Midterm 1

Solutions

1)

$$\frac{b-a}{b} = \frac{b}{b} - \frac{a}{b} = 1 - \frac{a}{b}$$

because $1 < a < b$

we know that $0 < \frac{a}{b} < 1$

So, $1 - \frac{a}{b} = \frac{b-a}{b}$

$$\Rightarrow 0 < \frac{b-a}{b} < 1$$

"1 minus something positive & also less than 1"

$$\frac{b+a}{a} = \frac{b}{a} + \frac{a}{a} = \frac{b}{a} + 1$$

because $1 < a < b$

we know that $\frac{b}{a} > 1$

So, $\frac{b}{a} + 1 = \frac{b+a}{a}$

$$\Rightarrow \frac{b+a}{a} > 1$$

"1 plus something positive & also greater than 1"

So, from least to greatest...

$$\boxed{0 < \frac{b-a}{b} < 1 < \frac{b+a}{a}}$$

2)

$S=\frac{a}{b}$ where $a,b \in \mathbb{Z} \mid a \geq 0, b>0$ definition of fraction

$T=\frac{c}{d}$ where $c,d \in \mathbb{Z} \mid c \geq 0, d>0$ definition of fraction

Let $S<T$

$\underbrace{\text{Their difference}}_{T-S}$ $\underbrace{\text{is}}_{=}$ $\underbrace{1}_{1}$ $\underbrace{\text{more than}}_{+}$ $\underbrace{\tfrac{2}{3}}_{\tfrac{2}{3}}$ $\underbrace{\text{of}}_{\times}$ $\underbrace{S}_{S}$ and $\underbrace{T}_{T}$ $\underbrace{\text{is}}_{=}$ $\underbrace{\tfrac{5}{2}}_{\tfrac{5}{2}}$ $\underbrace{\text{of}}_{\times}$ $\underbrace{S}_{S}$

Rewritten, this is

$T-S=1+\frac{2}{3}\times S$ and $T=\frac{5}{2}\times S$

$T-S+S=1+\frac{2}{3}\times S+S \qquad (a=b \Leftrightarrow a+c=b+c)$

$T=1+\frac{2}{3}\times S+S \qquad (-x+x=0)$

$T=1+\frac{2}{3}\times S+1\times S \qquad (1\cdot x=x)$

$T=1+\frac{2}{3}\times S+\frac{3}{3}\times S \qquad (\frac{k}{k}=1, k\neq 0)$

$T=1+(\frac{2}{3}+\frac{3}{3})\times S \qquad$ Distributive

$T=1+\frac{5}{3}\times S \qquad$ Addition

$\frac{5}{2}\times S=1+\frac{5}{3}\times S \qquad$ Substitution

$\frac{5}{2}\times S-\frac{5}{3}\times S=1+\frac{5}{3}S-\frac{5}{3}S \qquad (a=b \Leftrightarrow a+c=b+c)$

$\frac{5}{2}\times S-\frac{5}{3}\times S=1 \qquad (x-x=0)$

$$\frac{5}{2} \times S - \frac{5}{3} \times S = 1$$

$$\left(\frac{5}{2} - \frac{5}{3}\right) S = 1$$ Distributive

$$\left(\frac{15}{6} - \frac{10}{6}\right) S = 1$$ $\frac{ak}{bk} = \frac{a}{b}$, $k \neq 0$

$$\frac{5}{6} \times S = 1$$ Sub.

$$\frac{6}{5} \times \frac{5}{6} \times S = \frac{6}{5} \times 1$$ $a = b \Rightarrow ac = bc$

$$S = \frac{6}{5}$$ $(x \cdot 1 = x$ & $x \cdot x^{-1} = 1)$

$$T = \frac{5}{2} \times S$$

$$T = \frac{5}{2} \times \frac{6}{5}$$ subsitution

$$T = \frac{\cancel{5} \times (3 \times \cancel{2})}{\cancel{2} \times \cancel{5}}$$ multiplication

$$T = 3 = \frac{3}{1}$$ $\frac{Ka}{Kb} = \frac{a}{b}$, $K \neq 0$

$$\boxed{\text{The fractions are } \frac{6}{5} \text{ and } 3}$$

3)

The cookie receipt is consisted of $4\frac{1}{2}$ pounds of batter or 72 ounces (convert lbs to ounce $4\frac{1}{2}\text{ lbs} \Rightarrow \frac{9}{2}\text{ lbs} \times \frac{16\text{ ounce}}{\text{lbs}} = \frac{144}{2}\text{ ounce} = 72\text{ ounce}$)

Since a cookie takes $3\frac{1}{3}$ ounce batter or $\frac{10}{3}$ ounces batter then we want to divide the 72 ounce of batter into a equal "segments" (or cookie in this case) of weight $3\frac{1}{3}$ ounce.

$$\frac{72 \text{ ounce batter}}{\frac{10}{3} \text{ ounce batter}} = \frac{72}{\frac{10}{3}}$$

$= \frac{72}{10} \times 3$ multiply invert

$= 7.2 \times 3$ conversion fraction to decimal w/ denominator power of 10

$= 21.6$ algorithm to multiply decimals.

in a 72 ounce batter we will have 21 whole cookies, in our calculation we have 21.6, the 0.6 or $\frac{6}{10}$ of the cookie cannot be made because we cannot make part of a cookie. Thus we can only make 21 whole cookies.

