Quantifier Rank Spectra of Scattered Sentences of $\mathcal{L}_{\omega_1,\omega}$

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Abstract

In this talk we show that, assuming the existence of a nice meta-language, there is a set $Z \subseteq \omega_1$ which is unbounded in ω_1 where for every $\alpha \in Z$ there is a scattered sentence $S_\alpha \in \mathcal{L}_{\omega_1,\omega}$ such that the quantifier rank of S_α is less than or equal $\omega * 2$ but the supremum of the quantifier ranks of countable models of S_α is α.

TALK SLOWLY AND WRITE NEATLY!!

1 Background on Vaught's Conjecture

In previous talks we went through the proof that given any sentence of $\mathcal{L}_{\omega_1,\omega}$ the collection of countable models which have quantifier rank α is either countable or has size 2^ω.

Definition 1.0.1. A sentence $\sigma \in \mathcal{L}_{\omega_1,\omega}$ is called *Scattered* if for all $\alpha \in \omega_1$ the number of countable models of σ with quantifier rank α is countable.

Definition 1.0.2. Let $\sigma \in \mathcal{L}_{\omega_1,\omega}$. Define the *Quantifier Rank Spectrum of* σ to be $qrs(\sigma) = \{\alpha : (\exists M) M \models \sigma, |M| = \omega, \mathrm{qr}(M) = \alpha\}$

Theorem 1.0.3. *$\sigma \in \mathcal{L}_{\omega_1,\omega}$ is a counterexample to Vaught's conjecture if and only if σ is scattered and the $sup(qrs(\sigma)) = \omega_1$.*

We can then ask the question "is it possible to find sentences $\sigma_\alpha \in \mathcal{L}_{\omega_1,\omega}$ such that $\alpha < sup(qrs(\sigma_\alpha)) < \omega_1$?".

We can think of such a σ_α as an "Approximation" to a counterexample to Vaught's conjecture.

2 Trees

We will want an infinitely branching tree in the background of the theory we are considering.

Background Trees

Definition 2.0.4. Let $L_T = \{r, <_1\} \cup \{\text{level}_n : n \in \omega\}$ where r is a constant, level_n are unary relations, and $<_1$ is a binary relation. We will write $x <_1 y$ instead of $<_1 (x, y)$.

Definition 2.0.5. $T_T \in \mathcal{L}_{\omega_1,\omega}(L_T)$, *the theory of infinitely branching trees*, consists of the conjunction of the following L_T sentences

- $(\forall x)\text{level}_0(x) \leftrightarrow x = r$

- $(\forall x, y) x <_1 y \to \bigvee_{i \in \omega} \text{level}_{i+1}(x) \wedge \text{level}_i(y)$
- $(\forall x, y, z)(x <_1 y) \wedge (x <_1 z) \to y = z$
- $(\forall x) x \neq r \to (\exists y) x <_1 y$
- $(\forall y)(\exists^\infty x) y <_1 x$

Definition 2.0.6. Let $\leq$ be the transitive closure of $<_1$.

$$x \leq y \Leftrightarrow \bigvee_{n \in \omega} (\exists z_0, z_1, \cdots, z_n)(z_0 = y) \wedge (z_n = x) \bigwedge_{0 \leq i < n} z_{i+1} <_1 z_i$$

Also define $x < y \Leftrightarrow x \leq y \wedge x \neq y$.

Definition 2.0.7. Let $M \models T_T$. We call r^M the *root* of M. If $M \models \text{level}_n(x)$ we say x is on *level* n ($\text{level}(x) = n$).

Closed Tuples

Definition 2.0.8. Let $M \models T_T$. We say $\mathbf{x} \subset M$ is *closed* if $M \models (\forall a \in \mathbf{x})(\forall b) a <_1 b \to b \in \mathbf{x}$. We say $M \models$ Closed$(()\mathbf{x})$ if $\mathbf{x}$ is closed.

Tree Partial Isomorphisms

Definition 2.0.9. Let $L_T \subseteq L_K$ be a language and let M, N be L_K structures such that $M, N \models T_T$. Then $p : M \to N$ is a *Tree Partial Isomorphism* if p is a partial isomorphism from M to N and $M \models \text{Closed}(()\text{dom}(p))$.

Definition 2.0.10. We say $\langle I_\gamma : \gamma < \alpha \rangle$ is a *Sequence of Tree Partial Isomorphisms* from M to N if

- $(\forall \gamma < \beta < \alpha) I_\beta \subseteq I_\gamma$.
- $(\forall \gamma < \alpha)(\forall p \in I_\gamma)$ p is a tree partial isomorphism from M to N.
- If $\beta + 1 < \alpha$, $p \in I_{\beta+1}$, $a \in M$ and $\text{dom}(p) \cup \{a\}$ is closed then there is a $b \in N$ such that $p \cup (a, b) \in I_\beta$
- If $\beta + 1 < \alpha$, $p \in I_{\beta+1}$, $b \in N$ and $\text{range}(p) \cup \{b\}$ is closed then there is a $a \in M$ such that $p \cup (a, b) \in I_\beta$

We call the last two conditions are *tree back and forth property* and if such a sequence exists we say $M \equiv^T_\alpha N$.

Theorem 2.0.11. *Let $\mathcal{I} = \langle I_i : i < \omega * \alpha \rangle$ be a sequence of tree partial isomorphisms from M and N. Then there is a sequence $\mathcal{J} = \langle J_i : i < \alpha \rangle$ of partial isomorphisms from M to N*

Proof. Let $s \in J_\beta$ if and only $(\exists p \in I_{\omega * \beta}) s \subseteq p$. Then for each $\gamma \leq \beta < \alpha$, $J_\beta \subseteq J_\gamma$ and if $p \in J_\beta$, p is a partial isomorphism.

Next suppose we have $s \in J_{\beta+1}$, $s \subseteq p \in I_{\omega*(\beta+1)}$, and $a \in M$. If $\text{level}_n(a)$ then $\overline{a} = \{a = a_n, a_{n-1}, \ldots, a_0 = r\}$ where $\text{level}_i(a_i)$ and $a_{i+1} <_1 a_i$. Because $p \in I_{\omega*\beta+n+1}$, we can find $b \in N$ such that $\overline{b} = \{b = b_n, b_{n-1}, \ldots, b_0 = r\}$ and $p' = p \cup \{(a_i, b_i) : i \leq n\} \in I_{\omega*\beta}$. So $s \cup (a, b) \in J_\beta$.

