Independence, the Continuum Hypothesis and Determinacy

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First Order Logic

Definition

A *1st order language* consists of a collection of

- Relations symbols $\langle R_i(\mathbf{x}) : i \in \kappa \rangle$
- Function symbols $\langle f_i(\mathbf{x}) : i \in \kappa \rangle$
- Constant symbols $\langle c_i : i \in \kappa \rangle$

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The set of *1st order formula* from a (countable) language L, $\mathcal{L}_{\omega,\omega}(L)$, is the smallest collection F such that

- F contains all relations in L
- If $\varphi, \psi \in F$ then $\varphi \wedge \psi, \varphi \vee \psi \in F$.
- If $\varphi \in F$ then $\neg\varphi, (\exists x)\varphi, (\forall x)\varphi \in F$.

Theories

Theorem

There is a "computable" encoding of $\mathcal{L}_{\omega,\omega}(L)$ in $\mathbb{N}$ (obtainable from bijection between L and $\mathbb{N}$).

If $G \in \mathcal{L}_{\omega,\omega}(L)$ we let $\hat{G}$ be the corresponding number. We call $\hat{G}$ the Gödel number of G.

Definition

A *theory*, T, for a language L is a set of 1st order sentences such that

- $T \subseteq \mathcal{L}_{\omega,\omega}(L)$ and contains all tautologies.
- If $\varphi \in T$ and $\varphi \to \psi \in T$ then $\psi \in T$.

Properties of a Theories

Definition

If T is a theory then we say

- T is *consistent* if it has a model. We let $Con(T)$ be the statement that T is consistent.
- T is *computable* if there is a computer program which outputs exactly the Gödel numbers of sentences in T.
- T is *complete* if for all $\varphi \in \mathcal{L}_{\omega,\omega}(L)$ either $\varphi \in T$ or $\neg\varphi \in T$.

A theory T is complete if it determines the truth or falsity of every 1st order sentence of its language.

Examples of Complete Theories

Lets look at some examples of complete theories.

Example

Let M be a model of the language L and let $Th(M) = \{\varphi \in \mathcal{L}_{\omega,\omega}(L)$: such that M makes φ true$\}$.

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Example

Let DLO be the theory in the language $L = \{<\}$ consisting of all consequences of the following axioms:

- $<$ is a linear ordering
- (Density) $(\forall x, y) x < y \rightarrow (\exists z) x < z$ and $z < y$
- (No Endpoints) $(\forall x)(\exists y^+, y^-) y^- < x$ and $x < y^+$

Gödel's Incompleteness Theorems

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If T is a computable, consistent theory which contains Peano Arithmetic, then T is not complete.

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Theorem (Gödel's First Incompleteness Theorem)

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Theorem (Gödel's Second Incompleteness Theorem)

If T is a computable theory which contains Peano Arithmetic then $Con(T) \in T$ if and only if T is inconsistent.

Proof of Gödel's First Incompleteness

For any computable theory T in the language L containing Peano arithmetic.

Theorem

There is a formula $Prov_T(\cdot, \cdot)$ such that T proves $Prov_T(x, y)$ if and only if

- *x is the Gödel number of a sentence $\varphi_x \in \mathcal{L}_{\omega,\omega}(L)$*
- *y encodes a sequence of Gödel numbers $\langle x_0, \ldots, x_n \rangle$ such that $\langle \varphi_{x_0}, \ldots, \varphi_{x_n} \rangle$ is a proof of φ_x from T.*

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Definition

If $G \in \mathcal{L}_{\omega,\omega}(L)$ but neither G nor $\neg G$ are provable from T then we say G is *independent from* T

Correct Axioms of Natural Numbers

This suggests several questions

Question: How do we know that Peano arithmetic describes "true" facts about natural numbers?

Question: Might there be a different theory of natural numbers, incompatible with Peano arithmetic, but which avoids this problem (i.e. is computable, complete, and still complicated enough to do all of mathematics in)?

To answer these questions we will "enlarge" the collection of objects we are talking about.