4)

(A_0) Since $0^* = 0$, $\boxed{A_0 = 0}$.

(A_1) $A_1 = (0^* + 1)^* = (0 + 1)^*$ (by above)

$= 0^* + 1^*$ since $(a^* = -a)$

$= 0 - 1$

thus $\boxed{A_1 = -1}$

$A_2 = ((0^* + 1)^* + 2)^* = ((0 + 1)^* + 2)^* = (0 + 1^* + 2)^* =$

$= 0^* + 1^{**} + 2^*$

$= 0 + 1 - 2$ (since $a^{**} = a$)

thus $\boxed{A_2 = -1}$ $= -1$

$A_3 = (((0^* + 1)^* + 2)^* + 3)^* = (A_2 + 3)^* = (-1)^* + 3^* = 1 - 3 = -2$

$\underbrace{(0^*+1)^*+2}_{A_2}$

thus $\boxed{A_3 = -2}$

$A_4 = ((((0^* + 1)^* + 2)^* + 3)^* + 4)^* = (A_3 + 4)^* = A_3^* + 4^*$

$= -2^* - 4 = 2 - 4 = -2$

$\boxed{A_4 = -2}$

so ranking $A_1 \to A_4$ from smallest to largest, we see that

$A_4 = A_3 < A_2 = A_1 < A_0$

5)

let V_A & V_C be the constant rates @ which Alan & Christina mow, respectively.

let L be the total area of the lawn.

if it takes $4\frac{1}{3}$ hr for Alan to mow L this can be expressed as

$$L = (4\tfrac{1}{3}\text{hr})V_A \Rightarrow L = \tfrac{13}{3}V_A \rightarrow \tfrac{3L}{13} = V_A$$

if it takes Christina t hrs to mow L we have

$$L = tV_C \Rightarrow \tfrac{L}{t} = V_C$$

When they work together it takes 2 hrs to mow L

so Alan will have mowed:

$$\ell_A = (2\text{hr})V_A = 2\left(\tfrac{3L}{13}\right) = \tfrac{6L}{13}$$

& Christina will have mowed:

$$\ell_C = (2\text{hr})V_C = 2\left(\tfrac{L}{t}\right) = \tfrac{2L}{t}$$

together they mow L so $\ell_A + \ell_C = L$

distribute

$$\frac{6L}{13} + \frac{2L}{t} = L$$

$$\left(\tfrac{6}{13} + \tfrac{2}{t}\right)L = L$$

cancel

$$\frac{6}{13} + \frac{2}{t} = 1$$

common denom

$$\frac{6t + 26}{13t} = 1$$

FFFP

$$6t + 26 = 13t$$

algebra

$$26 = 13t - 6t$$

$$26 = 7t$$

$$\boxed{t = \tfrac{26}{7}\text{hr}}$$

6)

(a) $1.2+0.005 = \frac{12}{10} + \frac{5}{1000}$ by def. of decimal

$= \frac{12\cdot 100}{10\cdot 100} + \frac{5}{1000} = \frac{1200+5}{1000} = \frac{1205}{1000} = \boxed{1.205}$

by FFFP by addition of fractions

(b) $1.2 - 0.005 = \frac{12\cdot 100}{10\cdot 100} - \frac{5}{1000}$ from above but instead of $(+)$, it is $(-)$

$= \frac{1200-5}{1000} = \frac{1195}{1000} = \boxed{1.195}$

(c) $(1.2)(0.005) = \left(\frac{12}{10}\right)\left(\frac{5}{1000}\right)$ by def. of decimal

$= \frac{60}{10000}$ by product rule $= \frac{6\cdot\cancel{10}}{1000\cancel{0}\cdot\cancel{10}} = \boxed{.006}$

(d) $\frac{1.2}{0.005} = \frac{\left(\frac{12}{10}\right)}{\left(\frac{5}{1000}\right)} = \frac{12}{10}\cdot\frac{1000}{5} = \frac{12000}{50} = \frac{12000\cdot 2}{50\cdot 2}$

by product rule

$= \frac{24000}{100} = \boxed{240}$

7)