We get the other direction (were we are given a $b \in N$ and find an $a \in M$) in exactly the same way. Hence $\mathcal{J}$ is a sequence of partial isomorphisms. □

Corollary 2.0.12. *If $M \equiv^T_{\omega*\alpha} N$ then $M \equiv_\alpha N$.*

Subtrees

Definition 2.0.13. Let $L_P = L_T \cup \{P\}$ where P is a unary relation.

Definition 2.0.14. Let $T_{sub}(P) \in \mathcal{L}_{\omega_1,\omega}(L_P)$ be the conjunction of

- T_T
- $(\forall x, y)P(y) \wedge y <_1 x \to P(y)$
- $(\forall x)(\exists^\infty y)y <_1 x \wedge \neg P(y)$

Countable Substructures

Definition 2.0.15. If $M \models T_{sub}(P)$ let $P^M = \{x \in M : M \models P(x)\}$

Theorem 2.0.16. *If* $(Tr, \preceq)$ *is a countable tree (see* **??** *for a definition) then there is a unique countable model* $M \models T_{sub}(P)$ *such that* $(P^M, \leq) \cong (Tr, \preceq)$.

Proof. Because any element of M has infinitely many elements extending it which don't satisfy P. □

Height

Definition 2.0.17. Let $M \models T_{sub}(P)$ and $b \in M$. We define the *P-Height of b* ($\text{height}()P(b)$) recursively as follows.

- If $M \models \neg P(b)$ then $\text{height}()P(b) = -\infty$
- If $M \models P(b)$ and $(\forall x <_1 b) M \models \neg P(x)$ then $\text{height}()P(b) = 0$
- $\text{height}()P(b) = \sup\{\text{height}()P(x) + 1 : x <_1 b \wedge P(x)\}$ if it is defined.
- $\text{height}()P(b) = \infty$ otherwise.

Spectrum

Definition 2.0.18. Let $M \models T_{sub}(P)$. We define the *P-spectrum of M* to be $\text{Spec}_P(M) = \{\text{height}()P(b) : b \in M\}$

Definition 2.0.19. Suppose $M \models T_{sub}(P)$ and suppose $\mathbf{b} = (b_1, \cdots, b_n) \in M$ is a tuple. We define *P-Height Type* of $\mathbf{b}$ to be a function $htype_P(\mathbf{b}) : (x_1, \ldots, x_n) \to \omega_1 \cup \{-\infty, \infty\}$ such that $(\forall i \leq n) M \models htype_P(\mathbf{b})(x_i) = \text{height}()P(b_i)$

Slant Line

We want to think of the height of an element as determining the complexity of its infinity type. However tuples of the same height which are on different levels differ in the complexity of their infinity type slightly.

We define the notion of a slant line to take into account these differences.

Definition 2.0.20. A function $f : \omega \to \omega_1 \cup \{-\infty, \infty\}$ is a *slant line* if for all $i \in \omega$

- $f(i) > f(i+1)$
- If $f(i)$ is a successor ordinal then $f(i) = f(i+1) + 1$

Here we consider $-\infty > -\infty$ and $\infty > \infty$ and we let $sl_{-\infty}, sl_{\infty}$ be slant lines which have the constant value of $-\infty, \infty$ respectively.

If f is a slant line then we say $f < \alpha$ (for an ordinal α) if $f(i) < \alpha$ for all $i \in \omega$

Definition 2.0.21. Let $M \models T_{sub}(P) \wedge \text{level}(a) = \text{level}(b) = n$. Let f be a slant line. We say that *a and b have the same P-height up to f* $(\text{height}()P(a)|f = \text{height}()P(b)|f)$ if either

- $\text{height}()P(a) = \text{height}()P(b)$ or
- $\text{height}()P(a) \geq f(n)$ and $\text{height}()P(b) \geq f(n)$

Height up to a Slant Line

Definition 2.0.22. Let $M, N \models T_{sub}(P)$, $\mathbf{a} = (a_0, \cdots, a_n) \in M$, $\mathbf{b} = (b_0, \cdots, b_n) \in N$ be closed tuples such that $M \models a_i <_1 a_j \Leftrightarrow N \models b_i <_1 b_j$ for all $i, j \leq n$ and let f be a slant line. We say that $\mathbf{a}, \mathbf{b}$ have the same *P-height type up to f* ($htype_P(\mathbf{a})|f = htype_P(\mathbf{b})|f$) if

$$(\forall 0 \leq i \leq n) \text{ height}()P(a_i)|f = \text{height}()P(b_i)|f$$

Models of height α

Definition 2.0.23. Let $L_P \subseteq L_K$ and let $T_K \vdash T_{sub}(P)$ and let $X \subseteq \{-\infty, \infty\} \cup \omega_1$. We define $\mathcal{M}_X(T_K) = \{M : M \models T_K, |M| = \omega, \text{Spec}_P(M) \subseteq X\}$. We will omit T_K when it is clear from context.

Definable Height

Definition 2.0.24. Let $L_H = L_P \cup \{H_\leq\}$ where $H_\leq$ is a binary relation. We also define as shorthand $H_=(a, b) \Leftrightarrow H_\leq(a, b) \wedge H_\leq(b, a)$ and $H_<(a, b) \Leftrightarrow H_\leq(a, b) \wedge \neg H_=(a, b)$.

Definition 2.0.25. The theory $T_{Height} \in \mathcal{L}_{\omega_1,\omega}(L_H)$ is the conjunction of the following:

Background Trees	$T_{sub}(P)$
Linearity	$(\forall x, y) H_{\leq}(x, y) \vee H_{\leq}(y, x)$
Transitivity	$(\forall x, y, z)[H_{\leq}(x, y) \wedge H_{\leq}(y, z)] \rightarrow H_{\leq}(x, z)$
Tree Ordering	$(\forall x, y, a)[a < x \wedge P(x) \wedge H_{\leq}(x, y)] \rightarrow H_{<}(a, y)$
Base Case	$(\forall x, y) \neg P(x) \rightarrow H_{\leq}(x, y)$

Height Defined Correctly

Theorem 2.0.26. *If* $M \models T_{Height}$ *then*

- $M \models (\forall a, b) H_{\leq}(a, b) \Rightarrow height(_) P(a) \leq height(_) P(b)$
- $M \models (\forall a, b)\ height(_) P(a) < height(_) P(b) \wedge height(_) P(a) \neq \infty \rightarrow H_{<}(a, b)$.

