Homework 1

Solutions

1)

$\frac{1}{1}$ 1) Rational numbers are equivalence classes.
If $(a,b)\sim(a',b')$ and $(c,d)\sim(c',d')$
then

$$\frac{a}{b}+\frac{c}{d}=\frac{a'}{b'}+\frac{c'}{d'}$$

Prove that addition is well defined on equivalence classes.

$$\frac{a}{b}=\frac{a'}{b'} \Rightarrow ab'=a'b \text{ and } \frac{c}{d}=\frac{c'}{d'} \Rightarrow cd'=c'd$$

Addition is defined as

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd}$$

and

$$\frac{a'}{b'}+\frac{c'}{d'}=\frac{a'd'+c'b'}{b'd'}$$

So the question is... Does

$$\frac{ad+cb}{bd}\stackrel{?}{=}\frac{a'd'+c'b'}{b'd'}$$

good

If it did then

$$b'd'(ad+cb)\stackrel{?}{=}bd\,(a'd'+c'b')$$

We start with the left side and make substitutions

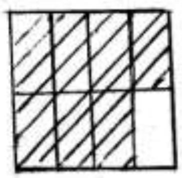
$$\begin{aligned} b'd'(ad+cb) &= ab'dd' + cd'bb' \\ &= (a'b)dd' + (c'd)bb' \\ &= a'd'(bd) + b'c'(bd) \\ &= bd\,(a'd'+c'b') \quad \blacksquare \end{aligned}$$

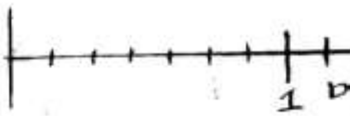
So they are equal, and addition is well defined on equivalence classes of $\mathbb{Q}$.

2)

(3/3) 2) Ch 1 Sec 1 Ex 2

Suppose the unit 1 on the number line is the area of the following shaded region obtained from a division of a given square into eight congruent rectangles. (rectangles have equal areas)





1 by Definition

Each rectangle which is $\frac{1}{8}$ of the square above is worth 1 step on this number line, where the unit has been defined as 7 rectangles from the above square. Each $\frac{1}{8}$ of the square has a value of $1/7$ on the number line.

good explanation

Write the fraction representing the shaded areas:

$A = 4/7$ $B = 3/7$ $C = 12/7$

Thanks for using a ruler, by the way. Those are some crisp drawings.

Find how many $\frac{1}{8}$ squares you have and that is how many steps you have in the sequence of 7-ths.

$A = 4$ rectangles of $\frac{1}{8} = \frac{4}{7}$ } $B = 3$ of $\frac{1}{8} = \frac{3}{7}$ } $C = 1 + \frac{1}{2} = \frac{8}{8} + \frac{4}{8} = 12$ of $\frac{1}{8}$

$C = 12/7$

3)

$\frac{1}{1}$

b) the area of one of these rectangles is $\frac{1}{478 \times 2043}$ because we think of the unit square as having area 1, then we break up the unit square into 478×2043 little

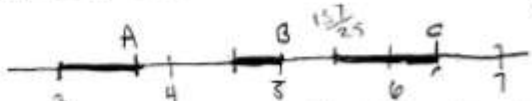
exactly rectangle each of equal area (if it were a number

right line each rectangle would be of equal length) meaning we broke up the square into nth in this case into 478×2043 th so one of these squares is $\frac{1}{478 \times 2043}$ just like on a number line each nth is $\frac{1}{n}$. Good.

4)

$\frac{1}{7}$

1.1 #8 It is known that the left segment has length $\frac{11}{16}$ middle $\frac{8}{17}$ and right = $\frac{23}{25}$
what is A, B, C?



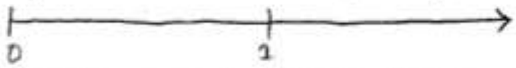
A: $\frac{3}{3}$ is can be broken into 16 pieces and have 48. making it $\frac{48}{16}$. A is $\frac{11}{16}$ to the right hence $A = \frac{59}{16}$.

B: the segment is $\frac{8}{17}$. we can convert 4 into 68 pieces of $\frac{1}{17}$ thus B is 9 $\frac{1}{17}$ths to the right so $B = \frac{77}{17}$.

