<u>Homework 9</u>

Solutions

1)

55-1 #5

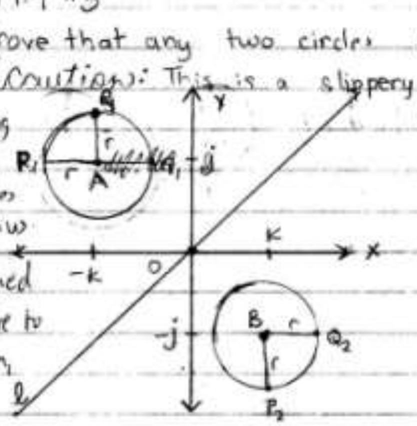
Math 151 Homework 9

(1.5/2) 1. §4.4 #3

Prove that any two circles of the same radius are congruent.

(Caution: This is a slippery proof. Be very precise.)

Let the following image hold for any 2 circles with same radius



Let circle A and circle B be any two circles s.t. they both have the same radius, r. Let φ be the reflection across the diagonal ℓ, $\{x=y\}$. Let the circles and diagonal be on a plane denoted as $\mathbb{R}^2$ with an x-and y-axis.

Since you leaned on your picture to solve the problem, your picture will be graded along with your problem.

By the definition of congruence, we must show that $\varphi(\text{circle } A) = \text{circle } B$ and $\varphi^{-1}(\text{circle } B) = \text{circle } A$. to prove that circle $A \cong$ circle B and circle $B \cong$ circle A. good.

Notice the following:

$$\varphi(A) = B \qquad \varphi^{-1}(B) = A$$
$$|\varphi(\overline{AQ_1})| = |\overline{BQ_2}| = r \qquad |\varphi^{-1}(\overline{BQ_2})| = |\overline{AQ_1}| = r$$
$$|\varphi(\overline{AP_1})| = |\overline{BP_2}| = r \qquad |\varphi^{-1}(\overline{BP_2})| = |\overline{AP_1}| = r$$

Because $\varphi(\overline{AQ_1}) = \overline{BQ_2}$ and $\varphi^{-1}(\overline{BQ_2}) = \overline{AQ_1}$ and $\varphi(A)=B$, $\varphi^{-1}(B)=A$, we can say $\varphi(Q_1) = Q_2$.

Similarly, because $\varphi(\overline{AP_1}) = \overline{BP_2}$ and $\varphi^{-1}(\overline{BP_2}) = \overline{AP_1}$ and $\varphi(A)=B$, $\varphi^{-1}(B)=A$ we can say $\varphi(P_1) = P_2$

This same process can be repeated, without losing generality, because $\varphi(\overline{AP_1}) = |\overline{BP_2}| = \varphi(\overline{AQ}) = |\overline{BQ_2}| = r$. for all points that lie on circle A. Same can be said about φ^{-1} that lie on circle B.

Hence, $\varphi(\text{circle } A) = \text{circle } B$ and $\varphi^{-1}(\text{circle } B) = \text{circle } A$.

which proves congruence by definition. You have the right idea, but it doesn't seem like you fully grasp reflections.

(2) 2. §...

2)

4.4.4 Prove Thm G7 (SAS): Given two triangles ABC and A'B'C' so that $|\angle A| = |\angle A'|$, $|AB| = |A'B'|$, and $|AC| = |A'C'|$. Then the triangles are congruent.

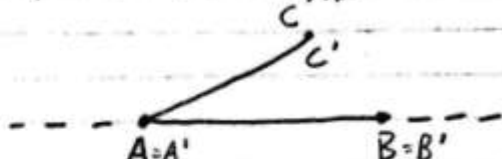
2/2

Consider the following cases
- Case $A = A'$ and $B = B'$
- Case $A = A'$
- The general case.

Case $A = A'$ and $B = B'$

This case has two subcases.
- subcase C and C' are on the same half-plane of L_{AB}.
- subcase C and C' are on opposite half-planes of L_{AB}
- Note: $C \notin L_{AB}$ or else we would not have triangles

- Subcase C and C' are on the same half-plane of L_{AB}

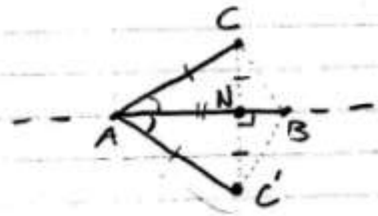


$|\angle CAB| = |\angle C'AB|$ & C and C' are on the same half-pla
$\Rightarrow R_{AC} = R_{AC'}$
$\Rightarrow C, C' \in R_{AC}$ together with $|AC| = |AC'|$
$\Rightarrow C' = C'$.

