Homework 4

Solutions

1)

1. §2.2 #1

1) Prove that for all $x, y \in \mathbb{Q}$ if $x+y=x$, then $y=0$

If $x+y=x \quad \forall\, x,y \in \mathbb{Q}$

$x^* + (x+y) = x + x^*$ By equality adding x^* to both sides

$(x^*+x)+y = x+x^*$ By associativity of addition

good. $0+y=0$ By (A3') assumption of addition

$y=0$ By (A2') assumption of addition

Note: (A3') is "If x is any rational number $x^*+x=0$" pg 100

(A2') is "$0+x=x$ for any rational number x."

2)

2.2.4

$\frac{2}{2}$

a) $(x-y)-z \overset{1}{=} (x+y^*)+z^* \overset{2}{=} x+(y^*+z^*) \overset{3}{=} x+(z^*+y^*) \overset{4}{=} (x+z^*)+y^* \overset{5}{=} (x-z)-y$

1 by def of subtraction 2 associativity 3 commutativity 4 associativity 5 Def of sub.

b) $x-(y-z) \overset{1}{=} x+(y+z^*)^* \overset{2}{=} x+(y^*+z) \overset{3}{=} (x+y^*)+z \overset{4}{=} (x-y)+z$

1 Def of Subtraction 2 Thm 1 3 associativity 4 Def of subtraction

c) $(x+y)-z \overset{1}{=} (x+y)+z^* \overset{2}{=} x+(y+z^*) \overset{3}{=} x+(y^*+z)^* \overset{4}{=} x+(z+y^*)^* \overset{5}{=} x-(z-y)$

1 Def of subtraction 2 associativity 3 Thm 1 4 commutativity 5 Def. of subtraction

3)

Ex. 3 (2,2) #5.

Compute: (a) $-4\frac{6}{7}+2\frac{2}{3}$

$= -\left(4+\frac{6}{7}\right)+\left(2+\frac{2}{3}\right)$ by def. of mixed fractions

$= -\left(4+\frac{6\cdot 3}{7\cdot 3}\right)+\left(2+\frac{2\cdot 7}{3\cdot 7}\right)$ by eq. fractions thm. equivalent

$= -\left(4+\frac{18}{21}\right)+\left(2+\frac{14}{21}\right)$

$= -\left(\frac{4\cdot 21+18}{21}\right)+\left(\frac{2\cdot 21+14}{21}\right)$ by fraction addition

$= -\left(\frac{102}{21}\right)+\left(\frac{56}{21}\right)$

$= -\frac{102}{21}+\frac{56}{21} = \frac{56}{21}-\frac{102}{21}$ by (A1)

$= \frac{56-102}{21} = \frac{-(102-56)}{21} = \boxed{-\frac{46}{21}}$

by Thm. 2^* on pg. 106 since $\frac{56}{21}<\frac{102}{21}$

good.

(2/2)

(b) $-7.1-22\frac{1}{3}$

$= -\left(\frac{7.1\cdot 10}{10}\right)-\left(22+\frac{1}{3}\right)$ by def. of decimals, mixed fraction

$= -\left(\frac{71}{10}\right)-\left(\frac{22\cdot 3+1}{3}\right)$ by fraction addition

$= -\left(\frac{71}{10}\right)-\left(\frac{67}{3}\right)$

$= -\left(\frac{71\cdot 3}{10\cdot 3}\right)-\left(\frac{67\cdot 10}{3\cdot 10}\right)$ by eq. fractions thm.

$= -\left(\frac{213}{30}\right)-\left(\frac{670}{30}\right)$

$= -\left(\frac{213}{30}+\frac{670}{30}\right)$ by Thm. 1^* on pg. 101

$= -\left(\frac{883}{30}\right)$ by fraction addition

$= \boxed{-\frac{883}{30}}$

(c) $7-\left(2.5-3\frac{2}{3}\right)$

$= 7-\left(\frac{2.5\cdot 10}{10}-\left(3+\frac{2}{3}\right)\right)$ by def. of decimals

$= 7-\left(\frac{25}{10}-\left(\frac{3\cdot 3+2}{3}\right)\right)$ by def. of mix #.