Proof. This is by induction on the height of a. □

Full Trees

Definition 2.0.27. $T_{Full} \in \mathcal{L}_{\omega_1, \omega}(L_H)$, *the theory of full trees*, is the conjunction of the following:

Comparing Heights	T_{Height}
Fullness	$(\forall x, y) H_{<}(x, y) \rightarrow (\exists^{\infty} z) H_{=}(x, z) \wedge z <_1 y$

If $M \models T_{Full}$ we say $M|_{L_P}$ is a *full* tree.

Theorem 2.0.28. *Suppose $M \models T_{Full}$ with $a \in M$. If $\beta <$ height$()P(a)$ then $M \models (\exists^\infty b) b <_1 a$ and height$()P(b) = \beta$*

Proof. If $\beta < \infty$ this follows from Theorem 2.0.26. If $\beta = \infty$ this follows from the fact that (height$()P(a) = \infty$) $\Leftrightarrow$ (the tree extending a in P^M is ill-founded) $\Leftrightarrow$ (($\exists^\infty b) b <_1 a$ and the tree extending b in P^M is ill-founded) □

Height Determines Model

Theorem 2.0.29. *If $M, N \models T_{Full}(P)$ and $Spec_P(M) = Spec_P(N)$ then $M|_{L_P} \equiv_\infty N|_{L_P}$.*

Proof. Let $I = \{p : p$ is a tree partial isomorphism in L_P and p preserves height$\}$. I then satisfies the tree back and forth property because both M and N are full. □

Condition on ∞ Equivalence

Notice even if $\mathrm{Spec}_P(M) = \mathrm{Spec}_P(N)$ and $M, N \models T_{Full}$ we do not in general have $M \equiv_\infty N$. The reason is that on ill-founded branches $H_\leq(x, y)$ does not describe heights accurately. Rather on ill-founded branches $H_{\leq}(x,y)$ introduces an arbitrary linear order. So, for any countable model $M \models T_{Full}$ with $\infty \in \mathrm{Spec}_P(M)$, there is a set of 2^ω many countable models M_i where $M_i \models T_{Full}$, $M_i \not\cong M_j$ if $i \neq j$ but $M_i|_{L_P} \cong M|_{L_P}$.

Corollary 2.0.30. *If $M, N \models T_{Full}$ and $Spec_P(M) = Spec_P(N) = \{-\infty\} \cup \alpha$ then $M \equiv_\infty N$.*

Proof. We know that $M|_{L_P} \equiv_\infty N|_{L_P}$ by Theorem 2.0.29. But by Theorem **??** we know that there is a formula $\varphi_\alpha \in \mathcal{L}_{\omega_1,\omega}(L_P)$ which defines $H_{\leq}$ in M and N. □

3 Base Predicates and Archetypes

Abstract Structure

Before we introduce the structure we want on our trees it is worth discussing "abstract structure" in general terms. In this section we will discuss two types of "abstract structure", collections of base predicates and collections of archetypes.

Base Predicates We will eventually need some way to relate the heights of our elements to our language. We do this in a very abstract way via something we call

"Base Predicates"

Definition 3.0.31. Let $L_T \subset L_K$ be a language and $T_K \in \mathcal{L}_{\omega_1,\omega}(L_K)$. We say $BP_K \subseteq L_K - L_T$ is a *collection of base predicates for* T_K if T_K satisfies the following. (We omit mention of T_K when it is clear from context)

(Truth on Base Predicates)

For all $A \in BP_K$ there is a complete quantifier free formula $\varphi_A \in \mathcal{L}_{\omega_1,\omega}(BP_K \cup L_T)$ such that

$$T_K \vdash (\forall \mathbf{x}) A(\mathbf{x}) \to \varphi_A(\mathbf{x})$$

(Uniqueness of Base Predicate)

For all $A, A' \in BP_K$, $T_K \vdash (\forall \mathbf{x}) A(\mathbf{x}) \wedge A'(\mathbf{x})$ or $T_K \vdash (\forall \mathbf{x})(\neg A(\mathbf{x}) \vee \neg A'(\mathbf{x}))$

(Closed Domains)

For all $A \in BP_K$, $T_K \vdash (\forall \mathbf{x}) A(\mathbf{x}) \to \text{Closed}(()\mathbf{x})$

(Uniqueness of Domains)

For all $A \in BP_K$, $T_K \vdash (\forall x_1, \dots, x_n) A(x_1, \dots, x_n) \to \bigwedge_{i \neq j} x_i \neq x_j$

(Completeness for Base Predicates)

$T_K \vdash (\forall \mathbf{x}) \bigvee_{n\in\omega} (\exists y_1, \ldots, y_n) \bigvee_{A\in BP, |\mathrm{dom}(A)|=n+|\mathbf{x}|} A(\mathbf{x}, y_1, \ldots, y_n)$

(Amalgamation for Base Predicates)

For all $A, B, C \in BP_K$ if $T_K \vdash (\forall \mathbf{x}, \mathbf{y}) A(\mathbf{x}, \mathbf{y}) \to C(\mathbf{y})$ and $T_K \vdash (\forall \mathbf{y}, \mathbf{z}) B(\mathbf{y}, \mathbf{z}) \to C(\mathbf{y})$ then there exists a $D \in BP_K$ such that $T_K \vdash (\forall \mathbf{x}, \mathbf{y}, \mathbf{z}, \omega^{<\omega}) D(\mathbf{x}, \mathbf{y}, \mathbf{z}, \omega^{<\omega}) \to (A(\mathbf{x}, \mathbf{y}) \wedge B(\mathbf{y}, \mathbf{z}))$

(Homogeneity for Base Predicates)

If $A \in BP_K$ and $T_K \vdash (\forall \mathbf{x}, \mathbf{y}) A(\mathbf{x}, \mathbf{y}) \to B(\mathbf{x})$ then $T_K \vdash (\forall \mathbf{x})[B(\mathbf{x}) \to (\exists^\infty \mathbf{y}) A(\mathbf{x}, \mathbf{y})]$

Theory of Base Predicates

Definition 3.0.32. If BP_K is a collection of base predicates for $T_K \in \mathcal{L}_{\omega_1,\omega}(L)$ then we define $Th(BP_K, T_K) \in \mathcal{L}_{\omega_1,\omega}(BP_K \cup L_T)$ to be the conjunction of

