Homework 8

Solutions

1)

HW 8

1) a) We pick a counter-example to show $F \circ G \neq G \circ F$.

$(F \circ G)(5,7) = F(G(5,7)) = F(35,7) = (35,8)$
$(G \circ F)(5,7) = G(F(5,7)) = G(5,8) = (40,8)$. good.

b) $F \circ G$ not injective b/c $(F \circ G)(5,0) = (F \circ G)(6,0) = (0,1)$

and $(5,0) \neq (6,0)$.

Not surjective b/c there is no (x,y) s.t. $(F \circ G)(x,y) = (1,1)$.
$(F \circ G)(x,y) = (xy, y+1)$. So given $(1,1)$, this implies $y=0$, but $xy = x \cdot 0 = 0 \neq 1$ for all x.

c) $G \circ G$ not injective b/c $(G \circ G)(5,0) = (G \circ G)(6,0) = (0,0)$ but $(5,0) \neq (6,0)$.

Not surjective b/c there is no (x,y) s.t. $(G \circ G)(x,y) = (5,0)$. This is because $(5,0)$ implies the y-coord has to be 0, but this would make the x-coord 0 for all x. great.

2)

Ex 2. (4.2.6) (a) Let $G(x,y)=(x^2,y)$. Is it injective? Surjective?

2/2

- Notice $G(1,0)=(1^2,0)=(1,0)$ and $G(-1,0)=((-1)^2,0)=(1,0)$
  Since for two distinct points $(1,0)$ & $(-1,0)$, G assigns to them the same point: $G(1,0)=G(-1,0)=(1,0)$ then $\boxed{G \text{ is not injective}}$ by T1 | since $(-1)^2=(-1)(-1)=1$ by Cor. on pg 119

- Notice that $G(x,y)=(x^2,y)$ cannot have a x-component smaller than 0 since $x^2 \geq 0$ for $\forall x \in \mathbb{Q} \Rightarrow G(x,y) \neq (-1,0)$ (let $y=0$)
  $\Rightarrow \exists Q=(-1,0)$ s.t. $G(P) \neq Q$ for some $P=(x,y)$ $\Rightarrow \boxed{G(x,y) \text{ is not surjective}}$ by T2

(b) Let $F(x,y)=(x,y^3)$. Is it injective? Surjective?

- We want to show that for any distinct y_1 and y_2 in (x,y_1) & (x,y_2), there exists distinct $F(x,y_1)=(x,y_1^3)$ & $F(x,y_2)=(x,y_2^3) \Rightarrow F$ is injective.
  Specifically, we want to show $y_1^3 \neq y_2^3$ when $y_1 \neq y_2$.
  By FASM, let $y_1=\frac{a}{b}$ and $y_2=\frac{c}{d}$. Assume $\frac{a}{b}$ & $\frac{c}{d}$ are in lowest reduced forms.
  Then $\left(\frac{a}{b}\right)^3=\left(\frac{c}{d}\right)^3 \Rightarrow \frac{a^3}{b^3}=\frac{c^3}{d^3} \Rightarrow a^3 \neq c^3$ or $b^3 \neq d^3$. ← why is this correct or necessary?
  Then $a^3d^3=b^3c^3 \Rightarrow \underbrace{s \cdot q}_{a^3} \cdot d^3 = b^3 \cdot \underbrace{s \cdot r}_{c^3} \Rightarrow q \cdot d^3 = b^3 \cdot r$ by dividing both sides by s (note: $\gcd(q,r)=1$) for some $s,q,r \in \mathbb{Z}$
  Assume $q>1$ and let a prime $p \mid q \Rightarrow p \mid b^3 \Rightarrow p \mid b$
  However, $p \nmid b$ since $\frac{a}{b}$ & $\frac{c}{d}$ are in lowest reduced forms. (Contradiction!)
  Hence $\left(\frac{a}{b}\right)^3 \neq \left(\frac{c}{d}\right)^3$ when $y_1 \neq y_2 \Rightarrow \boxed{F \text{ is injective}}$ by T1 ✓
  by proof of the contrapositive

- Since $\forall y, \exists r$ s.t. $r^3=y \Rightarrow \forall (x,r^3), \exists (x,r)$ s.t. $F(x,r)=(x,r^3)$
  Let $r=y \Rightarrow \forall (x,y^3), \exists (x,y)$ s.t. $F(x,y)=(x,y^3) \Rightarrow \boxed{F \text{ is surjective}}$ by T2

(c) What is $(F \circ G)(x,y)$ for any (x,y)? Injective? Surjective?

