Trees, Sheaves and Definition by Recursion

Nathanael Leedom Ackerman
University of Pennsylvania

Presheaves on a Topological Space

Definition

Suppose T is a topological space with open sets $\mathcal{O}(T)$. A *presheaf* on T is a functor from $\mathcal{O}(T)^{op}$ into the category of sets.

Specifically if F is a presheaf then

- For each open set $U \in \mathcal{O}(T)$ we have a set $F(U)$.
- For each pair of open sets $U \subseteq V$ with $U, V \in \mathcal{O}(T)$ we have a map $F(i_{U,V}) : F(V) \to F(U)$
- If $U \subseteq V \subseteq W$ then $F(i_{U,V}) \circ F(i_{V,W}) = F(i_{U,W})$

Presheaves on a Topological Space

Definition

If $a \in F(U)$ and $V \subseteq U$ we will write $a|_V$ for $F(i_{V,U})(a)$.

Definition

If A and B are presheaves on T then we say $A \subseteq B$ if $A(U) \subseteq B(U)$ for all open set $U \in \mathcal{O}(T)$ and $(\forall x \in A(U))A(i_{V,U})(x) = B(i_{V,U})(x)$ whenever $V \subseteq U$.

Sheaves on a Topological Space

Definition

Suppose F is a presheaf on $\mathcal{O}(T)$. We say that F is a *sheaf* if whenever

- $U = \bigcup_{i \in I} U_i$
- $a_i \in F(U_i)$ for all $i \in I$
- $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ for all $i, j \in I$

then there is a unique element $a \in F(U)$ such that $a|_{U_i} = a_i$ for all $i \in I$.

Examples

Example

An example of a sheaf on $\mathbb{R}$ is the collection C of functions to $\mathbb{R}$. That is

- $C(U) = \{f : U \to \mathbb{R}\}$
- If $f \in C(U)$ and $V \subseteq U$ then $f|_V$ is the function whose domain is V and which agrees with f on its domain.

Examples

Example

An example of a presheaf on $\mathbb{R}$ which is not a sheaf is the collection B of all bounded functions to $\mathbb{R}$. That is

- $B(U) = \{f : U \to \mathbb{R} \text{ s.t. } (\exists M \in \mathbb{R})(\forall x \in U)|f(x)| \leq M\}$
- If $f \in B(U)$ and $V \subseteq U$ then $f|_V$ is the function whose domain is V and which agrees with f on its domain.

To see this isn't a sheaf let $U_i = (-i, i)$ and let $f_i = id_{(-i,i)}$. Each f_i is bounded and hence $f_i \in B(U_i)$ for each i. Further, these functions are compatible. However there is no way to "glue" them together to get a bounded function on all of $\bigcup_{i\in\omega}(-i, i) = \mathbb{R}$.

ω As A Topological Space

Definition

Let $\hat{\omega}$ be the topological space where

- The underlying set is $\omega = \{0, 1, 2, \dots\}$
- Open sets are ordinals $\alpha \leq \omega$ (where $\alpha = \{\beta \in \omega : \beta < \alpha\}$)

Motivating Observations

We have two important observations about $\hat{\omega}$:

Theorem

There is a first order theory, TREE, of trees such that models of TREE are the "same thing" as presheaves on $\hat{\omega}$.

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Theorem

There is a sheaf $\mathcal{N}$ on $\hat{\omega}$ such that

- $\mathcal{N}(\omega) = \omega^\omega$
- *For every subset $A \subseteq \omega^\omega$ there is a subpresheaf A^* of $\mathcal{N}$*
- *A^* is a sheaf if and only if A is a closed subset of $\mathcal{N}$.*

Definition of Trees

Definition

The language of our first order theory *TREE* has two collections of relations, $Lev_n(x)$ and $Pred_n(x, y)$, and our theory says:

- $Lev_n(x)$ holds if and only if there are exactly n predecessors of x.
- If there are at least n predecessors of x then $Pred_n(x, y)$ holds if y is above x on the tree and y has exactly n predecessors.