- $(\forall \mathbf{x}) \bigvee_{n\in\omega} (\exists y_1, \ldots, y_n) \bigvee_{A\in BP', |\mathrm{dom}(A)|=n+|\mathbf{x}|} A(\mathbf{x}, y_1, \ldots, y_n)$
- $(\forall \mathbf{x})[B(\mathbf{x}) \to (\exists^\infty \mathbf{y}) A(\mathbf{x}, \mathbf{y})]$ where $A, B \in BP_K$ and

$T_K \vdash (\forall \mathbf{x}, \mathbf{y}) A(\mathbf{x}, \mathbf{y}) \to B(\mathbf{x})$

- $(\forall \mathbf{x}) A(\mathbf{x}) \leftrightarrow \varphi_A(\mathbf{x})$ for all $A \in BP$

Theory of Base Predicates is Categorical

Theorem 3.0.33. *If BP is a collection of base predicates for T and $T^* \in \mathcal{L}_{\omega_1,\omega}(L^*)$ where $BP \cup L_T \subseteq L^*$ and $T^* \vdash Th(BP, T)$ then $Th(BP, T) = Th(BP, T^*)$ and BP is a collection of base predicates for T^*.*

Proof. This is immediate from the definition of $Th(BP, T)$.

$\square$

So the theory is important because of how it interacts with the other relations.

Trivial Base Predicates

Definition 3.0.34. Let $BP_T = \{A_\varphi(\mathbf{x}) : \varphi(\mathbf{x}) \in \mathcal{L}_{\omega_1,\omega}(L_T)$ is a complete quantifier free formula and $\vdash (\forall \mathbf{x})\varphi(\mathbf{x}) \to$ Closed$(()\mathbf{x})\}$

Definition 3.0.35. Let $T_T^{BP} \in \mathcal{L}_{\omega_1,\omega}(L_T \cup BP_T)$ be the conjunction of

- T_T
- $(\forall \mathbf{x}) A_\varphi(\mathbf{x}) \leftrightarrow \varphi(\mathbf{x})$ for each $A \in BP_T$

Theorem 3.0.36. *BP_T is a collection of base predicates for T_T^{BP}.*

Proof. This is immediate from the definitions. □

Archetypes

Definition 3.0.37. For the rest of this paper we fix a language $L_T \subseteq L_K$ and a theory $T_K \in \mathcal{L}_{\infty,\omega}(L_K)$, and a collection of base predicates $BP_K \subseteq L_K - L_T$ for T_K. We also fix $\mathcal{M}_K \subseteq \{M \models T_K : |M| \leq \omega\}$.

Definition 3.0.38. If $\mathcal{M} \subseteq \{M \models T_K : |M| \leq \omega\}$ then define $S(\mathcal{M}) = \bigcup\{\mathrm{Spec}_P(M) : M \in \mathcal{M}\}$

Forces

Definition 3.0.39. Let $\varphi(\mathbf{x}, \mathbf{y}), \psi(\mathbf{y}, \mathbf{z})$ be "abstract properties" of elements of $\mathcal{M}_K$. We say that $\varphi(\mathbf{x}, \mathbf{y})$ *Forces*$_{\mathcal{M}_K}$ $\psi(\mathbf{y}, \mathbf{z})$ ($\varphi(\mathbf{x}, \mathbf{y}) \Vdash_{\mathcal{M}_K} \psi(\mathbf{y}, \mathbf{z})$) if

$$(\forall M \in \mathcal{M}_K)(\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in M) M \models \varphi(\mathbf{a}, \mathbf{b}) \Rightarrow M \models \psi(\mathbf{b}, \mathbf{c})$$

We say that the *domain* of $\varphi(\mathbf{x}, \mathbf{y})$ is $\mathbf{x}, \mathbf{y}$

Definition 3.0.40. Let $AT_K = AT(T_K, BP_K, \mathcal{M}_K)$ be a collection of "abstract properties" of elements of models of T_K. We say that AT_K is a *Collection of Archetypes (for T_K, BP_K and $\mathcal{M}_K$)* if it satisfies the following.

(Truth on Atomic Formulas)

For all $\phi \in AT_K$, $\phi(\mathbf{x}), \phi(\mathbf{y}) \Vdash_K (\forall$ quantifier free formula $\theta \in \mathcal{L}_{\omega_1,\omega}(L_K))[\theta(\mathbf{x}) \leftrightarrow \theta(\mathbf{y})]$

(Completeness of Archetypes)

For all $\sigma, \tau \in AT_K$, if $(\exists M \in \mathcal{M}_K) M \models (\exists \mathbf{a}, \mathbf{b})\sigma(\mathbf{a}) \wedge \tau(\mathbf{a}, \mathbf{b})$ then $\tau(\mathbf{x}, \mathbf{y}) \Vdash_K \sigma(\mathbf{x})$

(Existence of Empty Archetypes)

There exists $\varphi \in AT_K$ such that for all $M \in \mathcal{M}_K$, $M \models \varphi$ and $\mathrm{dom}(\varphi) = \emptyset$

(Trivial Amalgamation)

For each $\sigma, \tau \in AT_K$ there exists a $Trivial_{\sigma,\tau}(r, \mathbf{x}, \mathbf{y}) \in AT_K$ such that

- $\sigma(r, \mathbf{x}) \wedge \tau(r, \mathbf{y}) \Vdash_K (\exists^\infty \mathbf{z}) Trivial_{\sigma,\tau}(r, \mathbf{x}, \mathbf{y}, \mathbf{z})$

(recall that r is the root of the tree)

(Base Predicates Imply Archetypes)

For all $A \in BP$, $A(\mathbf{x}) \Vdash_K (\exists \sigma \in AT_K)\sigma(\mathbf{x})$

(Archetypes imply Base Predicates)

For all $\sigma \in AT_K$ there is an $A \in BP_K$ such that $\sigma(\mathbf{x}) \Vdash_K A(\mathbf{x})$

Lemma 3.0.41. *For all* $\phi \in AT$

$$\phi(\boldsymbol{x}) \Vdash_K (\forall \boldsymbol{y}) \bigvee_{n \in \omega} (\exists z_1, \cdots, z_n) \bigvee_{\psi(\boldsymbol{x}, \boldsymbol{y}, z_1, \ldots, z_n) \Vdash_K \phi(\boldsymbol{x})} \psi(\boldsymbol{x}, \boldsymbol{y}, z_0, \cdots, z_n)$$

Proof. This follows from (Extension to Base Predicates) and (Base Predicates Imply Archetypes). □

We eventually want our archetypes to be generalizations of actual $\mathcal{L}_{\infty,\omega}$ types. So we will want an archetype to completely determine all necessary properties of its domain.

