Homework 7

Solutions

1), 2)

Garin

4.1 2. Prove that the intersection of convex sets is convex.

$\frac{2}{2}$

Let $C_1 \ldots C_i \ldots$ be convex $\forall\, C_i$ (i whole number, $1 \le i \le \infty$). Then consider the possibly infinite set, $A = \{C_1 \cap C_2 \cap \ldots \cap C_i \cap \ldots\}$ of the intersection of some collection of convex sets. We want to show that A is convex, ie $\forall$ points $a, b \in A$, the segment $\overline{ab} \in A$. Let us start by choosing two points $a, b \in A$. Then a and b are both in each set $C_1, C_2, \ldots C_i, \ldots$. Thus, since each C_i is convex, the segment $\overline{ab}$ is in each C_i. Hence $\overline{ab}$ must be in the intersection of $C_1 \ldots C_i \ldots$ and thus $\overline{ab} \in A$ as needed. good.

3(a) Suppose we have an infinite number of convex sets C_i, where i is a whole number, and suppose that each C_i is contained in the next one, C_{i+1}. Prove that the union of these C_i's is also convex.

Let us consider the possibly infinite set of the union of C_i's, $B = \{C_1 \cup C_2 \cup \ldots \cup C_i \cup \ldots\}$. We want to show that for any two points $p, q \in B$, the segment $\overline{pq} \in B$. Let us first note that by assumption, $C_i \subset C_{i+1}$. Now, let us pick two points $a, b \in A$. Then if both a and b are in the same set $C_j \in A$, then since each C_i is convex, $\overline{ab} \in C_j$, thus $\overline{ab} \in B$ and we are done. So, let's assume without loss of generality that $a \in C_j$ and $b \in C_{j+n}$ for some positive whole number n. Then, since $C_j \subset C_{j+1} \subset C_{j+2} \subset \ldots \subset C_{j+n}$, $C_j \subset C_{j+n}$. Hence if elements $a \in C_j$, then $a \in C_{j+n}$. Now we know that both a and b are in C_{j+n} and since each C_i is convex, C_{j+n} is convex (letting $i = j+n$) so $\overline{ab} \in C_{j+n}$. Thus $\overline{ab} \in B$ as needed since $C_{j+n} \subset B$.

$\frac{2}{2}$

(b) Is the union of convex sets convex in general?

No, as a counter example, consider two convex sets $A = \{1, 2, 3\}$ and $B = \{5, 6, 7\}$ where $\{1, 2, 3\}$ and $\{5, 6, 7\}$ are points on a number line. Then choosing points 3, 5, we see that the segment $\overline{35}$ contains the point 4 which is neither in A nor B. Thus $\overline{35} \notin A \cup B$.

good.

3), 4a)

3. Suppose A, B, C are non-collinear points. By L1) we know for every 2 points
$\frac{2}{2}$ we can draw a unique line segment between them. And since A, B, C are
non-collinear, each distinct line does not intersect the third point. By
L2) we know every distinct line segment intersects at 1 point or not all
all. But obviously $\overline{AB} \cap \overline{BC} = \{B\}$, $\overline{BC} \cap \overline{CA} = \{C\}$ $\overline{CA} \cap \overline{AB} = \{A\}$ by
definition of line segment, so they only intersect at these points. ■
good.

4a) Assume $L_1 \parallel L_2$, $L_1 \parallel L_3$, where L_1, L_2, L_3 are distinct lines.
$\frac{2}{2}$ Suppose $L_2 \nparallel L_3$. Then $\exists \{p\} \in L_2 \cap L_3$ (By L2). Then $p \in L_2$ and $p \in L_3$.
By the Parallel postulate, $\exists!$ line going through p that is parallel to L_1
$\Rightarrow L_2 = L_3$ $\times$ L_2, L_3 are distinct.
So $L_2 \parallel L_3$ ■

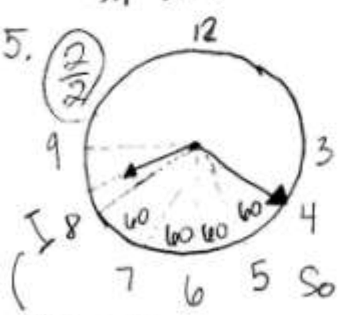
4b), 5), 6)

b. Let $L_1 \parallel L_2$, $\ell \neq L_1$. Assume $\exists p \in \ell \cap L_1$.

Suppose $\ell \cap L_2 = \emptyset$. Then $\ell \parallel L_2$. So by 4a) $\ell \parallel L_1$ or $\ell = L_1$ $\rightarrow\leftarrow$. Then $\ell \cap L_1 = \emptyset$ $\times$.

So $\ell \nparallel L_2$. good

5. 2/2



20 marks

The hour hand moves $\frac{1}{12}$ of the circle every hour. Since there are 60 min every hour it moves $\frac{1}{12} \times \frac{1}{60}$ every min.

define 1 mark = distance the hour hand moves / min.

So $360° = 720$ marks so $\frac{1}{2}° = 1$ mark

We know there are $4(60) + 20 = 260$ marks between the hands.

So $\boxed{130°}$ between the hands. good.

6. 2/2 Triangular Region $:= \angle A \cap \angle B \cap \angle C$. By definition $\angle A, \angle B, \angle C$ are convex subsets. By #1, the intersection of convex sets is convex.