- By def. of composite transformations, $(F \circ G)(x,y)=F(G(x,y))=F(x^2,y)=(x^2,y^3)$
  Hence, $\boxed{(F \circ G)(x,y)=(x^2,y^3)}$

- Notice $(F \circ G)(1,0)=(1^2,0^3)=(1,0)$ and $(F \circ G)(-1,0)=((-1)^2,0^3)=(1,0)$,
  Since $(F \circ G)$ assigns 2 distinct points to $(1,0)$, since $(-1)^2=1$ by Cor. on pg 119
  then $\boxed{(F \circ G) \text{ is not injective}}$ by T1

  good.

- Notice that $(F \circ G)=(x^2,y^3)$ cannot have a x-component smaller than 0 since $x^2 \geq 0$ $\forall x \in \mathbb{Q} \Rightarrow (F \circ G) \neq (-1,0)$ (let $y=0$)
  $\Rightarrow \exists Q=(-1,0)$ s.t. $(F \circ G)(P) \neq Q$ for some $P=(x,y) \Rightarrow \boxed{(F \circ G) \text{ is not surjective}}$ by T2

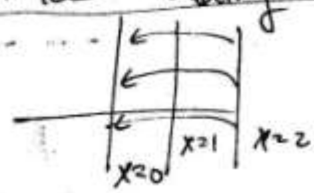
3)

(3) Not injective b/c $F(3,5) = F(1,5)$ but $(3,5) \neq (1,5)$.

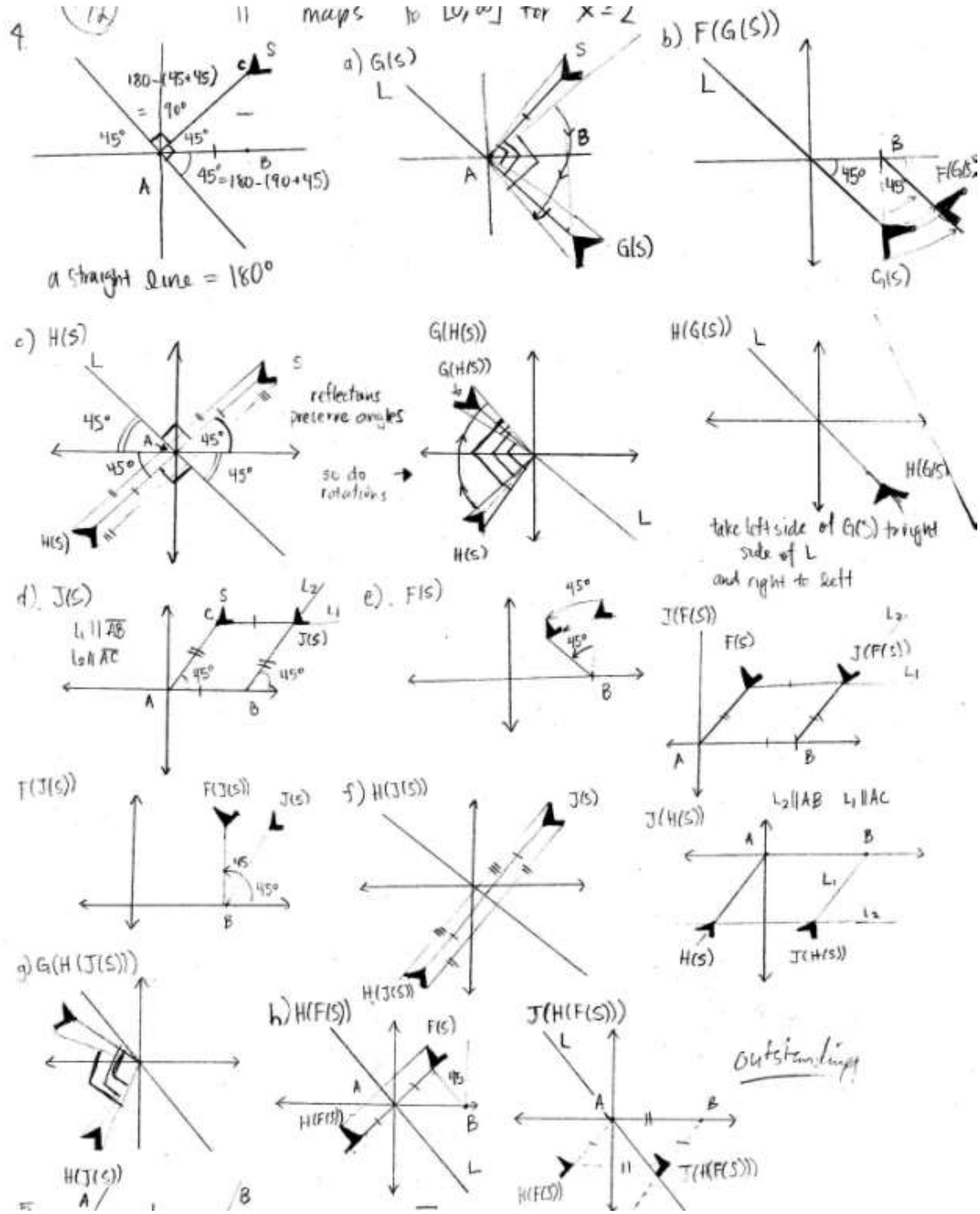
It is surjective.
Given any (x,y), if $x \leq 1$, then $F(x,y) = (x,y)$.
if $x > 1$, then $F(x+2,y) = (x,y)$.
if $0 \leq x \leq 1$, then $F(2-x,y) = (x,y)$.