(a) Let $A,B,C \in \mathbb{Q} \mid A=\frac{1}{2}, B=\frac{1}{3}, C=\frac{3}{2}$

Claim: $\exists A,B,C \in \mathbb{Q}$ s.t. $(A \div B) \div C \neq A \div (B \div C)$

$(A \div B) \div C = \left(\frac{1}{2} \div \frac{1}{3}\right) \div \frac{3}{2} = \left(\frac{1}{2} \times \frac{3}{1}\right) \div \frac{3}{2} = \frac{3}{2} \div \frac{3}{2} = \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} = 1.$

$A \div (B \div C) = \frac{1}{2} \div \left(\frac{1}{3} \div \frac{3}{2}\right) = \frac{1}{2} \div \left(\frac{1}{3} \times \frac{2}{3}\right) = \frac{1}{2} \div \frac{2}{9} = \frac{1}{2} \times \frac{9}{2} = \frac{9}{4} \neq 1$

Therefore $\div$ on $\mathbb{Q}$ is <u>not associative.</u>

Claim: $\exists A,B \in \mathbb{Q}$ s.t. $A \div B \neq B \div A$

Excelent!

$A \div B = \frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$

$B \div A = \frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3} \neq \frac{3}{2} = A \div B$

Therefore $\div$ on $\mathbb{Q}$ is <u>not commutative</u>

(b) Let $A,B \in \mathbb{Q} \mid \frac{A}{B} = \frac{B}{A}$

$\frac{A}{B} = \frac{B}{A} \Leftrightarrow AA = BB \Leftrightarrow A^2 = B^2 \Rightarrow |A| = |B|$

which means sign doesn't matter so $A = \pm B$.

If we want $\dfrac{a}{\frac{b}{c}} = \dfrac{\frac{a}{b}}{\frac{c}{1}}$ Nice!

first invoke $\dfrac{\frac{x}{y}}{\frac{z}{w}} = \dfrac{x \cdot w}{y \cdot z}$ which is true, since $\dfrac{\frac{x}{y}}{\frac{z}{w}} = A$

$\iff \frac{x}{y} = \frac{z}{w} A$ mult by $\frac{z}{w}$

$\iff w \cdot \frac{x}{y} = zA$ mult by w

the expression is undefined if $z = 0$, so assume $z \neq 0$, and divide by it

$\iff \frac{w}{z} \cdot \frac{x}{y} = A$

$$\frac{\frac{a}{1}}{\frac{b}{c}} = \frac{\frac{a}{b}}{\frac{c}{1}}$$

$$\iff \frac{ac}{b} = \frac{a}{bc}$$

$ac = \frac{a}{c}$ mult. by b

<u>Case 1</u> if $a = 0$, then b, c can be anything, $b \neq 0$

else, divide by a

$$c = \frac{1}{c} \iff c^2 = 1$$

<u>Case 2</u> if c is $+1$ or -1, then a, b can be anything

8)

$2\left|\frac{2}{3}x-2\right|+3<4$

$\iff 2\left|\frac{2}{3}x-2\right|+3-3<4-3$ by thm in book

$2\left|\frac{2}{3}x-2\right|<1$

$\iff \frac{1}{2}\cdot 2\left|\frac{2}{3}x-2\right|<1\cdot\frac{1}{2}$ by thm in book

$\left|\frac{2}{3}x-2\right|<\frac{1}{2}$

$\iff -\frac{1}{2}<\frac{2}{3}x-2<\frac{1}{2}$ by thm in book

We will now split the inequality into 2 separate inequalities and solve each one independently

$-\frac{1}{2}<\frac{2}{3}x-2$ and $\frac{2}{3}x-2<\frac{1}{2}$

$\iff 2+-\frac{1}{2}<\frac{2}{3}x-2+2$ (by thm)

$\iff \frac{4-1}{2}<\frac{2}{3}x$ (adding fractions)

$\iff \frac{3}{2}<\frac{2}{3}x$ (simplified)

$\iff \frac{3}{2}\cdot\frac{3}{2}<\frac{2}{3}\cdot\frac{3}{2}x$ (by thm)

$\iff \frac{9}{4}<x$ (multiplying fractions)

$\frac{2}{3}x-2<\frac{1}{2}$

$\frac{2}{3}x-2+2<\frac{1}{2}+2$ (by thm)

$\frac{2}{3}x<\frac{5}{2}$ (adding fractions)

$\frac{3}{2}\cdot\frac{2}{3}x<\frac{5}{2}\cdot\frac{3}{2}$ (by thm)

$x<\frac{15}{4}$ (multiplying fractions)

Combining the inequalities, we have

$$\boxed{\frac{9}{4}<x<\frac{15}{4}}$$