$= 7-\left(\frac{25}{10}-\frac{11}{3}\right)$

$= \frac{7\cdot 30}{30}-\left(\frac{25\cdot 3}{10\cdot 3}-\frac{11\cdot 10}{3\cdot 10}\right)$ by eq. fractions thm.

$= \frac{210}{30}-\left(\frac{75}{30}-\frac{110}{30}\right)$

$= \frac{210}{30}-\frac{75}{30}+\frac{110}{30}$ by Thm. 1^*

$= \left(\frac{210}{30}-\frac{75}{30}\right)+\frac{110}{30}$ by (A1)

$= \frac{135}{30}+\frac{110}{30} = \boxed{\frac{245}{30}}$ by fraction addition.

(d) $(-703.2+689.4)-\left(\frac{1}{5}-3\frac{2}{3}\right)$

$-\left(\frac{703.2\cdot 10}{10}\right)+\left(\frac{689.4\cdot 10}{10}\right)$ def. of decimals

$-\left(\frac{7032}{10}\right)+\left(\frac{6894}{10}\right)$

$-\left(\frac{7032}{10}-\frac{6894}{10}\right)$ by Thm. 2^* since $\frac{7032}{10}>\frac{6894}{10}$

$-\left(\frac{138}{10}\right)$ by fraction addition

$-\left(\frac{69\cdot 2}{5\cdot 2}\right)$ by eq. fractions thm.

$-\frac{69}{5} = -\left(\frac{69\cdot 3}{5\cdot 3}\right)$

(I) $-\frac{207}{15}$

$\frac{1\cdot 3}{5\cdot 3}-\left(\frac{3\cdot 3+2}{3}\right)$ by eq. fractions thm.

$\frac{3}{15}-\frac{11}{3}$

$\frac{3\cdot 3}{15\cdot 3}-\frac{11\cdot 15}{3\cdot 15}$ by eq. fractions thm.

$\frac{9}{45}-\frac{165}{45}$

$-\left(\frac{165}{45}-\frac{9}{45}\right)$ by Thm. 2^* since $\frac{9}{45}<\frac{165}{45}$

$-\left(\frac{156}{45}\right) = -\left(\frac{52\cdot 3}{15\cdot 3}\right)$ by eq. fractions

(II) $-\frac{52}{15}$

(I)−(II)

Hence $-\frac{207}{15}-\left(-\frac{52}{15}\right) = -\frac{207}{15}+\frac{52}{15}$

$= -\left(\frac{207}{15}-\frac{52}{15}\right) = -\left(\frac{155}{15}\right) = -\left(\frac{31\cdot 5}{3\cdot 5}\right) = \boxed{-\left(\frac{31}{3}\right)}$

by Thm. 2^* since $207>52$ by eq. fractions

Ex. 3 (2.2) #5e.

$\left(\frac{5}{6} - \left(1\frac{7}{18}\right)^*\right) + \frac{5}{24}$ since $1\frac{7}{18} = 1 + \frac{7}{18} = \frac{18+7}{18} = \frac{25}{18}$

$= \left(\frac{5}{6} - \left(\frac{25}{18}\right)^*\right) + \frac{5}{24}$

$= \frac{5}{6} - \frac{25}{18}^* + \frac{5}{24}$ by (A1)

$= \frac{5}{6} - \left(-\frac{25}{18}\right) + \frac{5}{24}$ since $y^* = -y$ on pg. 105 $y \in \mathbb{Q}$

$= \frac{5}{6} + \frac{25}{18} + \frac{5}{24}$ since $-(-x) = x$ on pg. 106 $x \in \mathbb{Q}$

$= \frac{5 \cdot 3 + 25}{18} + \frac{5}{24}$ by addition of fractions

$= \frac{40}{18} + \frac{5}{24}$

$= \left(\frac{20 \cdot 2}{9 \cdot 2}\right) + \frac{5}{24}$ by eq. fractions thm.