However, while working with as much generality as we are in this section it is hard to pin down exactly what this would mean. As such the requirements on a collection of archetypes are there to capture as much as can be said without knowing substantially more about our theory.

Collection of Archetype for Full Trees

Definition 3.0.42. Lets consider a collections of archetypes on $T_{Full} \cup T_T^{BP} = T'_{Full}$.

Definition 3.0.43. Let $AT_{Full} = \{(f, A_\varphi)(\mathbf{x}) : (\exists M \models T'_{Full})M \models f = htype_P(\mathbf{a}) \wedge \varphi(\mathbf{a}) \wedge \text{Closed}(()\mathbf{a})$ (where $A_\varphi \in BP_T)\}$. If $\psi(\mathbf{x}) = (f, A_\varphi) \in AT_{Full}$ then

- When $B \in \mathcal{L}_{\omega_1,\omega}(L_H \cup BP_T)$ is quantifier free, $\psi(\mathbf{x}) \Vdash_K B(\mathbf{x})$ if and only if $\vdash (\forall \mathbf{x})\varphi(\mathbf{x}) \rightarrow B(\mathbf{x})$
- $\psi(\mathbf{x}) \Vdash_{\{-\infty,\infty\}\cup\omega_1} htype(\mathbf{x}) = f$

It is not hard to see that this is a collection of archetypes relative to BP_T up to $M_{\omega_1\cup\{-\infty,\infty\}}(T_{Full})$.

4 Definable Collections of Archetypes

Extra Information

Definition 4.0.44. Let Y be a countable set not previously mentioned. We say $\mathcal{EI} = \langle(\mathcal{EI}_K, \prec_{EI}), ei_K, \varphi^{\mathcal{EI}} : \varphi \in AT_K\rangle$ is *Extra Information* (For AT_K, BP_K up to $\mathcal{M}_K$) if

- $\mathcal{EI}_K \subseteq S(\mathcal{M}_K) \times Y \cup \{\emptyset\}$
- $\prec_K$ is a partial quasi-order on $\mathcal{EI}_K$
- $ei_K : \mathcal{M}_K \cup AT_K \to Powerset(\mathcal{EI}_K)$
- For all $\varphi \in AT_K$, $\varphi^{\mathcal{EI}} : \mathrm{dom}(\varphi) \to \mathcal{EI}_K$

and $\mathcal{EI}_K$ satisfy the following properties

Archetypes

- For all $\varphi \in AT_K$, $ei_K(\varphi) = \mathrm{range}(\varphi^{\mathcal{EI}})$
- $(\forall \varphi \in AT_K)\varphi(\mathbf{x}) \Vdash_K \mathrm{height}()P(x) = \alpha \Leftrightarrow (\exists y \in Y)\varphi^{\mathcal{EI}}(x) = (\alpha, y)$
- If $M \in \mathcal{M}_K$, $\phi \in AT_K$, then $M \models (\exists \mathbf{x})\phi(\mathbf{x}) \Leftrightarrow ei_K(\phi) \subseteq ei_K(M)$.

Models

- $(\forall M \in \mathcal{M}_K)ei_K(M)$ is linearly quasi-ordered

- $(\forall M \in \mathcal{M}_K)(\forall x, y \in \mathcal{EI}_K) x \prec_{\mathcal{EI}} y \wedge y \in ei_K(M) \rightarrow x \in ei_K(M)$
- $(\forall M \in \mathcal{M}_K)(\exists(\alpha, y) \in \mathcal{EI}_K) ei_K(M) = \{x : x \prec_{\mathcal{EI}} (\alpha, y)\}$
- $(\forall(\alpha, y) \in \mathcal{EI}_K)(\exists M \in \mathcal{M}_K) ei_K(M) = \{x : x \prec_{\mathcal{EI}} (\alpha, y)\}$

Order

- If $(\omega * \alpha + n, x), (\omega * \beta + m, y) \prec_{\mathcal{EI}} z$ and $\beta < \alpha$ then $(\omega * \beta + m, y) \prec_{\mathcal{EI}} (\omega * \alpha + n, x)$ and $(\omega * \alpha + n, x) \not\prec_{\mathcal{EI}} (\omega * \beta + m, y)$

Summary

Up until now the only properties of elements which we have discussed are their height and their level. However, as our theories become increasingly complicated it is not in general the case that the height and level of an element completely determine its $\mathcal{L}_{\infty,\omega}$-type. Extra Information gives us a way to talk about other properties elements of our models might have.

Archetypes up to a Slant Line

Definition 4.0.45. Let $\sigma, \tau \in AT_K$. We say that σ and τ are *compatible* ($\sigma \| \tau$) if $(\exists M \in \mathcal{M}_K) ei_K(\sigma) \cup ei_K(\tau) \subseteq ei_K(M)$.

Definition 4.0.46. Let $\sigma(x_1, \ldots, x_n), \tau(x_1, \ldots, x_n) \in AT$ and let sl be a slant line. We say that σ and τ are the same up to a slant line sl ($\sigma|sl = \tau|sl$) if

- $\sigma(x_1, \ldots, x_n), \tau(y_1, \ldots, y_n) \Vdash_K A(x_1, \ldots, x_n) \leftrightarrow A(y_1, \ldots, y_n)$ for all quantifier free formulas $A \in \mathcal{L}_{\omega_1,\omega}(L_K)$.

and for all $i \leq n$ either

- $\sigma^{\mathcal{EI}}(x_i) = \tau^{\mathcal{EI}}(x_i)$ or
- $\sigma(x_1, \ldots, x_n), \tau(y_1, \ldots, y_n) \Vdash_K \text{height}(_{)}P(x_i) \geq sl(\text{level}(x_i))$ and $\text{height}(_{)}P(y_i) \geq sl(\text{level}(y_i))$

We define $ei_K(\sigma|sl) = \{(\alpha, y) : (\exists x \in \text{dom}(\sigma))\sigma^{\mathcal{EI}}(x) = (\alpha, y)$ and $\sigma(\mathbf{x}) \Vdash_K \alpha = \text{height}(_{)}P(x) < sl(\text{level}(x))\}$

Summary

Intuitively two elements which are on different levels of a tree contain the same information about the spectrum of their model if they are on the same slant line. So two archetypes σ and τ are the same up to a slant line sl if they give the same answers to questions which only require information below sl.

