Homework 11

Solutions

1)

6a $\frac{2}{2}$ 1) Prove $x = b/a$ is a solution of $ax=b$ where $a \neq 0$ and that any soln of the eqzn is b/a.

To prove $x_0 = b/a$ is a solution of $ax=b$, it's sufficient to let $x = x_0 = b/a$ in $ax=b$ and see if we get a true statement. So, $ax = ax_0 = a(b/a) = b$ so we get $ax=b$ when substituting x for b/a, thus $x_0 = b/a$ is a solution of $ax=b$. To show that this solution is unique let x' also be a solution to $ax=b$ such that $ax'=b$ we want to show that $x' = b/a$. By (F2) we can multiply both sides of the equation by $1/a$ to get $\frac{1}{a} \cdot ax' = \frac{1}{a} b$. Hence we have that $\frac{a}{a} x' = \frac{1}{a} \cdot b \leftrightarrow 1x' = \frac{b}{a} \Leftrightarrow x' = \frac{b}{a}$ as needed.

2)

6.2, #2 ② if $a-c \neq 0$ then...

$\frac{7}{8}$

$$a\left(\frac{d-b}{a-c}\right)+b=c\left(\frac{d-b}{a-c}\right)+d$$

$$a\left(\frac{d-b}{a-c}\right)-c\left(\frac{d-b}{a-c}\right)+b=d \quad \left[\text{subtract}: c\left(\frac{d-b}{a-c}\right)\right]$$

$$\frac{d-b}{a-c}(a-c)+b=d \quad \left[\text{factor out}: \left(\frac{d-b}{a-c}\right)\right]$$

$$(d-b)+b=d \quad \left[\text{cancel out}: (a-c)\right]$$

$$d=d \quad \underline{\text{True!}}$$

Thus our equation works for all b & d.

3)

2/2

1) Formulate a precise statement about a linear eqzn in x, $ax+b=cx+d$, where a,b,c,d are constants so that the eqzn (a) has a unique soln, (b) has no soln (c) has infinite solutions

(a) By the theorem on page 302, if $a-c \neq 0$. Then the unique solution is $\frac{d-b}{a-c}$ and by problem two, $x=\frac{d-b}{a-c}$ is a solution to the equation.

(b) Let $a=c$ and $d-b \neq 0$. Then $ax+b=cx+d \Leftrightarrow$ $ax+b=ax+d$ where $d \neq b$. Then using E1 and E2 we can solve for x to get $x=\frac{d-b}{(a-a)}$. So if $\exists$ a solution, this would be it. But since $a-a=0$, you cannot divide by zero so there is no solution

(c) Let $a=c$ and $b=d$. Then $ax+b=cx+d=ax+b$ so any x would satisfy this equation. Thus, the equation has infinite solutions.

4)

4) 6,2,8

Given three consecutive whole numbers.

IF the sum of the smallest plus twice the next number plus three times the largest is 110, what are the numbers

First we should declare our variables which are 3 consecutive whole numbers.

$X_1 :=$ smallest of 3 whole numbers

$X_2 := X_1 + 1$

$X_3 := X_2 + 1 = X_1 + 2$

So I have my 3 consecutive whole numbers

$X_1, X_2 = X_1 + 1, X_3 = X_1 + 2$

Now read the problem in small chunks and make sure your equation is set up as stated

We have the sum of 3 things

the smallest $= X_1$

twice the next $= 2X_2$

three times the largest $= 3X_3$

and the sum of the 3 is 110. So

$X_1 + 2X_2 + 3X_3 = 110$

We know $X_2 = X_1 + 1$ and $X_3 = X_1 + 2$

so substituting in

$X_1 + 2(X_1 + 1) + 3(X_1 + 2) = 110$

$\Rightarrow X_1 + 2X_1 + 2 + 3X_1 + 6 = 110$

$\overset{(E1)}{\Rightarrow} 6X_1 = 102 \overset{(E2)}{\Rightarrow} X_1 = 17$

Substituting into $X_2 = X_1 + 1$ and $X_3 = X_1 + 2$

$\boxed{X_1 = 17, X_2 = 18, X_3 = 19}$ Now check

$17 + 2(18) + 3(19) = 17 + 36 + 57$

$= 53 + 57 = 110$ ✓

5)

Section 6.3 1, 7, 8, 10, 11 $\frac{2}{2}$

1) Line is a graph of both $ax+by=c$ and $a'x+b'y=c'$
$\rightarrow \exists k$ s.t. $(a',b',c') = \lambda(a,b,c)$ $(a'=ka, b'=kb; c'=kc)$

Case 1 $b, b'=0$ So $ax=c$, $a'x=c'$ So $x=\frac{c}{a}$

So $a'\left(\frac{c}{a}\right)=c'$ so $\frac{a'}{a}c=c'$ So let $k=\frac{a'}{a}$ then $ck=c'$

and by cross multiplication $ak=a'$ and since $bk=b'$ but b and b' are 0 So this holds as well
Case 1 is proven.