$= \frac{20}{9} + \frac{5}{24}$ by cancellation law

$= \frac{20 \cdot 24 + 5 \cdot 9}{9 \cdot 24}$ by addition of fractions

$= \frac{525}{216} = \frac{175 \cdot 3}{72 \cdot 3} = \boxed{\frac{175}{72}}$ ☑

by cancelation law

4)

4a. Let $a, b, \ldots, z, w \in \mathbb{Q}$. Prove:

$$(a+b+c+d)-(x+y+z+w)=(a-x)+(b-y)+(c-z)+(d-w)$$

Pf Let $a, b, c, d, x, y, z, w \in \mathbb{Q}$.

$$
\begin{aligned}
&(a+b+c+d)-(x+y+z+w) \\
&=(a+b+c+d)+(x+y+z+w)^{*} && \text{Definition} \\
&=(a+b+c+d)+[(x+y)+(z+w)]^{*} && \text{associativity} \\
&=(a+b+c+d)+[(x+y)^{*}+(z+w)^{*}] && \text{Thm: } (s+t)^{*}=s^{*}+t^{*} \\
&=(a+b+c+d)+[(x^{*}+y^{*})+(z^{*}+w^{*})] && '' \\
&=(a+x^{*})+(b+y^{*})+(c+z^{*})+(d+w^{*}) && \text{associative \& commutative} \\
&=(a-x)+(b-y)+(c-z)+(d-w) && \text{Definition. } \square
\end{aligned}
$$

4b. Prove:

$$(a_1+a_2+\cdots+a_n)-(x_1+x_2+\cdots+x_n)=(a_1-x_1)+(a_2-x_2)+\cdots+(a_n-x_n)$$

Pf. Case $n=1$

$(a_1)-x_1=(a_1-x_1)$

Assume the conjecture is true for n-1. Does this imply it is true for n?

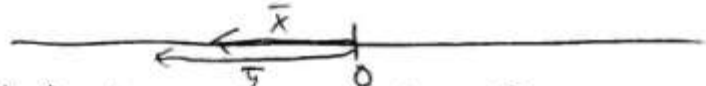
$$
\begin{aligned}
&(a_1+a_2+\cdots+a_{n-1}+a_n)-(x_1+x_2+\cdots+x_{n-1}+x_n) \\
&=(a_1+a_2+\cdots+a_{n-1}+a_n)+(x_1+x_2+\cdots+x_{n-1}+x_n)^{*} \\
&=(a_1+a_2+\cdots+a_{n-1}+a_n)+[(x_1+x_2+\cdots+x_{n-1})+(x_n)]^{*} && \text{perfect.} \\
&=(a_1+a_2+\cdots+a_{n-1}+a_n)+(x_1+x_2+\cdots+x_{n-1})^{*}+(x_n)^{*} \\
&=(a_1+a_2+\cdots+a_{n-1})+(x_1+x_2+\cdots+x_{n-1})^{*}+(a_n+x_n^{*}) \\
&=(a_1-x_1)+(a_2-x_2)+\cdots+(a_{n-1}-x_{n-1})+(a_n-x_n) && \square \text{ By Induction}
\end{aligned}
$$

5)

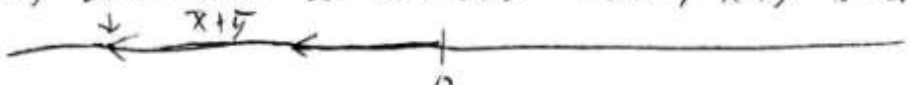
Section 2.3 4

4) Case (ii) Both x and y are negative $\bar{x}+\bar{y}$

(2/2) This means we have two vectors $\bar{x}$ and $\bar{y}$ left pointing as follows



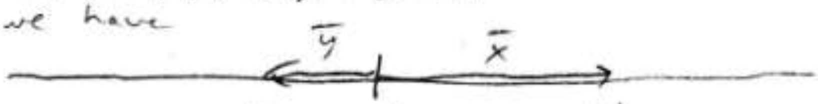
So by definition of a vector sum, $\bar{x}+\bar{y}$ is left pointing



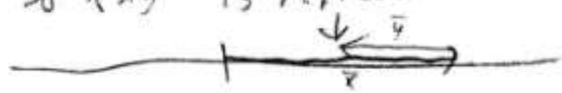
and is the length of x and y in the negative direction, namely $\bar{x}+\bar{y} = -(x+y)$ where x is $[x,0]$ as a length and y is $[y,0]$ as a length. Same as adding fractions that are negative.

Case (iv) x is positive but y is negative, and the length of $\bar{y}$ is less than the length of $\bar{x}$.

So we have



So $\bar{x}+\bar{y}$ is represented



So $x+y$ is positive and the length $[0, x+y]$ is the length $\bar{x}$ minus the length of $\bar{y}$ which is the definition of $\bar{x}+\bar{y}$ in this case. good.