Fix Extra Information

Theorem 4.0.47. *For the rest of this paper we fix Extra Information* $\mathcal{EI}_K = \langle(\mathcal{EI}_K, \prec_{EI}), ei_K, \varphi^{\mathcal{EI}} : \varphi \in AT_K\rangle$ *for* AT_K, BP_K *up to* $\mathcal{M}_K$.

Definable Collection of Archetypes

Definition 4.0.48. We say AT_K is a *Definable Collection of Archetypes (with respect to* $T_K, BP_K, \mathcal{EI}_K$ *up to* $\mathcal{M}_K$*)* if it satisfies

(Prediction)

For all $\sigma, \tau \in AT_K$ with $\tau(\mathbf{x}, \mathbf{y}) \Vdash_K \sigma(\mathbf{x})$ there is an $\eta_\tau(\mathbf{a})$ such that

- $(\forall M \in \mathcal{M}_K)\ M \models (\exists \mathbf{x}, \mathbf{y})\tau(\mathbf{x}, \mathbf{y}) \Rightarrow M \models (\exists \mathbf{a})\eta(\mathbf{a})$

and there is a $A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \omega^{<\omega}, \mathbf{z}, \mathbf{a}) \in BP_K$ such that

- $T_K \vdash (\forall \mathbf{x}, \mathbf{y}, \omega^{<\omega}, \mathbf{z}, \mathbf{a})A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \omega^{<\omega}, \mathbf{z}, \mathbf{a}) \rightarrow A_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ (where $Trivial_{\eta_\tau,\sigma}(\mathbf{a}, \mathbf{x}, \mathbf{z}) \Vdash_K A_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ and $A_\eta \in$

BP_K)

- $Trivial_{\eta_\tau,\sigma}(\mathbf{a},\mathbf{x},\mathbf{z}) \wedge A_{\sigma,\tau}(\mathbf{x},\mathbf{y},\omega^{<\omega},\mathbf{z},\mathbf{a}) \Vdash_K \tau(\mathbf{x},\mathbf{y})$

<u>(Prediction Up To A Slant Line)</u>

For all $\sigma,\sigma',\tau \in AT_K$ such that $\tau(\mathbf{x},\mathbf{y}) \Vdash_K \sigma(\mathbf{x})$, $\sigma' \| \tau$ and $\sigma|sl = \sigma'|sl$ there is an $\eta_{\tau|sl}(\mathbf{a})$ where

- $(\forall M \in \mathcal{M}_K) ei_K(\tau|sl) \cup ei_K(\sigma') \subseteq ei_K(M) \Rightarrow ei_K(\eta_{\tau|sl}) \subseteq ei_K(M)$

and there is a base predicate $A_{\sigma|sl,\tau|sl}(\mathbf{x},\mathbf{y},\omega^{<\omega},\mathbf{z},\mathbf{a})$ such that

- $T_K \vdash A_{\sigma|sl,\tau|sl}(\mathbf{x},\mathbf{y},\omega^{<\omega},\mathbf{z},\mathbf{a}) \to A_\eta(\mathbf{a},\mathbf{x},\mathbf{z})$ (where $Trivial_{\eta_{\tau|sl},\sigma'}(\mathbf{a},\mathbf{x},\mathbf{z}) \Vdash_K A_\eta(\mathbf{a},\mathbf{x},\mathbf{z})$ and $A_\eta \in BP_K$)
- $Trivial_{\eta_{\tau|sl},\sigma'}(\mathbf{a},\mathbf{x},\mathbf{z}) \wedge A_{\sigma|sl,\tau|sl}(\mathbf{x},\mathbf{y},\omega,\mathbf{z},\mathbf{a}) \Vdash_K (\exists \tau' \in AT_K)\tau'(\mathbf{x},\mathbf{y}) \wedge \tau'|sl = \tau|sl$.

<u>(Truth on Height)</u>

For all $\phi \in AT_K$, $\phi(\mathbf{x}),\phi(\mathbf{y}) \Vdash_K htype(\mathbf{x}) = htype(\mathbf{y})$

We don't want our base predicates to be able to tell us anything explicitly about the height of our elements (because we want every base predicate to be realized in every model). But, the base predicates can give us information about the relative heights of two elements (in some way). So in the statement of prediction we can think of η_τ as some archetype disjoint from σ but with elements with heights which are the same as those in τ. Then $A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \omega^{<\omega}, \mathbf{z}, \mathbf{a})$ is the base predicate which says "look at the tuple $\mathbf{a}$ and use it as a guide for what the heights of the tuple $\mathbf{y}$ should be".

As we will see (Prediction) allow us to show that a model is determined by the archetypes it realizes. (Prediction Up To A Slant Line) though, while similar to (Prediction), will be used to show that any two models which realize similar archetypes are in fact similar. This will allow us to get a lower bound on the quantifier rank of our models.

Extra Information Homogeneity

Theorem 4.0.49. *Suppose* $\sigma(\boldsymbol{x}), \tau(\boldsymbol{x}, \boldsymbol{y}) \in AT_K$*,* $\tau(\boldsymbol{x}, \boldsymbol{y}) \Vdash_K \sigma(\boldsymbol{x})$ *and* $ei_K(\tau) \subseteq ei_K(M)$*. Then* $M \models (\forall \boldsymbol{x})\sigma(\boldsymbol{x}) \rightarrow (\exists^\infty \boldsymbol{y})\tau(\boldsymbol{x}, \boldsymbol{y})$

Proof. This follows immediately from (Prediction) and (Homogeneity of Base Predicates) □

ATYPE

Definition 4.0.50. Define $ATYPE(M) = \{\phi \in AT_K : ei_K(\phi) \subseteq ei_K(M)\}$

If $M, N \in \mathcal{M}_K$ and sl is a slant line, we say $ATYPE(M)|sl = ATYPE(N)|sl$ if

$$(\forall \phi \in ATYPE(M))(\exists \psi \in ATYPE(N))(\phi|sl = \psi|sl)$$

and

$$(\forall \psi \in ATYPE(N))(\exists \phi \in ATYPE(M))(\phi|sl = \psi|sl)$$

Lemma 4.0.51. $(\forall \phi \in AT)\phi \in ATYPE(M) \Leftrightarrow M \models (\exists \boldsymbol{x})\phi(\boldsymbol{x})$

Proof. This follows from our definition of $\mathcal{EI}_K$. □

ATYPE $\equiv$

Theorem 4.0.52. *If $M, N \in \mathcal{M}_K$ and $ATYPE(M) = ATYPE(N)$ then $M \cong N$.*

Proof. First notice that as M and N are countable it suffices to show that $M \equiv_\infty N$.