C: C is $\frac{23}{25}$ beyond $\frac{137}{25}$ thus $C = \frac{160}{25}$

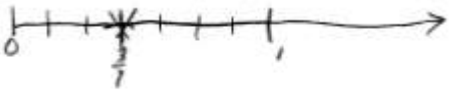
5)

$\frac{1}{1}$ 2) $\frac{6}{14} = \frac{3}{7}$ Lets look at the number line.



If we break 0 to 1 into 7 equal pieces

and then take 3 of them (starting at 0) we represent $\frac{3}{7}$.

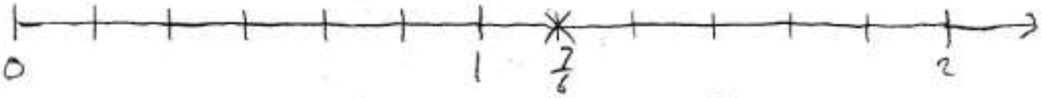


Now take that same line and break it into 14 pieces, notice how each piece is split in half.

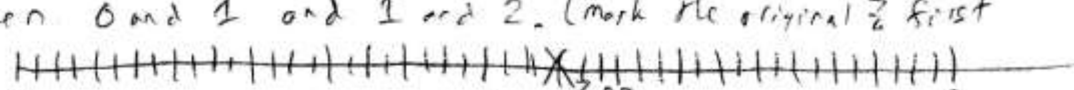
now count 6 spaces in. Where do you land? Exactly, right on the same place as $\frac{3}{7}$. So $\frac{3}{7}$ and $\frac{6}{14}$ are equivalent meaning the same or equal. This is why $\frac{6}{14} = \frac{3}{7}$

good

$\frac{28}{24} = \frac{7}{6}$ Lets look at a number line from 0 to 2 broken into 6 equal pieces between 0 and 1 and 1 and 2.

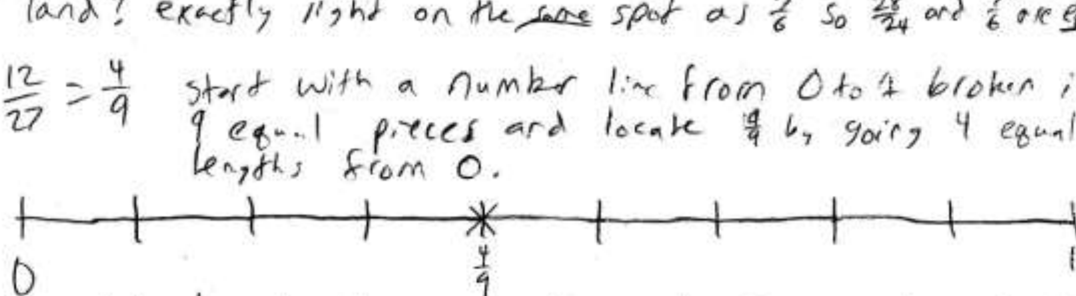


Now count 7 equal segments to locate $\frac{7}{6}$ above.
now lets break this same number line into 24 equal pieces between 0 and 1 and 1 and 2. (mark the original $\frac{7}{6}$ first

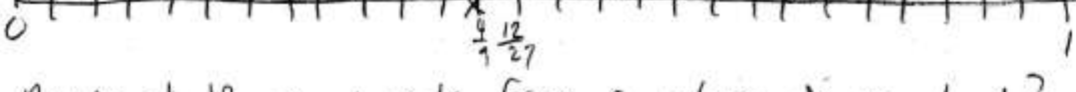


now count 28 equal lengths starting at 0, where do you land? exactly right on the same spot as $\frac{7}{6}$ so $\frac{28}{24}$ and $\frac{7}{6}$ are equal.

$\frac{12}{27} = \frac{4}{9}$ start with a number line from 0 to 1 broken into 9 equal pieces and locate $\frac{4}{9}$ by going 4 equal lengths from 0.



now lets break this same line into 27 equal parts, leaving $\frac{4}{9}$ where it is

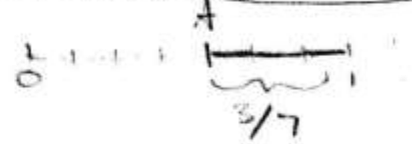


Now count 12 equal parts from 0, where do you land? exactly, right at the same place as $\frac{4}{9}$ so $\frac{4}{9} = \frac{12}{27}$.