Notice for simplicity we will sometimes write $x <_1 y$ if

$$Lev_n(x) \text{ and } Lev_{n+1}(y) \text{ and } Pred_n(y, x)$$

for some $n \in \omega$.

Trees as Presheaves

We can then associate to every tree T a presheaf T^p on $\hat{\omega}$ where

- $T^p(n) = \{x : T \models Lev_n(x) \text{ for each } n \in \omega\}$
- $T^p(\omega) = \{x : T \models \bigwedge_{n\in\omega} \neg Lev_n(x)\}$
- If $m \subseteq n$ and $x \in T^p(n)$ then $x|_m$ is the unique element of T less than x and on level m.

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- If $m \subseteq n$ and $x \in T^p(n)$ then $x|_m$ is the unique element of T less than x and on level m.

Similarly to every presheaf P on $\hat{\omega}$ we can associate a tree P^t where

- $P^t = \bigcup_{n\leq\omega}(n)$
- $P^t \models Lev_n(x)$ if and only if $x \in P(n)$
- $P^t \models Pred_n(x, y)$ if and only if $x \in P(m)$ for some $m \geq n$ and $x|_n = y$.

Baire Space

Definition

Let $\mathcal{N}$ be the sheaf on $\hat{\omega}$ where

- If $m \leq \omega$ let $\mathcal{N}(m) = \{f : m \to \omega\}$
- If $m \leq n \leq \omega$ and $f \in \mathcal{N}(n)$ then $f|_m$ is the restriction of f to domain m.

In particular we then have $\mathcal{N}(\omega) = \omega^\omega$ or Baire space.

Baire Space and Subsheaves

Definition

(a) For every set $A \subseteq \omega^\omega$ let A^* be the presheaf where $A^*(m) = \{f : m \to \omega \text{ such that } (\exists x \in A) x|_m = f\}$.

(b) For every presheaf $\mathcal{A} \subseteq \mathcal{N}$ let $\mathcal{A}^o = \mathcal{A}(\omega) \subseteq \omega^\omega$

For every subpresheaf of $\mathcal{N}$ we get a subset of Baire Space and for every subset of Baire space we get a subpresheaf of $\mathcal{N}$.

Baire Space and Subsheaves

We then have

Theorem

(1a) *For every presheaf* $\mathcal{A} \subseteq \mathcal{N}$ *we have the set* $((\mathcal{A}^o)^*)^o = \mathcal{A}^o$

(1b) *For every set* $A \subseteq \omega^\omega$ *we have the presheaf* $((A^*)^o)^* = A^*$

(2a) *For any presheaf* $\mathcal{A} \subseteq \mathcal{N}$ *we have that* $\mathcal{A}$ *is a sheaf if and only if the set* $\mathcal{A}^o \subseteq \omega^\omega$ *is closed.*

(2b) *For any set* $A \subseteq \omega^\omega$ *we have* A *is closed if and only if* A^* *is a sheaf.*

Definition of Well-Founded Trees

Definition

We call the tree with a unique element at every level and a unique element at no level the *terminal tree* and denote it by 1_{TREE}

Definition

We say a tree T is *well-founded* if there is no map from the terminal tree 1_{Tree} into T. If P is a presheaf on $\hat{\omega}$ such that P^t is a well-founded tree then we also say that the presheaf P is *well-founded*

Well-Founded Trees and Baire Space

We then have the following important results concerning trees and Baire space

Theorem

If T is a tree such that the corresponding presheaf, T^p, is a sheaf, then then T^p is a well-founded sheaf if and only if $T^p(\omega) = \emptyset$

Theorem

If T is a countable well-founded tree then there is a monic $m : T^p \to \mathcal{N}$

Classical Transfinite Recursion

We can now translate the notion of "definition by transfinite recursion" into the language of sheaves.