Case 2 $a, a'=0$ This is similar to case 1. Just replace x by y and a by b and a' by b' and the same argument holds.

Case 3 This is when none of $a, a', b, b'=0$ So $a, a', b, b' \neq 0$
So take the two lines. $ax+by=c$ and $a'x+b'y=c'$. Let $x=0$ since there must be an $x=0$ on the line. Then $by=c$ $b'y=c'$
so $y=\frac{c}{b}$ $y=\frac{c'}{b'}$ so $\frac{c}{b}=\frac{c'}{b'}$ and $\frac{c}{c'}=\frac{b}{b'}$. Now let $y=0$, so
$x=\frac{c}{a}$ and $x=\frac{c'}{a'}$ so $\frac{c}{a}=\frac{c'}{a'}$ and $\frac{c}{c'}=\frac{a}{a'}$ Now we know
$\frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}$ let all 3 equal k. So now $\frac{a'}{a}(a)=a'$ So $a'=a'$
and similarly $\frac{b'}{b}(b)=b'$ So $b'=b'$ and $\frac{c'}{c}(c)=c'$ so $c'=c'$ and this is proved for the final case.

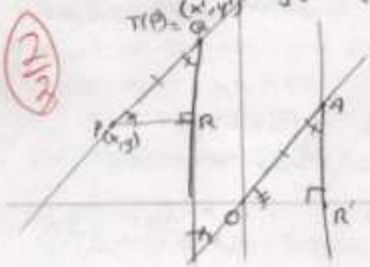
The converse is a trivial proof.
$ax+by=c$ and $kax+kby=kc$ now factor out k
$k(ax+by)=k(c)$ now since $k\neq 0$ multiply both sides by $\frac{1}{k}$ so
$ax+by=c$ and so the two lines are the same.

6)

7) Let A be the point with coords, (a_1, a_2) in the coord plane and let translation along the vector $\overline{OA}$ (O being the origen denoted by T.

(a) Prove that $T(x,y) = (x+a_1, y+a_2)\ \forall\ (x,y)$.

2/2



Consider the point $P=(x,y)$ then let $T(P) = Q$ and since the translation preserves distance $\overline{PQ} = \overline{OA}$. extending the R_Q vertically and R_P horiz. and R_A vertically such that $R_{AP} = R$ and $R_{AO} = R'$ we have two right triangles. More precisely, we have that line $AR' \parallel QR$ and $OR' \parallel PR$ thus since we have corresponding congruent angles $\angle PQR = \angle OAR'$ similarly $\angle QPR$ is congruent to $\angle AOR'$ thus we have congruent triangle. Now $|PR| = x'-x$ and $|QR| = y'-y$ and $|OR'| = a_1$ and $|AR'| = a_2$

since the triangles are congruent $y'-y = a_2$ and $x'-x = a_1$

by translation $T(x,y) = (x',y')$ and $y' = a_2 + y$ and $x' = a_1 + x$

plugging this in $T(x,y) = (x+a_1, y+a_2)$ thus proving what we want.

b) If L is the vertical line defined by $x = \frac{2}{7}$ what is the eqtation of T(L)?

T(L) is defined by $T(\frac{2}{7}, 0) = (x+a_1, 0+a_2)$ since we have a vertical line.

plugging this in $T(\frac{2}{7}, 0) = \frac{2}{7} + a_1, a_2) = T(L)$.

$x = \frac{2}{7} + a_1$

c) If L is the horizontal line defined by $y = 51$, what is the equation of T(L).

with similar reasoning

$T(x,y) = T(0, 51) = (0, 51 + a2)$ is our final equations.