So $A = A'$, $B = B'$, $C = C'$. Let I be the identity function
$\Rightarrow I(ABC) = A'B'C'$ & I is a basic isometry
$\Rightarrow ABC \cong A'B'C'$

4.4.4 continued.

- Case $A = A'$ and $B = B'$ (continued)
  - subcase C and C' are on opposite half-planes of L_{AB}

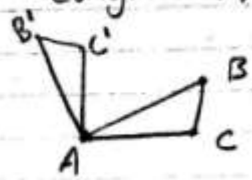


Given
$|\angle CAB| = |\angle C'AB|$
$|AC| = |AC'|$

Let R be the reflection over L_{AB} function.
$|\angle CAB| = |\angle C'AB| \Rightarrow R(C) \in R_{AC'}$
$R(C) \in R_{AC'}$ and $|AC| = |AC'| \Rightarrow R(C) = C'$

$R(A) = A'$; $R(B) = B'$; $R(C) = C'$
$\Rightarrow R(ABC) = A'B'C'$
$\Rightarrow ABC$ and $A'B'C'$ are congruent.

- Case $A = A'$



Let ρ be rotation.
There exists a $\rho(B')$ such that $\rho(B') = B$.

$\rho(A'B'C')$ will make $A' = A$, $B' = B$, at which point we have our previous Case $A = A'$ and $B = B'$.
$\Rightarrow A'B'C'$ and ABC are congruent.

- The general case.
  Let T be translation along $A'A$.
  $T(A') = A$. So $T(A'B'C')$ will put us in our previous case $A = A'$.
  $\Rightarrow A'B'C'$ and ABC are congruent

perfect.

$\Rightarrow A'B'C'$ and ABC are congruent in all cases.

3)

$\frac{2}{3}$

1 2 3

1. Prove $\frac{|AB|}{|AD|} = \frac{|AC|}{|AE|} \iff \frac{|AD|}{|DB|} = \frac{|AE|}{|EC|} \iff \frac{|AB|}{|DB|} = \frac{|AC|}{|EC|}$

$\Rightarrow$ By solving for one length and plugging it into another equation, you derive one of the other equations. This shows that each imply each equation.

From $\frac{|AB|}{|AD|} = \frac{|AC|}{|AE|} \Rightarrow |AD| = \frac{|AB||AE|}{|AC|}$ solving for $|AD|$

$(1) \Rightarrow (3)$

Plugging into $\frac{|AD|}{|DB|} = \frac{|AE|}{|EC|} \Rightarrow \frac{|AB||AE|}{|AC||DB|} = \frac{|AE|}{|EC|}$

by cross multiplication : $|AB||AE||EC| = |AE||AC||DB|$

by cancelation law $|AE|$'s cancel

$|AB||EC| = |AC||DB| \Rightarrow \frac{|AB|}{|DB|} = \frac{|AC|}{|EC|}$

From $\frac{|AD|}{|DB|} = \frac{|AE|}{|EC|} \Rightarrow |DB| = \frac{|AD||EC|}{|AE|}$

$(2) \Rightarrow (1)$

Plug into : $\frac{|AB|}{|DB|} = \frac{|AC|}{|EC|} \Rightarrow \frac{|AB||AE|}{|AD||EC|} = \frac{|AC|}{|EC|}$

by cross multiplication : $|AB||AE||EC| = |AC||AD||EC|$

by cancelation law : $|EC|$'s cancel.

$|AB||AE| = |AC||AD| \Rightarrow \frac{|AB|}{|AD|} = \frac{|AC|}{|AE|}$

From $\frac{|AB|}{|DB|} = \frac{|AC|}{|EC|} \Rightarrow |AB| = \frac{|AC||DB|}{|EC|}$

$(3) \Rightarrow (2)$

Plug into : $\frac{|AB|}{|AD|} = \frac{|AC|}{|AE|} \Rightarrow \frac{|AC||DB|}{|EC||AD|} = \frac{|AC|}{|AE|}$

by cross multiplication $|AC||DB||AE| = |AC||EC||AD|$

by cancelation law $|AC|$'s cancel

good

$|DB||AE| = |EC||AD| \Rightarrow \frac{|AE|}{|EC|} = \frac{|AD|}{|DB|}$

It's traditional – not necessary, but considered good form – to write out what you're proving (like I did on the side). This is helpful to you, too – you can make sure you're including everything in the "equivalence chain".