6)

7)

(2/5) 3) (a) Find a simple proof [illegible] $(-1)n = -n$ for a whole number n also using [illegible]

Using the hint we know that $(-1)n = (-1)[\underbrace{1+\dots+1}_{n\ 1's}] = [\underbrace{-1-\dots-1}_{n\ -1's}]$

where the last equality holds by the distributive law. Now, if we can add this to n and get 0, we will have $(-1)n = -n$ by lemma*. That is, we want to show that $n + (-1)n = 0$ because then we would have $(-1)n = -n$. Let's see if this works: $n + (-1)n = [\underbrace{1+1+\dots+1}_{n\ 1's}] + [\underbrace{-1-\dots-1}_{n\ -1's}] =$

lets do it!

$= \underbrace{(1-1)+\dots+(1-1)}_{n} = \underbrace{0+\dots+0}_{n} = 0$ where the second equality holds by #6 in 2.2. Hence we have $n + (-1)n = 0$ as needed and $(-1)n = -n$.

nice.

b) Prove $(-1)(-1) = 1$.

We can make lemma* [illegible] of 1 as $-(-1)$. Then if we can show that $(-1)(-1) + (-(1)) = 0$, then by lemma* we will have that [illegible] $= -(-(1)) = 1$. Hence

$(-1)(-1) + (-(1)) = (-1)(-1) + (-1)$ by [illegible]

$= (-1)((-1) + (1))$ by distributive law

$= (-1)(0)$ by A3*

$= 0$

Hence we have [illegible] $= -(-(1))$, as needed.

c) Explain why $(-3)(-2) = 6$ by using $(-1)(-1) = 1$.

Lets first observe that $-3 = (-1)\cdot 3$ and [illegible] which we can see to be true by [illegible] multiplication of whole numbers. Now we can see that $(-3)(-2) = [(-1)\cdot 3][(-1)\cdot 2]$ by [illegible] associative. Now we can [illegible] the commutative and associativity of multiplication to get $[(-1)\cdot 3][(-1)\cdot 2] = (-1)(-1)[3\cdot 2]$ and we know that [illegible] $1\cdot[3\cdot 2] = 1\cdot 6 = 6$ by our normal rules of multiplication of whole numbers.

perfect.

d) Prove $(-m)(-n) = mn$ using $(-1)(-1) = 1$.

We know $(-m) = (-1)m$ and $(-n) = (-1)n$ so we have that $(-m)(-n) = (-1)m\cdot(-1)n = (-1)(-1)m\cdot n = 1\cdot m\cdot n = mn$

8)

$= 1 \times (m \times n) = m \times n$

7/3

a) Show $\forall m,n \in \mathbb{Z}$ $m,n \geq 0$ $(-m)n = -(mn) =: P(m)$

Induction of m.

Let $P(m) := (-m)n = -(mn)$ $\forall n := (m^*)n = (mn)^*$

Base Case $m=0$ WTS $(-0)n = -(0n)$

$\Leftrightarrow 0n = -0$ $(0n = 0)$

True ✓ $\Leftrightarrow 0 = 0$

Inductive Hypothesis $P(m) \Rightarrow P(m+1)$

WTS $(-(m+1))n = -[(m+1)n]$

$\Leftrightarrow (m+1)^* n = (mn+n)^*$ $s^* + t^* = (s+t)^*$

$\Leftrightarrow (m^* + 1^*)n = (mn)^* + n^*$ distribute

$\Leftrightarrow m^* n + 1^* n = mn^* + n^*$

WTS $1^* n = n^*$

$\Leftrightarrow 1^* n - n^* = 0$

$\Leftrightarrow -1n + n = 0$

$\Leftrightarrow n(-1+1)^0 = 0 \Leftrightarrow 0n = 0$ ✓

$\Leftrightarrow m^* n = mn^*$

True by Inductive Hypothesis.

WTS ?