Definition 4.0.53. Let $I(M, N) = \{f : M \to N$ s.t.

- f is a tree partial isomorphism
- There exists $\sigma_f \in AT_K$ and $\mathbf{a} \in M, \mathbf{b} \in N$ such that $M \models \sigma_f(\text{dom}(f), \mathbf{a}),\ N \models \sigma_f(\text{range}(f), \mathbf{b})\}$

We need to show that $I(M, N) \subseteq I(M, N)$ satisfies the tree back and forth condition. Let $f \in I(M, N)$, $a \in M$ such that $\text{dom}(f) \cup \{a\}$ is closed.

Let $\sigma' \in AT$ be such that $M \models \sigma'(\text{dom}(f), a, \mathbf{a}')$ with $\mathbf{a} \subseteq \mathbf{a}'$ (which we know must exist by Lemma 3.0.41). But then we have by (Completeness of Archetypes) that $\sigma'(\mathbf{x}, a, \mathbf{a}') \Vdash_K \sigma_f(\mathbf{x}, \mathbf{a})$. So by Theorem 4.0.49 $N \models (\exists b, \mathbf{b}')\sigma'(\text{range}(f), b, \mathbf{b}')$ and hence $f \cup (a, b) \in I(M, N)$.

The case where we are given a $b \in N$ and we need to find an $a \in M$ is done analogously. Hence $I(M, N)$ has the tree back and forth property and $M \equiv^T_\infty N$. So $M \equiv_\infty N$. $\square$

Corollary 4.0.54. *If $M, N \in \mathcal{M}_K$ and $ei_K(M) = ei_K(N)$ then $M \cong N$.*

Sup of Spectrum

Theorem 4.0.55. $|S(\mathcal{M}_K)| \leq |\mathcal{M}_K| \leq \max\{\omega, |S(\mathcal{M}_K)|\}$

Proof. First notice that $|\{ei_K(M) : M \in \mathcal{M}_K\}| = |\mathcal{EI}_K|$ by the conditions on extra information. Further, by Corollary 4.0.54 if $ei_K(M) = ei_K(N)$ then $M \cong N$, so $|\mathcal{M}_K| = |\mathcal{EI}_K|$. Because $|Y| \leq \omega$ (in the definition of Extra Information) we know that $|S(\mathcal{M}_K)| \leq |\mathcal{EI}_K| \leq |S(\mathcal{M}_K) \times Y| \leq \max\{\omega, |S(\mathcal{M}_K)|\}$. □

Restricted Archetypes

Theorem 4.0.56. *If $ei_K(M) \subseteq ei_K(N)$ and sl is a slant line with $sl < \omega * \gamma \subset Spec_P(M)$ then $ATYPE(M)|sl = ATYPE(N)|sl$*

Proof. First notice that $ATYPE(M) \subseteq ATYPE(N)$. So it suffices to show that for every $\varphi \in ATYPE(N)$ there is a $\psi \in ATYPE(M)$ such that $\psi|sl = \varphi|sl$.

Let $\hat{\emptyset} \in AT_K$ be such that $\mathrm{dom}(\hat{\emptyset}) = \emptyset$ and $ei_K(\hat{\emptyset}) \subseteq ei_K(M)$. Then $\varphi(\mathbf{x}) \Vdash_K \hat{\emptyset}$. Now by the Other conditions on extra information we know that for all $\phi \in ATYPE(N)$ $ei_K(\phi|sl) \subseteq ei_K(M)$. So, by (Prediction up to a Slant Line) we then know there is a ψ satisfying $ei_K(\psi) \subseteq ei_K(M)$ and

$\psi|sl = \varphi|sl.$ □

α-Equivalence

Theorem 4.0.57. *If $M, N \in \mathcal{M}_K$, $ei_K(M) \subseteq ei_K(N)$ and $\omega * \gamma \subset Spec_P(M) \subseteq Spec_P(N)$ then $M \equiv^T_{\omega*\gamma} N$*

Proof. We need to define a sequence of partial tree isomorphisms from M to N of length at least $\omega * \gamma$.

Definition 4.0.58. Define $I_\zeta(M, N) = I_\zeta$ as follows:
$I_{\omega*\zeta+n} = \{f : M \to N s.t.$

- f is a tree partial isomorphism
- There exists a slant line sl, $\sigma_f, \tau_f \in AT_K$ and $\mathbf{a} \in M, \mathbf{b} \in N$ such that
  - $sl < \omega * (\zeta + 1)$ and $sl(|\text{dom}(f)| + n) \geq \omega * \zeta\}$
  - $M \models \sigma_f(\text{dom}(f), \mathbf{a})$, $N \models \tau_f(\text{range}(f), \mathbf{b})$
  - $\sigma_f|sl = \tau_f|sl$

We need to show that $\langle I_\mu : \mu < \omega * \gamma\rangle$ satisfies the tree back and forth property. Let $\omega * \zeta + n + 1 < \omega * \gamma$ and let $f \in I_{\omega*\zeta+n+1}$, $a \in M$ such that $\text{dom}(f) \cup \{a\}$ is closed.

Let $\sigma' \in AT$ be such that $M \models \sigma'(\text{dom}(f), a, \mathbf{a}')$ with

$\mathbf{a} \subseteq \mathbf{a}'$ and $\sigma'(\mathbf{x}, a, \mathbf{a}') \Vdash_K \sigma_f(\mathbf{x}, \mathbf{a})$. We know $\sigma' || \tau_f$ by assumption so (Prediction up to a Slant Line) tells us there is an archetype $\eta_{\sigma'|sl}$ and a base predicate $A_{\sigma|sl,\sigma'|sl}$ such that whenever

$$(*)\ N \models Trivial_{\eta_{\sigma'|sl},\tau_f}(\mathbf{c}, \text{range}(f), \mathbf{b}, \mathbf{d}) \wedge A_{\sigma|sl,\sigma'|sl}(\text{range}(f), b, \mathbf{b}', \mathbf{c}, \mathbf{d}, \mathbf{e})$$

then $N \models \tau'(\text{range}(f), b, \mathbf{b}')$ for some $\tau' \in AT_K$ where $\mathbf{b} \subseteq \mathbf{b}'$ and $\tau'|sl = \sigma'|sl$.