4)

5.1.4 Given quadrilateral ABCD.

Draw segments AC and DB.
Since H and E are midpoints of AB and AD, by G15 on $\triangle ADB$, we have $HE \parallel DB$.
Since G and F are midpoints of DB, then by G15 on $\triangle DCB$, we have $GF \parallel DB$.
By a previous hw, $HE \parallel DB$ and $GF \parallel DB \Rightarrow HE \parallel GF$.
Also note that $HE \neq GF$ b/c endpoints are distinct.

Similarly, by G15 on $\triangle ABC$, $EF \parallel AC$.
Applying G15 again on $\triangle ADC$, $HG \parallel AC$.
So $EF \parallel HG$, and $EF \neq HG$.
EF, HG, HE, GF are segments connected at each other's endpoints and hence a quadrilateral.
Since it has a pair of parallel sides,
HEFG is a parallelogram.

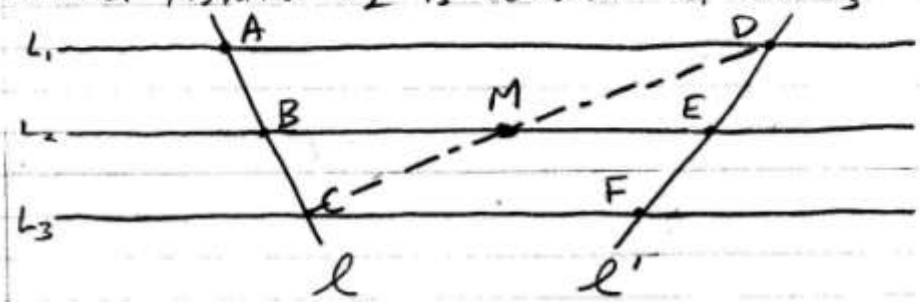
3/3

5)

5.1.6 Let L_1, L_2, and L_3 be mutually parallel lines. Let ℓ and ℓ' be two distinct traversals. If the two segments intercepted on ℓ by L_1, L_2, and by L_2, L_3 are the same length, then prove the same is true of the corresponding segments intercepted on ℓ'.

$\frac{2}{3}$

WLOG Assume L_2 is between L_1 and L_3



Let $A = L_1 \cap \ell$; $\quad D = L_1 \cap \ell'$;
$B = L_2 \cap \ell \quad E = L_2 \cap \ell'$,
$C = L_3 \cap \ell \quad F = L_3 \cap \ell'$
$M = L_2 \cap L_{CD}$

$\triangle ACD$ has B as the midpoint of AC; $AD \| BM$
$\Rightarrow M$ is the midpoint of CD — Thm G15*

$\triangle CDF$ has M as the midpoint of CD; $CF \| ME$
$\Rightarrow E$ is the midpoint of DF — Thm G15*
$\Rightarrow |DE| = |EF|$ — Def of midpoint

Thus the segments intercepted on ℓ' by L_1, L_2, and L_3 have equal length. □

I like it. Very clever.
You should make note, though, that it's possible for C=F or A=D.
You then should make explicit what happens in these situations.

6)

2/2 8) (a) Prove $FTS^* \Rightarrow FTS$

FTS^*: Let ΔABC be given, and D on AB. Suppose a line parallel to BC and passing through D intersects AC at E. Then $\frac{|AB|}{|AD|} = \frac{|AC|}{|AE|} = \frac{|BC|}{|DE|}$.

FTS: Given ΔABC, let D' and E' be points on AB and AC respectively. Then if $\frac{|AB|}{|AD'|} = \frac{|AC|}{|AE'|} = r$, then $D'E' \parallel BC$ and $\frac{|BC|}{|D'E'|} = r$.

$FTS^* \Rightarrow FTS$ If we can show that $D'E' \parallel BC$ then we will have, by FTS^* that $\frac{|AB|}{|AD'|} = \frac{|AC|}{|AE'|} = \frac{|BC|}{|D'E'|}$ which we can call r. Hence, it is sufficient to show that $D'E' \parallel BC$. Let $L_{D'E}$ be the unique line passing through D' and parallel to BC. And this line intersect L_{AC} at E. Then D'=D as in FTS^*. Thus we have that $\frac{|AD'|}{|AB|} = \frac{|AD'|}{|AB|} = \frac{|AE'|}{|AC|} = r = \frac{|AE|}{|AC|}$. Thus, $|AE'| = |AE|$ so $E = E'$ since both E and E' are on L_{AC} and are the same distance away from A. Thus since $D'E \parallel BC$ by construction, then $D'E' \parallel BC$ as needed. good.