Zermelo Frankel Set Theory and Natural Numbers

All of modern mathematics can be "interpreted" in Zermelo Frankel Set Theory with choice.

However we also have that ZFC is part of modern mathematics and so any system which can "interpret" modern mathematics must also interpret ZFC.

Zermelo Frankel Set Theory and Natural Numbers

All of modern mathematics can be “interpreted” in Zermelo Frankel Set Theory with choice.

However we also have that ZFC is part of modern mathematics and so any system which can “interpret” modern mathematics must also interpret ZFC.

Further ZFC proves the axioms of Peano arithmetic.

As such any system which can “interpret” all modern mathematics must prove Peano arithmetic.

So, in some Platonic sense, the axioms of Peano arithmetic “true” for the natural numbers.

Types of Incompleteness

Gödel's incompleteness theorems gives us two types of sentences which are independent of Peano Arithmetic:

- Gödel Sentences
- Sentences of the form $Con(T)$ for an appropriate T.

Gödel sentences are independent because they were, in some sense, artificially constructed to be so. Consistency sentences on the other hand, while formally independent, are in a Platonic sense "true".

Types of Incompleteness

This suggests the question:

Question: Are there any sentences independent from Peano arithmetic which don't reduce to this form.

Types of Incompleteness

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Question: Are there any sentences independent from Peano arithmetic which don't reduce to this form.

Answer: We haven't found any yet.

Hence, Peano Arithmetic is in some sense "complete" for formulas we care about (as far as we can tell).

Size of a Set

Definition

The *cardinality* of a set X is less than or equal to the cardinality of a set Y, $|X| \leq |Y|$, if there is a one-to-one function from X to Y.

If $|X| \leq |Y|$ and $|Y| \leq |X|$ then we say $|X| = |Y|$ and that X and Y have the same *cardinality*.

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Definition

The *Continuum Hypothesis* (CH) is the statement:

For all sets X, if $|\mathbb{N}| \leq |X| \leq |\mathbb{R}|$ then either $|\mathbb{N}| = |X|$ or $|X| = |\mathbb{R}|$

The Continuum Hypothesis is the first of Hilbert's famous problems.

Independence of Continuum Hypothesis

Theorem (Gödel, 1940)

If ZFC is consistent then so is ZFC + CH

Theorem (Cohen, 1963)

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Independence of Continuum Hypothesis

Theorem (Gödel, 1940)

If ZFC is consistent then so is $ZFC + CH$

Theorem (Cohen, 1963)

If ZFC is consistent then so is $ZFC + \neg CH$

As such we have

Theorem

If ZFC is consistent then CH is independent of ZFC

CH was the first example of a sentence which is independent of a theory we care about but which isn't a statement of consistency or a Gödel sentence.

Is the Continuum Hypothesis true

We have seen that in some sense Peano arithmetic is "true". As such, even though $Con(PA)$ is independent of PA, $Con(PA)$ should also be "true" and $\neg Con(PA)$ should be "false".

i.e. Peano Arithmetic + $\neg Con(PA)$, while consistent isn't compatible with the larger systems we care about as mathematicians.

This suggests the following question:

Question: Is there a larger sense in which either CH or $\neg CH$ is "true"?

Transitive Closure

Definition

Let X be a set. We define the *transitive closure of* X,

$$tc(X) = X \cup \bigcup X \cup \bigcup\bigcup X \cup \ldots$$

Definition

If κ is an infinite cardinal, we say a set X is *hereditarily of size* κ if $|tc(X)| = \kappa$.

We denote by $H(\kappa)$ the collection of all sets hereditarily of size κ.

$H(\omega_2)$ and CH

Theorem

CH is equivalent to the statement

$$(\exists R \in H(\omega_2))(\forall x \in H(\omega_2))x \subseteq \mathbb{N} \leftrightarrow x \in R$$

In particular CH is a statement about $H(\omega_2)$ and hence to study CH we need to study $H(\omega_2)$.

A good place to start might be to study $H(\omega)$ and $H(\omega_1)$.