But we also know $ei_K(\sigma'|sl) \cup ei_K(\tau_f) \subseteq N$ by Theorem 4.0.56 and so $N \models (\exists^\infty \mathbf{c})\eta_{\sigma'|sl}(\mathbf{c})$. Hence $N \models (\exists \mathbf{c}, \mathbf{d}) Trivial_{\eta_{\sigma'|sl}(\mathbf{c}),\tau_f}(\mathbf{c}, \text{range}(f),$ and by (Homogeneity of Base Predicates) N satisfies $(*)$ for some $b, \mathbf{b}', \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$. So if we let b be as above then $g = f \cup (a, b) \in I_{\omega*\eta+n}$ (because $sl(|\text{dom}(f)| + n + 1) = sl(|\text{dom}(g)| + n)$).

We do the case where we are given $b \in N$ and we find $a \in M$ analogously. Hence we have proved that $\langle I_\mu : \mu < \omega * \gamma \rangle$ has the tree back and forth property and $M \equiv^T_{\omega*\gamma} N$. $\square$

Quantifier Rank of Models

Theorem 4.0.59. *If $\omega * \gamma \in Spec_P(M)$ then the quantifier* *rank of M is at least γ ($qr(M) \geq \gamma$)*

Proof. For all $\beta < \gamma$ there is a $\sigma_\beta \in AT_K$ such that

- $\sigma_\beta(\mathbf{x}) \Vdash_K (\exists x)\omega * \beta \leq \text{height}()P(x) < \omega * (\beta + 1)$
- $M \models (\exists \mathbf{x})\sigma_\beta(\mathbf{x})$

By the definition of Extra Information there must be a $(\omega * \beta + n, y) \in ei_K(M)$. Let M_β be the model such that $ei_K(M_\beta) = \{i \in \mathcal{EI}_K : i \prec_K (\omega * \beta + n, y)\}$. Then by Theorem 4.0.57 $M_\beta \equiv^T_{\omega*\beta} M$ and $M \equiv_\beta M_\beta$ so $\text{qr}(M) > \beta$ (as $M_\beta \neq M$). But as β was arbitrary we have $\text{qr}(M) \geq \gamma$. □

Vaught's Conjecture

Summary

Notice that if $X \subseteq X' \subseteq \omega_1 \cup \{-\infty, \infty\}$ then any definable collection of archetypes up to $\mathcal{M}_{X'}(T)$ yields a definable collection of archetypes up to $\mathcal{M}_X(T)$ by restricting $(\mathcal{EI}_K, \prec_{\mathcal{EI}})$ to $\{(\alpha, y) : \alpha \in X\}$. So, as X gets larger, the statement "there exists a definable collection of archetypes for $\mathcal{M}_X(T)$" becomes stronger. Hence the strongest statement we can make of this form is "there exists a definable collection of archetypes for $M_{\omega_1 \cup \{-\infty, \infty\}}(T)$ (for some T)". This statement is significantly stronger what we will need for two reasons.

First, it assumes that there is a single definable collection of archetypes which works for all $\{-\infty\} \cup \alpha$. But as we will see, to construct the scattered sentence S_α we will only need a theory with a definable collection of archetypes up to $\{-\infty\} \cup \alpha$ and we will not care what the models look like that have larger spectra. Hence, we do not require a single collection of archetypes for all $\{-\infty\} \cup \alpha$.

Second, the existence of a definable collection of archetypes for an $\mathcal{M}_K$ where $\infty \in X$ is a very strong assumption. Much of the work in studying trees as they relate to Vaught's conjecture comes from trying to deal with the ill-founded branches.

In what follows we will develop a method that will allow us to ignore ill-founded branches all together. However the cost will be a strict upper bound on our quantifier rank.

Vaught's Conjecture

Theorem 4.0.60. *Suppose T has a definable collection of* *archetypes for* $\mathcal{M}_{\{-\infty,\infty\}\cup\omega_1}(T)$*. Then* T *has* ω_1 *many countable models.*

Proof. We know that every countable model M must have $\mathrm{Spec}_P(M) \subseteq \{-\infty,\infty\}\cup\omega_1$ and hence $\mathcal{M}_{\{-\infty,\infty\}\cup\omega_1}(T)$. By Theorem 4.0.55 we therefore know that T has $|\{-\infty,\infty\}\cup\omega_1| = \omega_1$ many countable models. □

The central argument of the proposed counterexample of Robin Knight **??** is (essentially) a construction of a definable collection of archetypes for $\{-\infty,\infty\}\cup\omega_1$ (that these condition follow from slight modifications to the theory Θ in **??** is proved in **??**(my thesis)).

Extra Information for Full Trees

Definition 4.0.61. Define $\mathcal{EI}_{Full} = \langle(\mathcal{EI}_{Full}, \prec_{Full}), ei_{Full}, \varphi^{\mathcal{EI}} : \varphi \in AT_{Full}\rangle$ as follows

- $\mathcal{EI}_{Full} = \{(\alpha, n) : \alpha \in \{-\infty\}\cup\omega_1, n \in \omega\}$
- $(\beta, m) \prec_{\mathcal{EI}} (\alpha, n) \Leftrightarrow \beta + m \leq \alpha + n$ (where $\infty + n = \infty$ and $-\infty + n = -\infty$ for all $n \in \omega$).
- $\varphi^{\mathcal{EI}}(x) = (\alpha, n)$ such that $\varphi(\mathbf{x}) \Vdash_K \mathrm{height}()P(x) = \alpha$ and $\mathrm{level}(x) = n$.

- $ei_K(M) = \{(\alpha, n) : (\exists a \in M)\text{height}()P(a) = \alpha, \text{level}(a) = n\}$.

Theorem 4.0.62. *$\mathcal{EI}_{Full}$ is extra information for $T_{Full}(P)$ for $\mathcal{M}_{\{-\infty\}\cup\omega_1}(T_{Full})$*

Proof. The only conditions which aren't self evident are those in Models. To see that these are satisfied notice that if $M \models (\exists a)\text{height}()P(a) = \alpha$ and $\text{level}(a) = n$ then for all $\beta < \alpha$ such that $\beta + m \leq \alpha + n$ (with $m > n$), $M \models (\exists b < a)\text{height}()P(b) = \beta$ and $\text{level}(b) = m$. □

Notice here, for the first time, we actually need that our trees are full (and not just that we can compare heights). Also notice that $\mathcal{EI}_{Full}$ is not extra information up to $M_{\{-\infty,\infty\}\cup\omega_1}(T_{Full})$. This is because $ei_K(M)$ only determines $M|_{L_P}$, and $M|_{L_P}$ only determines M if $\infty \notin \text{Spec}_P(M)$.

5 Pairs of Archetypes