Therefore, the triangular region is convex.

8)

(8) (a) We'll construct an inverse G.
Given each $Q \in \pi$, there exist a $P \in \pi$ such that $F(P) = Q$. because F is surjective. This P is unique because F is injective. Define $G(Q) = P$.

Then $(F \circ G)(Q) = F(G(Q)) = F(P) = Q$. $\forall Q \in \pi$, and $(G \circ F)(P) = G(F(P)) = G(Q) = P$ $\forall P \in \pi$. Hence G is an inverse.

(b) Assume G_1 and G_2 are inverses of F.

Then $G_1 \circ (F \circ G_2) = G_1 \circ I = G_1$,
$= (G_1 \circ F) \circ G_2 = I \circ G_2 = G_2$.
This implies $G_1 = G_2$, so the inverse of a transformation F is unique.

good.

9)

B) Composition of 2 surjections is a surjection and composition of two injections is an injection.

2 surjections, $F, G \rightarrow F \circ G$ is a surjection.
Let F and G, without loss of generality be surjective functions. Let $G(A) = B$ for some $A, B \in \Pi$, and let $F(B) = C$ for some $C \in \Pi$. Since G is surjective there exists an A such that $G(A) = B$ and since F is surjective there exists a B such that $F(B) = C$. Then $F \circ G(A) = F(G(A)) = F(B) = C$. So for there exist an A such that $F \circ G(A) = C$ so $F \circ G$ is a surjection. QED

2 injections $F, G \longrightarrow F \circ G$ is an injection.
So G takes points X, y to distinct points since G is injective so $F(x) = x'$ and $F(y) = y'$ where $x \neq y$ and $x' \neq y'$. Then some function F takes x' and y' to distinct points (since injective) so $F(x') = x''$ and $F(y') = y''$ (all points in Π) and so $F \circ G(x)$ by definition is $F G(x) = F(x') = x''$ and $F \circ G(y) = F(y') = y''$ and $x'' \neq y''$ so $F \circ G(x)$ is injective and thus QED

c) If F, G are bijection, prove inverse of $F \circ G \rightarrow G^{-1} \circ F^{-1}$
Something $\cdot$ $F \circ G = I$ We know inverse of F is F^{-1} since F is a bijection, and we know G^{-1} is the inverse of G since G is a bijection. So $(G^{-1} \circ F^{-1}) F \circ G$. We know $F^{-1} F = I$

So $G^{-1} \circ I \circ G$ well $G^{-1} \circ I = G^{-1}$ since I is the identity
so $G^{-1} \circ G = I$ and we have shown this since inverses are unique.
So $(G^{-1} \circ F^{-1}) F \circ G = I$ and hence $G^{-1} \circ F^{-1}$ is the inverse.

B) 2) a) Prove composition of two isometries is an isometry.
F, G are isometries so if $A, B \in \Pi$ $dist(A, B) = dist(F(A), F(B))$ $= dist(G(A), G(B))$. If we compose F and G say $F \circ G$. Let $G(A) = A'$ $(F \circ G)(A) = A''$ and $F(A') = A''$
And same for B. $G(B) = B'$ $F(B') = B''$ so we need to show $dist(A, B) = dist(F \circ G(A), F \circ G(B))$ well we know $dist(A, B) = dist(G(A), G(B))$ by definition. We also know $F(G(A)) = F(A')$, $F(G(B)) = F(B')$. But since the $dist(A, B) = dist(G(A), G(B))$ or $dist(A', B')$ then $dist(A, B) = dist(A', B')$. So $dist(F(A'), F(B')) = dist(A', B')$ since F is an isometry. Therefore, we already showed $dist(A', B') = dist(A, B)$ so the composition of two isometries is an isometry.

10)

4.2.3 $\frac{2}{2}$

a) Lets show by contrapositive, i.e. G not inj, F∘G not inj.
Assume G not inj. i.e $G(x) = y$ $G(x') = y$ but $x \neq x'$
so $F \circ G(x) = F(y)$ and $F \circ G(x') = F(y)$ Thus $F \circ G(x) = F \circ G(x')$
but $x \neq x'$ thus F∘G not Injective. Thus if G is inj
then F∘G is inj.

b) Sometimes true/false.
True when $F(x) = x$ $G(x) = x$, Then $F \circ G(x) = x$ is 1-1
false, when $G(x) = \arctan x$ $F(x) = \begin{cases} 0 & \text{if } x > \pi/2,\ x < -\pi/2 \\ x+1 & \text{if } -\pi/2 < x < \pi/2 \end{cases}$
F∘G 1-1, but F not 1-1

c) Sometimes true/false
True when $F(x) = x$ $G(x) = x$ then $F \circ G(x) = x$ onto
False when $F(x) = \ln x$ $G(x) = e^x$ $F \circ G(x)$ onto but G not onto
there are no $x < 0$ that $G(x) < 0$

d) contrapositive, if F not onto, F∘G not onto
If F not onto then $\exists q \in \mathbb{R}$ st $F(p) \neq q$ for all $p \in \mathbb{R}$
let $G(r) = p$ then $F \circ G(r) = F(p) \neq q$ for $q \in \mathbb{R}$ thus
F∘G not onto. Hence if F∘G onto then G onto.