$H(\omega)$ and $\langle \mathcal{N}, +, *, 0, 1\rangle$

Theorem

The structure $\langle H(\omega), \in\rangle$ is bi-interpretable with the first order structure $\langle \mathbb{N}, +, \times, 0, 1\rangle$.

Proof.

One direction is because every finite number can be represented as a hereditarily finite set.

The other direction is because every hereditarily finite set can be represented as a finite tree. □

$H(\omega_1)$ and $\langle P(\mathcal{N}), \mathcal{N}, +, *, 0, 1\rangle$

Similarly we have

Theorem

The structure $\langle H(\omega_1), \in\rangle$ is bi-interpretable with the second order structure $\langle Powerset(\mathbb{N}), \mathbb{N}, +, \times, 0, 1\rangle$.

Proof.

One direction is because every subset of $\mathbb{N}$ is hereditarily countable.

The other direction is because every hereditarily countable set can be represented as a countable tree. This tree can then be encoded as a binary relation on $\mathbb{N}$ (which is a subset of $\mathbb{N} \times \mathbb{N}$). □

Questions about $H(\omega_1)$

There are several questions about $H(\omega_1)$ which *ZFC* is not equipped to answer.

This suggests:

Question: Does *ZFC* prove all "true" statements about $H(\omega_1)$ that we care about? (in a way similar to how Peano Arithmetic proves all "true" statements about $H(\omega)$ that we care about)

To answer this we first have to determine what type of questions we are interested in.

Definition of Projective Sets

Definition

Consider the following basic operation on $\mathbb{R}^n$

- (Projection) Suppose $X \subseteq \mathbb{R}^{n+1}$. The *projection of X on* $\mathbb{R}^n$ is the image of X under the projection map $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by $\pi(a_1, \ldots, a_n, a_{n+1}) = (a_1, \ldots, a_n)$.
- (Complements) Suppose $X \subseteq \mathbb{R}^n$. The *Complement of X* is the set $X^* = \{(a_1, \ldots, a_n) | (a_1, \ldots, a_n) \notin X\}$

Definition

A set $X \subseteq \mathbb{R}^n$ is *projective* if for some integer k, X can be generated from a closed subset of $\mathbb{R}^{n+k}$ in finitely many steps by taking projections and complements.

Definability of Projective Sets

Definition

A set $A \subseteq H(\omega_1)$ is said to be *definable* if there is a formula $\varphi(x, y)$ of $L_\in = \{\in, =\}$ and an element $b \in H(\omega_1)$ such that

$$A = \{a \in H(\omega_1) : H(\omega_1) \text{ models } \varphi(a, b)\}$$

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Recall, we can associate $\langle Powerset(\mathbb{N}), \mathbb{N}, +, \times, 0, 1\rangle$ with $H(\omega_1)$.

Theorem

Under this association a set $A \subseteq H(\omega_1)$ is definable if and only if the corresponding subset of $\mathbb{R}$ is projective.

Theory of Projective Sets

Question: Does *ZFC* "correctly" describe the projective sets?

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Theorem

ZFC Proves there is a non-Lebesgue measurable set of real numbers

Theorem (Paradoxical Decomposition of the Sphere)

ZFC proves that it is possible to break the unit sphere in $\mathbb{R}^3$ into finitely many pieces and then reassemble those pieces into two unit spheres.

Theory of Projective Sets

The proof of both of the previous theorems is very non-constructive and makes fundamental use of the axiom of choice.

As such, because the projective sets are the “definable” subsets of $\mathbb{R}^n$, our intuition tells us that if we restrict ourselves to projective subsets of the reals then these theorems “shouldn't” hold.

Theory of Projective Sets

Theorem (Gödel)

It is consistent with ZFC that there is a non-Lebesgue measurable projective set and a Paradoxical Decomposition of the Sphere consisting of projective sets.

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Theorem (Gödel)

It is consistent with ZFC that there is a non-Lebesgue measurable projective set and a Paradoxical Decomposition of the Sphere consisting of projective sets.

So *ZFC* isn't strong enough to prove everything which our intuition tells us "should" be true about the projective sets. This begs the question:

Question: Is there an extension of *ZFC* which answers all "interesting" questions about the projective sets correctly and which is "canonical" (i.e. which is "true" in a way similar to how Peano arithmetic is true)?