F does nothing to the plane if $x < 1$. If $x > 2$, then F moves everything to the right of the line $x=2$ to the left by 2. If $1 \leq x \leq 2$, then F reflects the points across the line $x=1$.
So it is like folding ~~the~~ plane at $x=2$ to $x=0$.

good.



4)

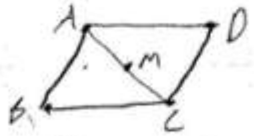

4
180-(45+45)
= 90°
45°=180-(90+45)
a straight line = 180°
a) G(S)
G(S)
b) F(G(S))
c) H(S)
reflections preserve angles
so do rotations
G(H(S))
H(G(S))
take left side of G(S) to right side of L
and right to left
d) J(S)
J(S)
e) F(S)
J(F(S))
F(J(S))
f) H(J(S))
H(J(S))
J(H(S))
g) G(H(J(S)))
h) H(F(S))
J(H(F(S)))
outstanding

5)

3) Prove: The angles of a parallelogram at opposite vertices are equal.

Given a parallelogram ABCD we need to show $\angle A = \angle C$ and $\angle B = \angle D$. (See figure below)

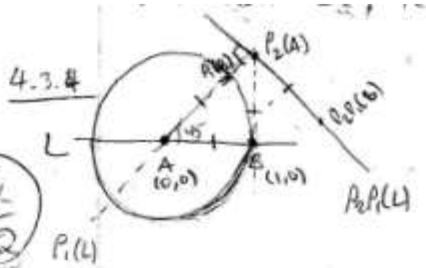
2/3



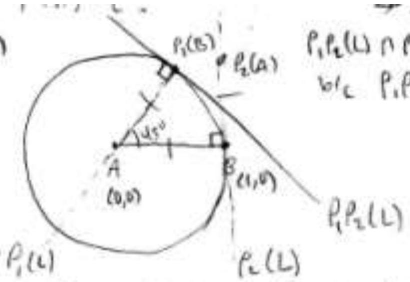
Let M be the midpoint of the diagonal Line AC. Let R be the rotation of 180° around M. In the proof of Theorem G4 we concluded $R(C) = A$ and $R(B) = D$ and $R(BC) = AD$. This also means $R(BAD) = DCB$. Since R is an isometry (preserves angles) $\angle BAD$ (also denoted $\angle A$) has to equal $\angle DCB$ (also denoted $\angle C$), so $\angle A = \angle C$. Similarly $R(CBA) = ADC$ so $\angle B = \angle D$ as this rotation is an isometry as well, hence the corollary to theorem G4 is proven. QED

6)

4.3.4



$P_2P_1(L) \cap P_1(L) = P_2(A)$
b/c $P_2(A) = (1,1)$



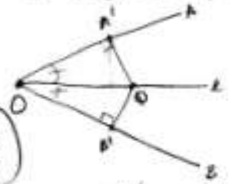
$P_1P_2(L) \cap P_1(L) = P_1(B)$
b/c $P_1P_2(B) = P_1(B)$