$1^* n^* = n$

$\Leftrightarrow -1n^* + n^* = 0$

$\Leftrightarrow n^*(-1+1) = 0$

$\Leftrightarrow n^*(0) = 0$

$\Leftrightarrow 0 = 0$ ✓

b) Show $\forall m,n, \mathbb{Z}$ $m,n > 0$ $(-m)(-n) = mn =: P(m)$ $\forall n$

Induction on m

Base Case $m=0$ WTS $(-0)(-0) = (0)(0)$

$\Leftrightarrow 0 \cdot 0 = 0 \cdot 0$

True ✓ $\Leftrightarrow 0 = 0$ ✓

Inductive Hypothesis $P(m) \Rightarrow P(m+1)$

WTS $(-(m+1))(-n) = (m+1)n$

$\Leftrightarrow (m+1)^* n^* = mn + n$ $(-x) = x^*$

$\Leftrightarrow (m^* + 1^*)n^* = mn + n$ $s^* + t^* = (s+t)^*$

$\Leftrightarrow m^* n^* + 1^* n^* = mn + n$ distributive

$\Leftrightarrow m^* n^* = mn$

True by Inductive hypothesis

good.

9)

2.4, #6 (9) a.) $3x < x$

2/2

Claim: Sometimes true.

This is true only for $x < 0$.

Examples:

x	$3x < x$	
-2	$-6 < -2$	True
-1	$-3 < -1$	
$-1/2$	$-3/2 < -1/2$	
$1/2$	$3/2 < 1/2$	False
1	$3 < 1$	
2	$6 < 2$	

b.) $\frac{1}{10}x > x$

Claim: Sometimes true

this is only true for $x < 0$

Examples:

x	$\frac{1}{10}x > x$	
-2	$-1/5 > -2$	True
-1	$-1/10 > -1$	
$-1/2$	$-1/20 > -1/2$	
$1/2$	$1/20 > 1/2$	False
1	$1/10 > 1$	
2	$1/5 > 2$	

good.

10)

$\left(\frac{2}{2}\right)$ 9 a) $\frac{3}{4}(-x) = x+49$

$\frac{3}{4}(-1)(x) = x+49$ $\qquad -x = (-1)x$

$-\frac{3}{4}x = x+49$ $\qquad$ assoc/comm $\quad (-1)n = -n$

$+(-x) \quad +(-x)$ $\qquad$ add $(-x)$ to both sides

$-\frac{3}{4}x + (-x) = x+49+(-x)$

$-\frac{3}{4}x + (-1)x = x + (-x) + 49$ commut

$\left(-\frac{3}{4} - 1\right)x = (x + (-x)) + 49$ distributivity + associativity

$-\frac{7}{4}x = 0 + 49 = 49$ $x + x^{*} = 0$

Multiply both sides by $\left(-\frac{4}{7}\right)$

$\left(-\frac{7}{4}\right)\left(-\frac{4}{7}\right)x = 49\left(\frac{-4}{7}\right)$

$x = \frac{7 \times 7 \times (-4)}{\cancel{7}} = 7 \times (-4) = -28$

b) $2 \cdot t = t^2 + \frac{4}{7}t$

$-2t \qquad -2t$ subtract $2t$ from both sides

$0 = t^2 + \frac{4}{7}t - 2t$

$0 = t^2 + \left(\frac{4}{7} - 2\right)t$ distributivity

$0 = t^2 + \left(-\frac{10}{7}\right)t$ $\frac{4}{7} - 2 = \frac{4}{7} - \frac{2 \times 7}{7} = \frac{4 - 14}{7} = -\frac{10}{7}$

$0 = t \cdot t + \left(-\frac{10}{7}\right)t$ factor t^2

$0 = t\left(t + \left(-\frac{10}{7}\right)\right)$ distributivity

$0 = t\left(t - \frac{10}{7}\right)$ definition of subtraction/negative

Since we know $0 \cdot x = 0 = x \cdot 0 \quad \forall x \in \mathbb{Q}$

Either t or $t - \frac{10}{7}$ must be zero

If t is zero, twice definitely exceeds $0^2(0)$ by any fraction of zero (zero again)

If $t - \frac{10}{7} = 0 \quad t = \frac{10}{7} \quad \left[\frac{10}{7} - \frac{10}{7} = 0\right)$

Lets check whether

$2t = \frac{20}{7} = t^2 + \frac{4}{7}t = \frac{100}{49} + \frac{4}{7}\frac{10}{7} = \frac{140}{49}$

good.

$\frac{20}{7} = \frac{20 \times 7}{7 \times 7} = \frac{140}{7}$ correct