Definition (Transfinite Recursion)

Suppose $T \subseteq \mathcal{N}$ is a well-founded sheaf, X is a set, and G, F are partial functions such that:

- $G : \bigcup_{n \leq \omega} \mathcal{N}(n) \to X$ and is total on $\bigcup_{n \leq \omega} \mathcal{N}(n) - \mathcal{T}(n)$
- F takes two arguments and returns a value in X. The first argument is an element of $\bigcup_{n \leq \omega} \mathcal{N}(n)$ and the second is a partial function $I^* : \bigcup_{n \leq \omega} \mathcal{N}(n) \to X$
- Further if $x \in \mathcal{N}(n)$ and I^* is defined on $\{y \in \mathcal{N}(n+1) : y|_n = x\}$ then $F(x, I^*)$ is defined.

Classical Transfinite Recursion

Definition (Transfinite Recursion Cont.)

We then define the partial functions I_α for $\alpha \in \mathrm{ORD}$ as follows:

- $I_0(x) = G(x)$ if $x \in \bigcup_{n \leq \omega} \mathcal{N}(n) - T(n)$ and undefined otherwise.
- $I_{\omega \cdot \alpha} = \bigcup_{\gamma < \omega \cdot \alpha} I_\gamma$
- $I_{\alpha+1}(x)$ breaks into three cases:
  1. If $I_\alpha(x)$ is defined then $I_{\alpha+1}(x) = I_\alpha(x)$
  2. Otherwise if $x \in \mathcal{N}(n)$ and $(\forall y \in \mathcal{N}(n+1)$ s.t. $y|_n = x) I_\alpha(y)$ is defined then $I_{\alpha+1}(x) = F(x, I_\alpha)$
  3. Otherwise $I_{\alpha+1}(x)$ is undefined.

We then let $I = \bigcup_{\alpha \in \mathrm{ORD}} I_\alpha$

Classical Transfinite Recursion

Theorem

I is a total function on $T(0)$

Proof.

Assume not.

Then there is some $x_0 \in T(0)$ such that $I(x_0)$ is undefined.

Then because of how I was defined at successor stages we must have some $x_1 \in T(1)$ such that $x_1|_0 = x_0$ and $I(x_1)$ is undefined (because otherwise $I(x_0)$ must be defined). Similarly for all $n \in \omega$ we get $x_{n+1} \in T(n+1)$ such that $x_{n+1}|_n = x_n$. □

Classical Transfinite Recursion

Proof.

Now because T is a sheaf and $\langle x_n : n < \omega \rangle$ is a compatible collection of elements there must be some element $x^* \in T(\omega)$ such that $x^*|_n = x_n$.

But this contradicts the fact that T is well-foundness of T □

Example of Rank

Example

As an example lets suppose $T \subseteq \mathcal{N}$ is a well-founded tree and we want to compute the rank of T, $\rho(r_T)$ (where $r_T \in T(0)$ is the root of T which is represented as the unique function from $\emptyset$ into ω).

To do this we say

- If $x \in \bigcup_{n\in\omega} \mathcal{N}(n) - T(n)$ then $G(x) = -1$.
- If $x \in T(n)$ and $I^*(y)$ is defined for all $y \in \mathcal{N}(n+1)$ with $y|_n = x$ then we let $F(x, \rho^*) = \sup\{\rho^*(y) + 1 : y \in \mathcal{N}(n+1), y|_n = x\}$

Then applying the previous method we get a function ρ_T which is total on $T(0)$ and hence defines the rank of the tree T.

Second Order Tree

We have seen how trees can be thought of as sheaves on the topological space $\hat{\omega}$. This then suggests the following definitions

Definition

If T is a tree let $\hat{T}$ be the topological space such that

- The underlying set of $\hat{T}$ is T.
- The sets $\hat{x} = \{y \in T : y < x\}$, for $x \in T$, are a sub-basis $\mathcal{O}(\hat{T})$

Definition

If T is a tree and S is a sheaf on $\hat{T}$ then we say that S is a *Second Order Tree*.

Well-Founded Sheaves

The classical construction of transfinite recursion suggests the following important definitions.