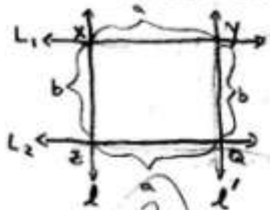
Now assume FTS is true, we want to show that FTS^* follows. Let $DE \parallel AC$. Let E' be the point on L_{AC} such that $\frac{|AD|}{|AB|} = \frac{|AE'|}{|AC|} = r$. Then by FTS, $DE' \parallel AC$. But $DE \parallel AC$, so by the parallel postulate, E must equal E' since both lie on the same unique line passing through D, parallel to AC and since the unique line can only intersect L_{AC} at one point, this one point must be at $E = E'$. Thus $\frac{|BC|}{|DE|} = r = \frac{|AB|}{|AD|} = \frac{|AC|}{|AE|}$ by FTS as needed. Excellent.

5.7.9?

7)

SECTION 5.1

9) Given positive numbers a & b prove that there exists a rectangle whose sides have lengths a & b.



2/2

Let L_1 & ℓ be perpendicular lines that intersect at point X. The transformation T_a along L_1 of distance a maps the point X to a point Y on L_1 & maps ℓ to a line ℓ' where $\ell \parallel \ell'$ by the property of translations on page 214. $\ell \perp L_1$ & $\ell \parallel \ell' \Rightarrow \ell' \perp L_1$ & $\ell' \cap L_1 = Y$ & $|XY| = a$. The transformation T_b along ℓ of distance b maps the point X to a point Z on ℓ & the line L_1 to a line L_2 where $L_1 \parallel L_2$ by the argument above, $L_2 \perp \ell$, $\ell \cap L_2 = Z$ & $|XZ| = b$.

We now have $L_1 \parallel L_2$, $\ell \parallel \ell'$, & $L_1 \perp \ell'$, $L_2 \perp \ell$

Claim: $L_2 \cap \ell'$ & $L_2 \perp \ell'$

assume $L_2 \not\cap \ell'$ then L_2 must be parallel to ℓ' but as ℓ' intersects L_1 & $L_1 \parallel L_2$ $L_2 \perp \ell'$ & therefore $L_2 \cap \ell' = Q$ where Q is a point, by G3.

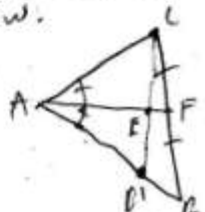
$XY \perp YQ$, $YQ \perp QZ$, $QZ \perp ZX \Rightarrow XYQZ$ is a rectangle by def. XYQZ is a parallelogram by the corollary to G2 & $|XY| = |ZQ| = a$, $|XZ| = |YQ| = b$ by G4.

8)

10) F midpoint of BC on $\triangle ABC$. AF is the angle bisector of $\angle A$ $\longleftrightarrow$ AB and AC are equal.

AF is the angle bisector of $\angle A$. Lets prove by contradiction. Assume WLOG that $|AC|<|AB|$ in the picture below.

2/3



Now assume AF is the angle bisector of $\angle A$. Define a B' that is a perpendicular bisector That goes through AF at E and connects C to B' Such that $|AC|=|AB'|$. Now by thm G7 since $|AB|$, $\angle EAB'$, and $|AE|$ is equal to $|AC|$, $|\angle CAE|$, $|AE|$ respectively, by SAS, $\triangle CAE$ and $\triangle EAB'$ are congruent so $|CE|=|EB'|$. So E is the midpoint between CB'. But by thm G15* Since E is the midpoint of CB' and F is the midpoint of CB then for $\triangle CB'B$, $L_{BB'} \parallel L_{AF}$, but $A \in L_{AF}$ and $A \in L_{BB'}$ so $L_{AF} \cap L_{BB'} = A$ which contradicts thm G15* so $|AC|=|AB|$.

$|AB|=|AC| \longrightarrow$ AF is angle bisector of $\angle A$.

Assume AF is not the angle bisector of $\angle A$ but that $|AB|=|AC|$ as assumed. Then there is another point F' on CB such that AF' is the angle bisector of $\angle A$. Then $\angle CAF' = \angle BAF'$ so by Thm G7 $\triangle CAF' \cong \triangle F'AB$ So $|CF'|=|F'B|$, but that would mean F' is the midpoint of CB, which F also is, so $F=F'$ and so by contradiction

good.

it must be the angle bisector and this statement is proved. QED.