$y = 51 + a_2$

d) If L is defined by $2x - 3y = 1$ what is the equation of T(L)

If we apply ϵ_1 and ϵ_2 $y = \frac{2}{3}x - \frac{1}{3}$. Thus from Ex 3 $(y-0) = \frac{2}{3}(x - \frac{1}{2})$

and we have $(\frac{1}{2}, 0)$ and $(0, -1/3)$ as points for line L.

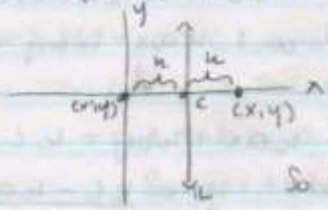
T(L) would have $(1/2 + a_1, a_2)$ and $a_1, -1/3 + a_2)$ as $y - a_2 = \frac{a_2 - (-\frac{1}{3} + a_2)}{1/2 + a_1 - a_1} \cdot (x - \frac{1}{2} + a_1))$ $\Rightarrow y - a_2 = \frac{2}{3}(x - (\frac{1}{2} + a_1))$

good

7)

2/2

8. let $x=c$ be a vertical line. Prove the reflection with respect to L is the transformation $R(x,y)=(2c-x,y)$



(a) let (x,y) be k distance from c. Then reflection, (x',y') is k distance from c. So the point $(x,y)-(c)=k$

So if $R(x,y)=(2c-x,y)$ then $c-R(x,y)=k$

$c-R(x,y)=c-(2c-x,y)$ => b/c we are only interested in the x value. look at $c-(2c-x)=-c+x=x-c=k$ so the distance from $(2c-x,y)$ and c is k $\therefore$ $R(x,y)$ is the reflection.

(b) Formulate the same statement for $y=d$.

=> So because this is a horizontal line the x value will not be affected, but the y value will.

$\therefore R(x,y)=(x,2c-y)$.

8)

6.3.10 (a) We have two points (x,y) and (z,w).
If $x=z$, then the one of the points are above the other. Hence we have a vertical line going through them.
$x = x$.

If $y=w$, then the points are horizontal. Hence, the horizontal line going through them is $y=y$.

If $(x,y) \neq (z,w)$, then the slope of the the line going through them is $\frac{w-y}{z-x}$.

So the line has the form

$$y = \left(\frac{w-y}{z-x}\right)x + k.$$

Since the point (z,w) lies on the line, we have $w = \left(\frac{w-y}{z-x}\right)z + k$.

So $k = w - \left(\frac{w-y}{z-x}\right)z$

So $y = \left(\frac{w-y}{z-x}\right)x + w - \left(\frac{w-y}{zy}\right)z$

(b) The line has the form

$$y = Ax + k.$$

Since $(0,B)$ is on the line, we have $B = A(0) + k$.

So $k = B$.

So $y = Ax + B$.

9), 10)

6.3.11 using Ex 3 p 327, $y - d^3 = \frac{d^3 - c^3}{d - c}(x - d)$

2/2

$\Rightarrow y - d^3 = \left(\frac{d^3 - c^3}{d - c}\right)x - \left(\frac{d^3 - c^3}{d - c}\right)d$

$\Rightarrow y = \left(\frac{d^3 - c^3}{d - c}\right)x - \left(\frac{d^3 - c^3}{d - c}\right)d + d^3$

$\Rightarrow y = (d^2 + dc + c^2)x - (d^2 + dc + c^2)d + d^3$ by identity p283

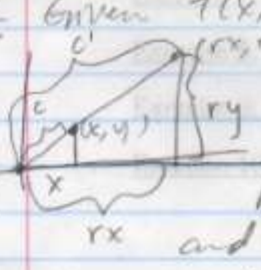
$\Rightarrow y = (d^2 + dc + c^2)x - d^3 - d^2c - c^2 + d^3$

$\Rightarrow y = (d^2 + dc + c^2)x - d^2c - c^2$.

$\Rightarrow \boxed{y = d^2(x - c) + c^2(x - 1) + dcx}$

6.4.2 Given $T(x, y) = (rx, ry)$

1.5/2



By pythagorean theorem
$c = \sqrt{x^2 + y^2}$ and
$c' = \sqrt{r^2x^2 + r^2y^2} = r\sqrt{x^2 + y^2}$

By FTS, $\{O, (x, y), (rx, ry)\}$ is colinear and thus is a dilation w/ center O and scale factor r.

Also need to show every point mapped to under the dilation is able to be mapped to under this transformation.