Wonderfully explained. Your 8th grade student now understands not only the problem, but that equal = equivalent = same place on the number line

6)

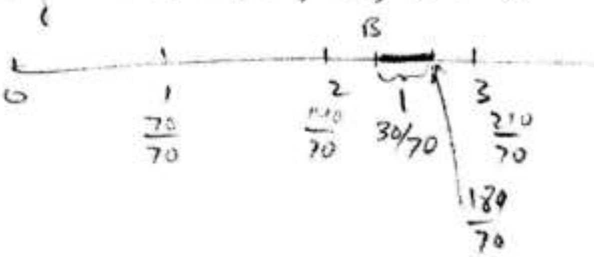
(6) If we split $[0,1]$ into 7 parts, we find that the thickened segments are $\frac{3}{7}$. good



To find B, we convert 2.7 into a fraction $- \frac{27}{10}$.
Second, convert $3/7$ and $27/10$ into same denominators.

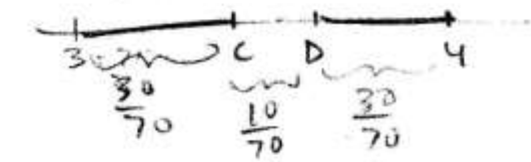
$$\frac{3}{7} = \frac{10 \times 3}{10 \times 7} = \frac{30}{70}, \quad \frac{27}{10} = \frac{7 \times 27}{7 \times 10} = \frac{189}{70}$$

Splitting $[0,1]$, $[1,2]$, $[2,3]$ into 70 parts,



we find that there are 19 and 21 parts of $\frac{1}{70}$ to the left and right of B respectively.
By concatenating $\frac{140}{70}$ and $\frac{19}{70}$, we get $B = 159/70$.

To get C, it is 30 parts of $\frac{1}{70}$ right of $3 = \frac{210}{70}$, so $C = 240/70$.



There are 10 parts of $\frac{1}{70}$ b/w C and D, so concatenating 10 parts of $\frac{1}{70}$ onto C, we get $D = 250/70$.

To find F, convert $\frac{13}{3}$ to $\frac{7 \times 13}{7 \times 3} = \frac{91}{21}$ and $\frac{3}{7} = \frac{3 \times 3}{3 \times 7} = \frac{9}{21}$. So by concatenating 9 parts of $\frac{1}{21}$ to $\frac{91}{21}$, we'd get $F = \frac{100}{21}$.

good

7)

Ex. 7) (1, 2, #8)

(a) $\frac{3}{7}$ of a fraction is equal to $\frac{5}{6}$. What's this fraction?

$\frac{3}{3}$

Step(1): Draw a # line with an unknown fraction F

(2) Partition $[0, F]$ into $\frac{1}{7}$ths and locate the 3^{rd} point from 0. The 3^{rd} point is $\frac{5}{6}$

(3) Force the numerator (5) of $\frac{5}{6}$ to be divisible by 3 so we can get the length of each sub-segment.

By equivalent fractions thm: $\frac{5}{6} = \frac{5\times 3}{5\times 3} = \frac{15}{18}$. (4) Then each segment is $\frac{5}{18}$, so (7) many such segments is $\frac{5\times 7}{18} = \boxed{\frac{35}{18}}$ ☑

(b) $\frac{m}{n}$ of a fraction is equal to $\frac{k}{l}$

$\Rightarrow$ Use Thm 4: if $\frac{k}{l} = \frac{K}{L}$ and $\frac{m}{n} = \frac{M}{N}$, then $\frac{k}{l}$ of $\frac{m}{n} = \frac{K}{L}$ of $\frac{M}{N}$

let the unknown fraction be $\frac{a}{b}$,

If $\frac{m}{n} = \frac{M}{N}$ and $\frac{a}{b} = \frac{A}{B}$,

then $\frac{m}{n}$ of $\frac{a}{b} = \frac{M}{N}$ of $\frac{A}{B}$ perfect.

then $\frac{ma}{nb} = \frac{MA}{NB}$ by proof of Thm 4 on pg. 39

then $\frac{ma}{nb} = \frac{MA}{NB} = \frac{k}{l} \Rightarrow mal = nbk$ by Thm. 2 (cross-X) $\Rightarrow \boxed{\frac{a}{b} = \frac{nk}{ml}}$ ☑

8)

Ex 8) (1.2, #12)

(a) For which fraction $\frac{m}{n}$ is it true that $\frac{m}{n} = \frac{m+1}{n+1}$?

