6 p_1's in prime factorization of x^2. Same process to prove that for q_1's. So $x^2 = p_1^{2a} q_1^{2b} = p_1^{6e} q_1^{6f}$ where $e, f \in$ positive natural #'s, so $r^6 = x^2$ means

where $x^2 = r^6 = p_1^{6e} q_1^{6f}$ $r = p_1^e q_1^f$ ✓

3)

Problem 3: For each of the following either prove it is true or give a counterexample. Remember, "proof by picture" is not enough!

(a) The intersection of two non-convex sets is always non-convex.

(b) The intersection of two non-convex sets is always convex.

(c) The intersection of a convex set and a non-convex sets is always non-convex.

(d) The intersection of a convex set and a non-convex sets is always convex.

together are not convex

a) False. take 2 parallel lines their intersection is $\emptyset$ which is convex.

b) False. Consider A_1, A_2 non convex and $A_1 \subseteq A_2$. then $A_1 \cap A_2 = A_1$ not convex

c) False let A_1 be convex, A_2 not be convex and $A_1 \subseteq A_2$ then $A_1 \cap A_2 = A_1$ convex.

d) False let A_1 be convex A_2 not be convex. $A_2 \subseteq A_1$ then $A_1 \cap A_2 = A_2$ not convex

4)

Problem 4: Consider the following transformation of the plane.

$$F(x, y) = (x^3, x + y)$$

Is F bijective? Is F injective? Is F surjective?

F is injective

let (x_1, y_1) & (x_2, y_2) be distinct points

assume $F(x_1, y_1) = F(x_2, y_2)$

$F(x_1, y_1) = (x_1^3, x_1 + y_1)$

$F(x_2, y_2) = (x_2^3, x_2 + y_2)$

$F(x_1, y_1) = F(x_2, y_2) \Rightarrow x_1^3 = x_2^3$ & $x_1 + y_1 = x_2 + y_2$

$x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ $\quad x_1 + y_1 = x_2 + y_2 \Rightarrow x_1 + y_1 = x_1 + y_2$ as $x_1 = x_2$

$x_1 + y_1 = x_1 + y_2 \Rightarrow y_1 = y_2$ & so as $x_1 = x_2$ & $y_1 = y_2$

$(x_1, y_1) = (x_2, y_2)$ contradiction $\therefore$ F is injective

F is surjective

let (x', y') be $F(x, y)$ for some x, y in Π

$F(x, y) = (x^3, x + y)$

$x' = x^3 \quad y' = x + y \Rightarrow x = \sqrt[3]{x'}$ & $y = y' - \sqrt[3]{x'}$

so for any x', y' in Π $F(\sqrt[3]{x'}, y' - \sqrt[3]{x'}) = (x', y')$

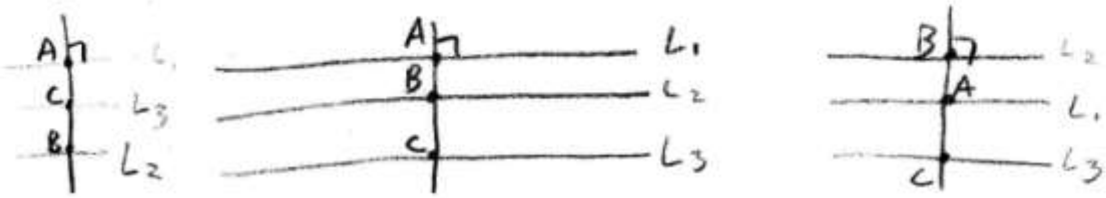
$\Rightarrow F$ is surjective

F injective & surjective $\rightarrow$ F bijective

5)

Problem 5: Suppose L_1, L_2 and L_3 are parallel lines. Suppose D_{12} is the distance from L_1 to L_2, D_{23} is the distance from L_2 to L_3 and D_{31} is the distance from L_3 to L_1. Prove $D_{12} + D_{23} \geq D_{31}$. Assuming $D_i > 0$

Done



Let line ℓ be $\perp$ to L_1. Because $L_1 \parallel L_2$ and $L_1 \parallel L_3 \Rightarrow \ell \perp L_2$, $\ell \perp L_3$.

Let $A = L_1 \cap \ell$; $B = L_2 \cap \ell$, $C = L_3 \cap \ell$.

$\Rightarrow |AB| = D_{12}$; $|AC| = D_{31}$; $|BC| = D_{23}$

$A, B, C \in \ell \Rightarrow A, B, C$ are colinear.