Two Player Games

Definition

Fix $A \subseteq [0,1]$ and define the game G_A which is an infinite game with two players.

Player I and Player II alternating in choosing $\epsilon_i \in \{0,1\}$ with Player I choosing all ϵ_{2i+1} and Player II choosing all ϵ_{2i}.

Player I wins the game if

$$\sum_{i=1}^{\infty} \epsilon_i 2^{-i} \in A$$

and Player II wins otherwise.

Winning Strategies

Definition

Let *SEQ* be the set of all binary sequences. A *strategy* τ is a function $\tau : SEQ \rightarrow \{0, 1\}$

Definition

A run $\langle \epsilon_i : i \in \mathbb{N} \rangle$ of the game G_A is *generated according to* τ *by Player I* if $\epsilon_1 = \tau(\emptyset)$ and for all $k \in \mathbb{N}$

$$\epsilon_{2k+1} = \tau(\epsilon_1, \ldots, \epsilon_{2k})$$

We define a run generated by τ for Player II in an analogous way.

Projective Determinacy

Definition

τ is a *winning strategy for Player I* in the game G_A if every run generated according to τ for Player I is winning for Player I.

τ is a *winning strategy for Player II* in the game G_A if every run generated according to τ for Player II is winning for Player II.

Projective Determinacy

Definition

τ is a *winning strategy for Player I* in the game G_A if every run generated according to τ for Player I is winning for Player I.

τ is a *winning strategy for Player II* in the game G_A if every run generated according to τ for Player II is winning for Player II.

Definition

We say a game G_A is *determined* (or briefly A is determined) if either Player I or Player II has a winning strategy.

Definition

The *Axiom of Projective Determinacy* (PD) says that every projective set is determined.

Properties of Projective Determinacy

Theorem

From $ZFC + PD$ we can prove

- *All projective sets are Lebesgue measurable*
- *There is no paradoxical decomposition of the sphere consisting of projective sets.*
- *There are no essential uses of the Axiom of Choice needed in the analysis of $H(\omega_1)$*

Further, the only sentences about projective sets which are known to be independent of
$ZFC + PD$ are analogs of Gödel sentences or consistency statements.

Is Projective Determinacy true?

So $ZFC + PD$ is a theory which captures much of our intuition about $H(\omega_1)$ and the projective sets.

But is there a sense in which it is "true"?

New Axioms

We know that there are statements which *ZFC* can't prove. So we might try adding new axioms to *ZFC*, to get a theory T, in order to see what new results might follow from T.

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We know that there are statements which *ZFC* can't prove. So we might try adding new axioms to *ZFC*, to get a theory T, in order to see what new results might follow from T.

Notice that because T is an extension of *ZFC*, *ZFC* won't be able to prove T's consistent (for any T).

However, it is possible that for some T, T will prove the consistence of *ZFC*. In this case we say that T has more *consistency strength* than *ZFC*.

This notion induces partial order on theories: $T <_{Con} T'$ if the consistency of T' implies the consistency of T.

Large Cardinals

As it turns out, for (essentially) all known “interesting” extensions of ZFC, $<_{Con}$ is a linear order.

In particular there are, for most elements of this linear order, canonical representatives which we call *Large Cardinal Axioms*.

The fact that the consistency strengths are linear ordered (i.e. that there is only one path through the consistency strengths) strongly suggests that in some sense these large cardinal axioms are “true”

Large Cardinals and Projective Determinacy

Further we have

Theorem

If there exists infinitely many Woodin cardinals, then Projective Determinacy holds.

Projective Determinacy follows from "Large Cardinal Axioms" and hence there is a sense in which Projective Determinacy is "true".

$H(\omega_2)$

Now, we are finally ready to begin the study of $H(\omega_2)$ and the continuum hypothesis.

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Unfortunately though we are out of time.

But if you are interested in seeing more come to the talk next week (or you can also read “The Continuum Hypothesis, Part II” by W.Hugh Woodin).