B) $\varphi^{-1} D \varphi$ is a dilation. Need to show this $\frac{2}{3}$
$D' = \varphi^{-1} D \varphi$

For D' to be a dilation, it must take some point O' to itself. We know that by definition the only point in D that keeps the same point is the center O so $\varphi(O') = O$ then $D(O) = O$ and $\varphi^{-1}(O) = O'$ so there is indeed some point O' such that $D'(O') = O'$.

Now for points $P \neq O$, let $\varphi(P) = Q$ for some point Q now by definition D takes Q to a point such that $|OQ'| = r|OQ|$ so $D(Q) = Q'$ and by definition Q' is on R_{OQ} by definition. Now $\varphi^{-1}(Q') = R$. To prove the R is collinear to the original point P, we take 3 examples. We know O is collinear to P and $D(P)$. Now take $\varphi^{-1}(O)$, $\varphi^{-1}(P)$, $\varphi^{-1}(D(P))$. Then $\varphi^{-1}(O) = O' \in D'$, $\varphi^{-1}(P) = \varphi^{-1}(P) \in D'$, $\varphi^{-1}(D(P)) \in D'$, so these 3 are collinear as well. Therefore, P is collinear to R. So $\varphi^{-1} D \varphi$ is a dilation. good.

9)

9. Let D be a dilation w/ center O and scale factor r.
Consider $\varphi^{-1} \circ D \circ \varphi$ where φ is a congruence.

2/3

Claim: $\varphi^{-1}(O)$ is the center of $\varphi^{-1} \circ D \circ \varphi$.

$\varphi^{-1}(D(\varphi(\varphi^{-1}(O)))) = \varphi^{-1}(D(O)) = \varphi^{-1}(O)$ ∎

$\varphi(\varphi^{-1}(O)) = O$ $\quad$ $D(O) = O$ (center)

So $\varphi^{-1}(O)$ is the center of $\varphi^{-1} \circ D \circ \varphi$ b/c def of center of dilation.

Claim: r is the scale factor of $\varphi^{-1} \circ D \circ \varphi$.

Let x be an arbitrary element of the plane.

$\text{dist}(\varphi^{-1}(D(\varphi(x))), \varphi^{-1}(D(\varphi(\varphi^{-1}(O)))))$ ← Length of image

$= \text{dist}(D(\varphi(x)), D(\varphi(\varphi^{-1}(O))))$ $\quad$ φ^{-1} is also a congruence hence isometry

$= r\,\text{dist}(\varphi(x), \varphi(\varphi^{-1}(O)))$ $\quad$ Dilations scale distances by r

$= r\,\text{dist}(x, \varphi^{-1}(O))$ $\quad$ φ is isometry

$= r\,|\varphi^{-1}(O)x|$

$\Rightarrow r$ is the scale factor of $\varphi^{-1} \circ D \circ \varphi$ ∎

10)

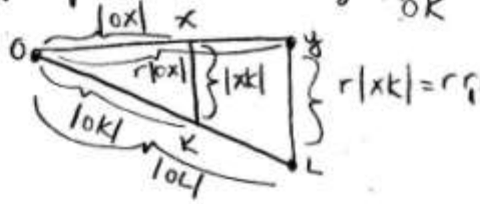
Ex. 10 (5.2.5) Prove that the image $D(C)$ is a circle, !
and that the center of $D(C)$ is the image under D of the center of C

PROOF Let O be an arbitrary point not on a circle $C=\{x \mid dist(K,x)=r_1\}$
Let D be the dilation with center O.
Let K be the center of C.

2/2

Assume $K \neq O$.

By definition of dilation, $D(O)=O$ and $D(x)=y$,
where y is the point on the ray R_{OX} s.t. $|Oy| = r|Ox|$.
Similarly, $D(K)=L$ where L is the point on the ray R_{OK}
s.t. $|OL| = r|OK|$. See figure:



By FTS, since $\dfrac{|Oy|}{|Ox|} = r = \dfrac{|OL|}{|OK|}$
$\Rightarrow |yL| = r|xk| = rr_1$

$\Rightarrow D(C) = \{y \mid dist(L,y) = rr_1\} \Rightarrow$ by def. of circle, $\boxed{\text{circle with center } L \text{ and radius } rr_1}$ ✓

Case(2) $K=O$
$D(O)=O \Rightarrow D(K)=K=O$
Since $C=\{x \mid dist(K,x)=r_1\} \Rightarrow D(C)=\{y \mid dist(O,k)=rr_1\} \Rightarrow$ circle with center O and radius rr_1 ✓

Good.