Definition

A sheaf S on a topological space T is *well-founded* if $S(T) = \emptyset$.

Definition

A sheaf S on a topological space T is *flabby* if for every element $a \in S(U)$ there is an element $a^* \in S(T)$ with $a^*|_U = a$

Notice that $\mathcal{N}$ is a flabby sheaf.

Well-Founded Sheaves

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A sheaf S on a topological space T is *well-founded* if $S(T) = \emptyset$.

Definition

A sheaf S on a topological space T is *flabby* if for every element $a \in S(U)$ there is an element $a^* \in S(T)$ with $a^*|_U = a$

Notice that $\mathcal{N}$ is a flabby sheaf. An important fact to notice is that

Theorem

If S is a second order tree then S is also a tree.

Example

Example

Let $T = 2^{<\omega}$ be a binary branching tree.

Let S be the sheaf on T generated as follows:

- $S(\emptyset)$ has a single point
- For any sequence $s \in 2^{<\omega}$, $S(s^\wedge\langle 0\rangle) = \emptyset$
- For any sequence s, if $x \in S(s)$ then there are ω many $y \in S(s^\wedge\langle 1\rangle)$ such that $y|_s = x$

We then have

- S is a well-founded second order tree.
- S, treated as a tree, is isomorphic to $\mathcal{N}$

Recursion on a Second Order Tree

We can give our definition of recursion on sheaves

Definition (Sheaf Recursion)

Suppose T is a tree, N is a flabby sheaf on $\hat{T}$, and $W \subseteq N$ is a well-founded subsheaf. Let Y be a set and G, F_V (for each $V \in T$) be partial functions such that:

- $G : \bigcup_{V \in T} N(V) \to Y$ and is total on $\bigcup_{V \in T} N(V) - W(V)$
- F_V takes two arguments and returns a value in Y. The first argument is an element of $N(U)$, where $U <_1 V$, and the second is a partial function $I^* : \bigcup_{V \in T} N(\hat{V}) \to Y$
- Further if $U <_1 V$, $x \in N(\hat{U})$ and I^* is defined on $\{y \in N(\hat{V}) : y|_{\hat{U}} = x\}$ then $F_{\hat{V}}(x, I^*)$ is defined.

Sheaf Recursion

Definition (Sheaf Recursion Cont.)

We next define the partial functions I_α for $\alpha \in \mathrm{ORD}$ as follows:

- $I_0(x) = G(x)$ if $x \in \bigcup_{V \in T} N(\hat{V}) - W(\hat{V})$ and undefined otherwise.
- $I_{\omega \cdot \alpha} = \bigcup_{\gamma < \omega \cdot \alpha} I_\gamma$
- $I_{\alpha+1}(x)$ breaks into three cases:
  1. If $I_\alpha(x)$ is defined then $I_{\alpha+1}(x) = I_\alpha(x)$
  2. Otherwise if
     - $x \in N(U)$, $U <_1 V$, $(\forall y \in N(\hat{V})$ s.t. $y|_V = x)I_\alpha(y)$ is defined

     then there is a $Q \in T$ with $U <_1 Q$ such that
     - $I_{\alpha+1}(x) = F_Q(x, I_\alpha)$
  3. Otherwise $I_{\alpha+1}(x)$ is undefined.

We then let $I = \bigcup_{\alpha \in \mathrm{ORD}} I_\alpha$

Sheaf Recursion

Theorem

I is a total function on $N(0)$

Proof.

Assume the theorem fails.

Then there is some $x_0 \in N(0)$ such that $I(x_0)$ is undefined. By construction we must have $x_0 \in W(0)$.

For each $V \in T(1)$ there must then be an $x_V \in W(\hat{V})$ such that $I(x_V)$ is undefined and $x_V|_0 = x_0$. (Otherwise, because of how I was defined, $I(x_0)$ must have a value). □

Sheaf Recursion

Proof.