2/2

$\frac{m}{n} = \frac{m+1}{n+1} \iff m(n+1) = n(m+1)$ by cross multiplication (Thm. 2)

(distributive prop.) $\iff mn + m = mn + n$

(note $(mn+n) > mn$) $\iff (mn+m) - mn = (mn+n) - mn$ (note: $mn = nm$)

$\iff (mn - mn) + m = (mn - mn) + n$

$\iff 0 + m = 0 + n$

$\iff m = n$

$\therefore$ $\boxed{\frac{m}{n} = \frac{m+1}{n+1} \text{ is only true if } m = n}$ ☑

(b) For which fraction $\frac{m}{n}$ is it true that $\frac{m}{n} = \frac{m+b}{n+b}$ for $b > 0, \in \mathbb{N}$?

$\frac{m}{n} = \frac{m+b}{n+b} \iff m(n+b) = n(m+b)$ by Thm. 2

$\iff mn + mb = mn + nb$

$\iff (mn + mb) - mn = (mn + nb) - mn$

$\iff (mn - mn) + mb = (mn - mn) + nb$

$\iff 0 + mb = 0 + nb$

$\iff mb = nb$

$\iff m = n$

$\therefore$ $\boxed{\frac{m}{n} = \frac{m+b}{n+b} \text{ is only true if } m = n}$

great. ☑

9)

Ex 9) Prove that between any 2 fractions A and C,
(1.2, HB) there is a fraction B (i.e. $A < B < C$)

PROOF. by construction. Let $A < C$

$\left(\frac{1}{1}\right)$ Let $A = \frac{a_1}{a_2}$, $C = \frac{c_1}{c_2}$ for $a_1, a_2, c_1, c_2 \in \mathbb{Z}^+, \neq 0$

Force A and C to have the same denominator by FFFP.
The most obvious choice is $a_2 c_2$,

so $A = \frac{a_1 c_2}{a_2 c_2}$ & $C = \frac{a_2 c_1}{a_2 c_2}$, $\left(\text{let } a_2 c_1 > a_1 c_2 \text{ since } A < C\right)$.

Note: For simplicity, let $a_1 c_2 = x$, $a_2 c_1 = y$, $a_2 c_2 = z$, then $A = \frac{x}{z}$ & $C = \frac{y}{z}$ for $x, y \in \mathbb{Z}^+$

Construct $B = \frac{b_1}{b_2}$ s.t. $A < B < C$,

Pictorially:

Case(1)
If $x+1 = y$, then notice $A = \frac{x}{z} = \frac{2x}{2z}$ by Thm. of equivalent fractions and $B = \frac{y}{z} = \frac{2y}{2z}$. Then by assumption, $x+1 = y$, $B = \frac{2(x+1)}{2z} = \frac{2x+2}{2z}$

By inspection, there exists a point in between $2x$ and $2x+2$ (namely $2x+1$). Since we know that $2x < 2x+1 < 2x+2$, then we partition each of these values into $\frac{1}{2z}$ ths: $\frac{2x}{2z} < \frac{2x+1}{2z} < \frac{2x+2}{2z}$. Or, equivalently, $A < B < C$ with $B = \frac{2x+1}{2z} = \frac{b_1}{b_2}$. $\boxed{\checkmark}$

good

Case (2)
If $x+1 \neq y$, then $x+1 < y$ since $x, y \in \mathbb{Z} \Rightarrow x < y$

so $x < x+1 < y \Rightarrow \frac{x}{z} < \frac{x+1}{z} < \frac{y}{z}$

$\Rightarrow A < \frac{x+1}{z} < C \Rightarrow B = \frac{x+1}{z} = \frac{b_1}{b_2}$ s.t. $A < B < C$

$\boxed{\checkmark}$

10)