3 cases: ① A is in the middle of B, C.
② B is in the middle
③ C is in the middle

Case A is in the middle:

$$D_{23} = D_{12} + D_{31} \Rightarrow D_{12} + D_{23} = D_{12} + D_{12} + D_{31} \geq D_{31}$$
$$\Rightarrow \boxed{D_{12} + D_{23} \geq D_{31}}$$

Case B is in the middle:

$$\boxed{D_{31} = D_{12} + D_{23}}$$

Case C is in the middle:

$$D_{12} = D_{31} + D_{23} \Rightarrow D_{12} + D_{23} = D_{31} + D_{23} + D_{23} \geq D_{31} \Rightarrow \boxed{D_{12} + D_{23} \geq D_{31}}$$

So in All cases: $D_{12} + D_{23} \geq D_{31}$ /

6)

Problem 6: Suppose D is a dilation and S is a convex set. Prove $D(S)$ is convex.

It's probably easiest to do a contradiction. If $D(S)$ is not convex, $\exists\, A', B', C' \in \Pi$. $A', B' \in D(S)$ and $C' \notin D(S)$.

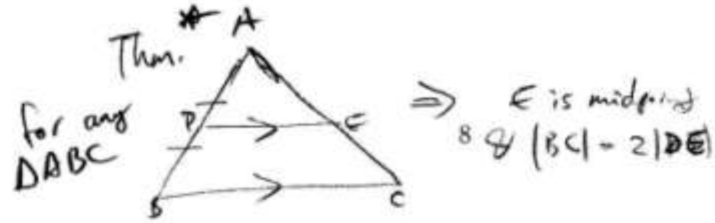
Now, since dilations have inverses, which are also dilations, let A, B, C be pts s.t. $A' \overset{D^{-1}}{\mapsto} A$, etc.

Then, since dilations map segments to segments, $C \in \overline{AB}$, but $C \notin S$, so S is not convex.

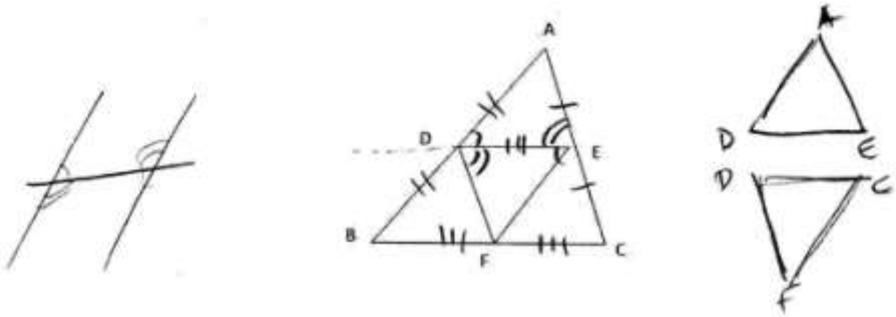
which is $= D^{-1}(D(S))$

7)

10/10



Problem 7: Consider the triangle



where D is the midpoint of AB, E is the midpoint of AC and F is the midpoint of BC. Prove that the triangles ΔADE and ΔFED are congruent (i.e. $\Delta ADE \cong \Delta FED$).

Note you *cannot* use the fact that two triangles with equal sides are congruent (SSS).

Since $|AD| = |BD|$ and $|AE| = |CE|$

$\Rightarrow \dfrac{|AB|}{|AD|} = \dfrac{|AC|}{|AE|} \Rightarrow DE \parallel BC$ by FTS

Similarly, $FE \parallel AB$ by FTS

Then DE intersects 2 parallel lines L_{AB} and L_{FE}

$\Rightarrow |\angle ADE| = |\angle FED|$ and

$|\angle AED| = |\angle FDE|$ by alternate interior angles & opposite angles thm(s)

By ASA $\Rightarrow \Delta ADE \cong FED$

8)

Problem 8: Using the triangle from Problem 7, show that ΔABC is similar to ΔFED (i.e. $\Delta ABC \sim \Delta FED$).

Show $\Delta FED \sim \Delta ABC$

We know $\frac{|AB|}{|AD|} = \frac{|AC|}{|AE|}$

by #7 we know $|AD| = |FE|$ and $|AE| = |\overline{DF}|$

So $\frac{|AB|}{|FE|} = \frac{|AC|}{|DF|}$ and we also know $|\angle F| = |\angle A|$

So by SAS for similarity $\Delta ABC \sim \Delta FED$

9)

10)