Repeating this process we have for each $n \in \hat{\omega}$, $U \in T(n)$ and $V \in T(n+1)$ such that $V|_n = U$, there must then be an $x_V \in W(\hat{V})$ such that $I(x_V)$ is undefined and $x_V|_{\hat{U}} = x_U$. (Otherwise, because of how I was defined, $I(x_U)$ must have a value).

We then have constructed $\langle x_V : V \in T \rangle$ where $x_V \in W(\hat{V})$ for each $V \in T$ and $x_V|_{\hat{U} \cap \hat{V}} = x_{U \cap V} = x_U|_{\hat{U} \cap \hat{V}}$. □

Sheaf Recursion

Proof.

So $\langle x_V : V \in T \rangle$ is a compatible collection of elements and hence, because W is a sheaf, there must be an element $x \in W(\bigcup_{V \in T} \hat{V}) = W(\hat{T})$.

But this contradicts the well-foundedness of W. Hence I must be total on $N(0)$. □

κ-Suslin and κ-Borelian Sets

Definition

The κ-*Borelian* subsets of ω^ω, $B_\kappa(\omega^\omega)$, is the smallest collection of sets such that

- All closed subsets of ω^ω are in $B_\kappa(\omega^\omega)$
- $B_\kappa(\omega^\omega)$ is closed under κ-unions and κ-intersections.

κ-Suslin and κ-Borelian Sets

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Definition

A set $X \subseteq \omega^\omega$ is said to be κ-*Suslin* if there is a closed set $D \subseteq \kappa^\omega \times \omega^\omega$ with $X = \{x : (\exists y \in \kappa^\omega)(y, x) \in D\}$

Suslin-Kleene Separation Theorem

Theorem

Given any two disjoint sets $X, Y \subseteq \omega^\omega$, there is a $|\omega^\omega|$-Borelian set B such that $X \subseteq B$ and $Y \cap B = \emptyset$

Theorem (Suslin-Kleene Separation Theorem)

Given any two disjoint κ-Suslin $X, Y \subseteq \omega^\omega$, there is a κ-Borelian set B such that $X \subseteq B$ and $Y \cap B = \emptyset$

Hence given two sets X and Y, the size of κ such that a κ-Borelian set can separate X from Y is a measure of how complicated X and Y are.

Suslin And Borelian Presheaf

These notions suggest the following definitions

Definition

Suppose N is a sheaf. The κ-*Borelian* subpresheaves of N, $B_\kappa(N)$, is the smallest collection of presheaves such that

- All subsheaves of N are in $B_\kappa(N)$
- $B_\kappa(N)$ is closed under κ-unions and κ-intersections (taken in the category of presheaves).

Suslin And Borelian Presheaf

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Suppose N is a sheaf. The κ-*Borelian* subpresheaves of N, $B_\kappa(N)$, is the smallest collection of presheaves such that

- All subsheaves of N are in $B_\kappa(N)$
- $B_\kappa(N)$ is closed under κ-unions and κ-intersections (taken in the category of presheaves).

Definition

A presheaf $X \subseteq N$ is said to be K-*Suslin* if there is a sheaf $D \subseteq K \times N$ with $X(U) = \{x : (\exists y \in K(U))(y, x) \in D(U)\}$

Splitting Numbers

Definition

Suppose S is a topological space, $T \subseteq \mathcal{O}(S)$ is a tree such that $\bigcup T = S$, and N a sheaf on S. We then define the *splitting number of N relative to T* to be

$$Split_T(N) = \sup\{|\{f \in N(V) : f|_U = g\}| \text{ such that } U <_1 V, U, V \in T, \text{ and } g \in N(U)\}$$

Suslin-Kleene Separation Theorem on Recursion Sheaves

We then get the following variant of the Suslin-Kleene separation theorem.

Theorem

Suppose $X, Y \subseteq N$ are K-Suslin subpresheaves such that $X(S) \cap Y(S) = \emptyset$. Then there is a $Split_T(K \times N)$-Borelian presheaf B such that $X(S) \subseteq B(S)$ and $Y(S) \subseteq B(S)$