1.3 #1) a) Explain how to obtain an algorithm for adding 3 fractions

$\frac{k}{l} + \frac{m}{n} + \frac{p}{q}$

we know that $\frac{k}{l} + \frac{m}{n} = \frac{kn + lm}{ln}$ by the algorithm of adding 2 fraction

to add $\frac{p}{q}$ to $\frac{kn+lm}{ln}$ we can treat $\frac{kn+lm}{ln}$ as one fraction and repeat

the steps thus

$\frac{p}{q} + \frac{kn+lm}{ln} \boxed{= \frac{pln + qlm + knq}{lnq}}$

b) If a, b, c are non zero whole numbers, what is $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}$

simplify as much as possible

$\frac{1}{ab} + \frac{1}{bc} = \frac{bc + ab}{ab^2c} = \frac{\cancel{b}(c+a)}{\cancel{b}\, abc} = \frac{c+a}{abc}$

then $\frac{c+a}{abc} + \frac{1}{ac} = \boxed{\frac{c+a+b}{abc}}$ great

11)

$\boxed{1.3 \# 11}$

$\frac{1}{3}$ Define $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

Prove $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

good.

plugging into the definition $\binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!\,k!}$ and $\binom{n-1}{k-1} = \frac{(n-1)!}{((n-1)-(k-1))!\,(k-1)!}$

so $\frac{(n-1)!}{(n-1-k)!\,k!} + \frac{(n-1)!}{(n-k)!\,(k-1)!} = \frac{(n-k)(n-1)!}{(n-k)!\,k!} + \frac{(n-1)!\,k}{(n-k)!\,k!}$

since we are missing a $(n-k)$ on the first fraction and a k in the second we can add those into the top and bottom to get similar fractions.

now we can add

$$\frac{(n-k)(n-1)! + (n-1)!\,k}{(n-k)!\,k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

thus $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

12)

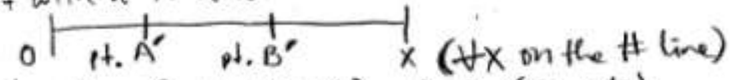
A

Ex 12) Prove the following statements for fractions A, B, C, D.
(1.3, #12)

(1) $A < B \Leftrightarrow \exists C$ s.t. $A + C = B$

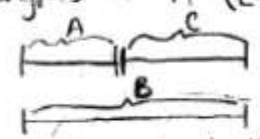
PROOF by the number line

- Start with a # line. Let $A<B$.



(2/2)

- the lengths of A ([0,A']) and B ([0,B']) are shown:

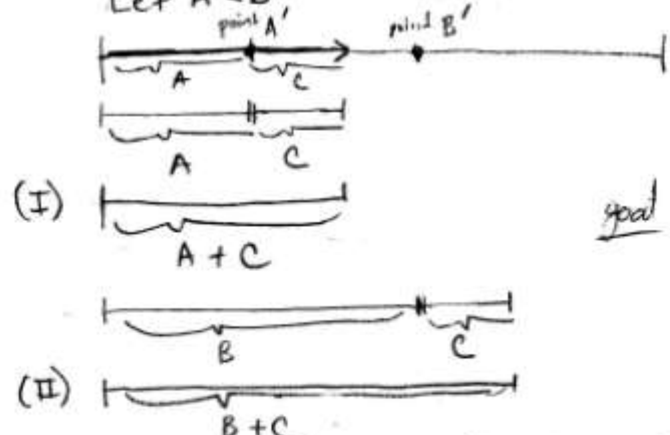


- Clearly, $\exists C$ such that the concatenation of A and B is exactly the length of B. ☑

(2) $A < B$ implies $A + C < B + C$ for every fraction C.

PROOF. by the # line

- Let A, B, and C be short segments on the # line: Let $A<B$

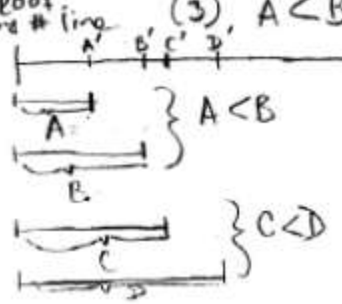


good

- Clearly (I) < (II), thus $A+C < B+C$ $\forall C$. ☑

PROOF by # line

(3), $A<B$ & $C<D$ implies $A+C<B+D$



- Let A, B, C, D be short segments on # line.

⟹ Clearly, $A+C<B+D$ by concatenating A and C and comparing it to the concatenation of B and D ☑