$\frac{2}{2}$

$P_2P_1(L) \perp P_1(L)$, b/c we rotate line $P_1(L)$ 90° about B
$P_1P_2(L) \perp P_1(L)$, b/c $P_2(L) \perp L$ and rotating $P_2(L)$ 45° about A, $P_1P_2(L) \perp l$ and l is line that is 45° w/ L, and $P_1(L)$ is 45° w/ L thus $\underline{l = P_1(L)}$ and thus $P_1P_2(L) \perp P_1(L)$
$P_2P_1(L) \cap P_1(L) = P_2(A)$, $P_1P_2(L) \cap P_1(L) = P_1(B)$
Since $P_2P_1(L)$ and $P_1P_2(L)$ $\perp$ to $P_1(L)$, if each share same point w/ $P_1(L)$ then $P_1P_2(L) = P_2P_1(L)$ or if they don't share same point then $P_1P_2(L) \parallel P_2P_1(L)$.
Thus $P_1(B) \stackrel{?}{=} P_2(A)$, $A = (0,0)$, $B = (1,0)$
$P_1(B) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $P_2(A) = (1,1)$ since $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \neq (1,1)$ then $P_1(B) \neq P_2(A)$ thus $P_2P_1(L) \parallel P_1P_2(L)$. good

7)

5. Prove that every point on the angle bisector of an angle is equidistant from both sides of the angle.



2/3

Let l be the angle bisector of $\angle AOB$. We want to show that $|A'Q| = |B'Q|$ $\forall A' \in \overrightarrow{OA}, B' \in \overrightarrow{OB}$ and $Q \in l$.

You need to quantify A', B' in terms of Q, since $\overline{QA'} \perp \overline{OA}$ and $\overline{QB'} \perp \overline{OB}$

Let ρ be the reflection around l. If we can show that any point $A' \in L_{OA}$ can be mapped to a point $B' \in L_{OB}$, Then since reflections are isometries and map segments to segments, and since $\rho(Q)=Q$, we would have that $|A'Q| = |\rho(A'Q)| = |B'Q|$ as needed. So, lets show this. Claim: $\rho(L_{OA}) = L_{OB}$. Let $|\angle AOQ| = \Theta$. Then since Reflections preserve angles, $|\rho\angle AOQ| = |\angle AOQ| = \Theta$. Specifically, the line $\rho(L_{OA})$ makes an angle Θ with the line l. By assumption, so does L_{OB}. Let $\alpha \in \rho(L_{OA})$. Then specifically we know that $\forall \alpha \in \rho(L_{OA})$, $|\angle QO\alpha| = \Theta$. We also know that $\forall B' \in L_{OB}$, $|\angle QOB'| = \Theta$. We want to show that $\alpha = B'$ $\forall \alpha \in \rho(L_{OA})$ and $B' \in L_{OB}$. Suppose not. Suppose $\alpha \notin L_{OB}$. Then we have 2 cases. Either α is in one half plane of L_{OB}, L_{OB}^{+} or in the other, L_{OB}^{-}. This is a contradiction, however since that would mean that $|\angle QO\alpha| \neq \Theta$ by definition of degrees of angles. Thus $\alpha = B'$ $\forall \alpha, B$ so $L_{OB} = \rho(L_{OA})$. Specifically, we can now see that $\rho(A') = B$ and since $\rho(Q) = Q$, then $\rho(|A'Q|) = |BQ|$ as needed.

8)

of the angle $\boxed{\checkmark}$.

Ex. 8 (4.3.6) If ABCD is a parallelogram, prove that $\angle ADB$ & $\angle CBD$ are equal.

↳ The distance from a point to a line is the length of the $\perp$ line from the point to the line (this is unique)

ab $\not\perp$ AD or BD, in general.

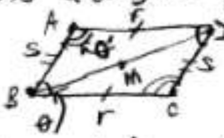
PROOF.

2/3

Let ABCD be a parallelogram.

Then by Thm. 64 & Cor. to 64, opposite sides of a parallelogram are equal and angles at opposite vertices are equal.

Let there exists a diagonal S_{BD} from point B to point D, (see figure)



and let m be the midpoint of S_{BD}.

Define φ as the rotation of 180° around m.

$\Rightarrow \varphi(A) = C$, $\varphi(D) = B$, $\varphi(B) = D$, and $\varphi(C) = A$

$\Rightarrow \varphi(ADB) = CBD$ by Remark on pg. 211

$\Rightarrow$ Since φ is an isometry, then φ preserves the degree of angles

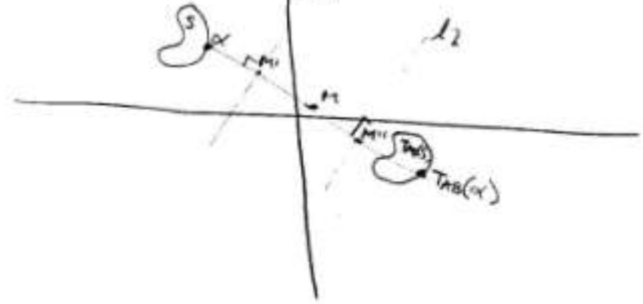
$\Rightarrow \varphi(\angle ADB) = \angle CBD$

$\Rightarrow \angle ADB = \angle CBD$ $\boxed{\checkmark}$

9)

2/3 7. (a) Prove any translation is the composition of 2 reflections.
Let T_{AB} be the translation along $\overline{AB}$. We want to show that for any shape S, $T_{AB}(S) = R_1R_2(S)$ where R_1 and R_2 are reflections. Choose any point $\alpha \in S$, let D be the distance traveled by from $\alpha \in S$ to same point $T_{AB}(\alpha) \in S'$ where $T_{AB}(S) = S'$. We want to show that the distance, D, and the direction traveled from α to $T_{AB}(\alpha)$ is the same as from α to $R_1R_2(\alpha)$ for some R_1, R_2. Let M be the midpoint from α to $T_{AB}(\alpha)$. Then let M' be the midpoint of $\overline{MT_{AB}(\alpha)}$ and M'' be the midpoint of $\overline{M\alpha}$. Let l_1 be the unique line perpendicular to the line passing through α and $T_{AB}(\alpha)$ and through M' and l_2 be the unique line through M'' and perpendicular to $L_{\alpha, T_{AB}(\alpha)}$. Then let R_1 be the reflection over l_1 and R_2 be the reflection over l_2. Then $R_1(\alpha) = M$ and $R_2R_1(\alpha) = R_2(M) = T_{AB}(\alpha)$ as needed.

good



b) Prove every rotation is a composition of two reflections.
We want to show that for any rotation ρ of shape S, that $\rho_{\alpha}(S) = R_1R_2(S)$ where R_1 and R_2 are reflections.
Let R_1 and R_2 be reflections over ℓ_1 and ℓ_2 respectively.
Let ℓ_1 and ℓ_2 intersect at an angle $\frac{\theta}{2}$ where θ is the degree of rotation for ρ_θ, and ℓ_1 and ℓ_2 intersect the circle with radius $\overline{O\alpha}$ in the interior of $\angle \alpha O \rho(\alpha)$. We want to show that $\forall$ points $\alpha \in S$, $\rho_\theta(\alpha) = R_1R_2(\alpha)$. Choose any point $\alpha \in S$.

Since the angle created between ℓ_1 and ℓ_2 is $\theta/2$, then the angle created between $L_{O\alpha}$ and ℓ_2, and ℓ_1 and $L_{O\rho(\alpha)}$ must each be $\theta/4$. Since reflections preserve distances, and α is $\theta/4$ away from β where β is the intersection of ℓ_2 and the circle, then $R_2(\alpha)$ must also be a distance of $\theta/4$ away from ℓ_2. Thus R_2 moves α a total distance of $\theta/2$ in the clockwise direction. Similarly, R_1 moves $(R_2(\alpha))$ a total distance of $\theta/2$ in the clockwise direction. Hence, $R_1R_2(\alpha)$ moves α a total distance of $\frac{\theta}{2}+\frac{\theta}{2}=\theta$. Next we must show that $R_1R_2(\alpha) \in$ Circle with radius $\overline{O\alpha}$. This is trivial, however, since reflections and rotations preserve distances, $|O\alpha| = |O\rho_\theta(\alpha)| = |OR_1R_2(\alpha)|$ as needed.

10)

10. $\boxed{4.4.1}$

Say $S \cong S'$ & $S' \cong S''$ → Prove $S'' \cong S$

2/3

Say the congruence φ_1 exists
s.t. $\varphi_1(S) = S'$
& φ_2 exists. s.t $\varphi_2(S') = S''$

By theorem G5 $\varphi_2 \circ \varphi_1$ is a
congruence, so $\varphi_2(\varphi_1(S)) = \varphi_2(S') = S''$
and therefore, $S \cong S''$ since
there is a congruence $\varphi(S) = S''$ ✓
good