Lecture Notes on Vaught's Conjecture at Logic Seminar(Fall 2007)

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TALK SLOWLY AND WRITE NEATLY!!

1 Vaught's Conjecture

1.1 Statement of Vaught's Conjecture

In 1961 Robert Vaught published a ground breaking paper on countable model theory entitled "Denumerable Models of Complete Theories". At the end of this paper he asked a question

> "Do all countable first order theories have either countably many or continuum many countable models"

The statement that this is true has become known as Vaught's Conjecture.

Dependence on CH

Now there is an obvious issue at the start. That is this conjecture isn't absolute.

This is because

Theorem 1.1.1. *Let σ be a countable first order language. Then there are at most 2^ω many countable models for the language σ.*

Proof. First notice without loss of generality we can assume that σ is a relational language. Let

$$\sigma = \{R_i(x_1, \ldots, x_{n_i}) : i \in \omega\}$$

Similarly if the model is countable we can assume it's domain is ω.

A model is then an assignment of tuples of ω which satisfy these relations. So a model is an element of

$$\text{Models}(\sigma) = \prod_{i \in \omega} 2^{\omega^{n_i}}$$

which as size 2^ω. $\square$

So we have

Corollary 1.1.2. *If $2^\omega = \omega_1$ the previously mentioned version of Vaught's Conjecture holds.*

So we want an absolute version of Vaught's conjecture.

1.2 Absolute Version of Vaught's Conjecture

It isn't enough to come up with an absolute version of Vaught's conjecture. What we really want is an way to characterize the number of countable models a first order has which is independent of the universe we are working in.

At first glance this looks hopeless because suppose the continuum is something very large like $2^\omega = \omega_{100}$ in V. Then if we have a theory T which has ω_{10} many countable models and we look at a generic extension $V[G]$ in which $2^\omega = \omega_5$ what can we say about the number of countable models of T?

Fortunately there is a result of Scott/Morely which makes this manageable but first we need a definition.

Definition 1.2.1. $A \subseteq 2^\omega$ is a *Perfect Kernel* if it is a closed subset with no isolated points.

Theorem 1.2.2. *If σ is a countable language and $\varphi \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$ then one of three things can happen*

1 φ has a perfect kernel of countable models

2 φ does not have a perfect kernel of models and has ω many countable models.

3 φ does not have a perfect kernel of models and has ω_1 many countable models.

Further, for any particular theory T, which of $1, 2$ *or* 3 *is satisfied is independent of the model of set theory we are in.*

Definition 1.2.3. We say a countable theory T is scattered if it does not have a perfect kernel of countable models.

So an absolute version of Vaught's conjecture is

Theorem 1.2.4 (Absolute Vaught's Conjecture)**.** *Every scattered theory has countably many models.*

2 Σ_1^1 Subsets of Reals

Definition 2.0.5. We say $A \subseteq X$ is Σ_1^1 if there is a Borel B such that $B \subseteq \omega^\omega \times X$ and $A = \pi_2[B]$

We will often talk interchangably about sets and formulas which define those sets. In this context a Σ_1^1 set is one which has an $(\exists r \in \omega^\omega)$ in front of arbitrary quantifiers over ω.

This theorem is an incredibly important and useful theorem and we will use it several times. However, its proof, while very interesting, would take us far off topic. So you will just have to take it on faith.

3 $\mathcal{L}_{\omega_1,\omega}$

3.1 Definition

Definition 3.1.1. Give definition of $\mathcal{L}_{\omega_1,\omega}$

Definition 3.1.2. Define a subformula recursively.

Definition 3.1.3. Define the subformula tree representation of a formula.

Now $\mathcal{L}_{\omega_1,\omega}$ is a very natural infinitary generalization of first order logic in which many of the nice properties of first order logic have analogs. Specifically, there are analogs of compactness, completeness and as we will see later, and omitting types theorem.

In the context of Vaught's conjecture, everywhere that we have mentioned a first order theory we can replace it by a sentence of $\mathcal{L}_{\omega_1,\omega}$ and all the results still hold.

3.2 Fragments

In general we don't want to have to deal with all of $\mathcal{L}_{\omega_1,\omega}$ at once as it is a very large uncountable object. So instead we deal with what are called fragments.

Definition 3.2.1. Define a fragment (close under first order operations and subformulas).

A fragment is determined by the subformula relation. And further there is a Σ^1_1 relation on models which determines which sentences a given model satisfies (and likewise on formulas)

Theorem 3.2.2. *Let σ be a countable language and let A be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\sigma)$ with $f : \omega \to A$ a bijection. Then there is a $\boldsymbol{\Sigma^1_1}$ relation on $\models \subseteq \omega \times \omega^{<\omega} \times Models(\sigma)$ such that*

$$\models (n, \boldsymbol{x}, M) \Leftrightarrow M \models f(n)[\boldsymbol{x}]$$

Proof. Let $\prec_A (a, b, c)$ hold if $f(a)$ is an immediate subformula of $f(b)$ and if

- $c = 0$ implies that the operation to get from $f(a)$ to $f(b)$ is $\neg$

- $c = 1$ implies that the operation to get from $f(a)$ to $f(b)$ is $(\exists y)$
- $c = 2$ implies that the operation to get from $f(a)$ to $f(b)$ is $(\forall y)$
- $c = 3$ implies that the operation to get from $f(a)$ to $f(b)$ is $\bigvee$
- $c = 4$ implies that the operation to get from $f(a)$ to $f(b)$ is $\bigwedge$

Notice that $\prec_A$ can be viewed as a single real number.

Now let

$T(\prec_A, g, n, \mathbf{x}, M) \Rightarrow$

- $g : \omega \times \omega^{<\omega} \to \{0, 1\}$
- If $f(n)$ is a relation then $g(n, \mathbf{x}) = 1$ if and only if $M \models R(\mathbf{x})$.

- If $(\exists m) \prec_A (m, n, 0)$ then $g(n, \mathbf{x}) = 1$ if and only if $g(m, \mathbf{x}y) = 0$
- If $(\exists m) \prec_A (m, n, 1)$ then $g(n, \mathbf{x}) = 1$ if and only if $(\exists y)g(m, \mathbf{x}y) = 1$
- If $(\exists m) \prec_A (m, n, 2)$ then $g(n, \mathbf{x}) = 1$ if and only if $(\forall y)g(m, \mathbf{x}y) = 1$
- If $(\exists m) \prec_A (m, n, 3)$ then $g(n, \mathbf{x}) = 1$ if and only if $(\exists m) \prec (m, n, 3)g(m, \mathbf{x}) = 1$
- If $(\exists m) \prec_A (m, n, 4)$ then $g(n, \mathbf{x}) = 1$ if and only if $(\forall m) \prec (m, n, 4)g(m, \mathbf{x}) = 1$

So we can let

$\models (n, \mathbf{x}, M) \Leftrightarrow (\exists g)T(\prec_A, g, n, \mathbf{x}, M)$ and g is total and $g(n, \mathbf{x}) = 1$ $\square$

Definition 3.2.3. Let $\prec_A$ be as above. Then we say $\prec_A$ is an encoding of the fragment A.

3.3 Polish Groups

Notice that we have the following theorem

Theorem 3.3.1. *Let* $\sigma \in \mathcal{L}_{\omega_1,\omega}$*. Then* $\{M \in Models : M \models \sigma\}$ *is a* Σ^1_1 *subset of Models and hence has either countably many or continuum many models.*

The issue though is that while we might have continuum many representations of models that doesn't mean we have continuum many non-isomorphic representaionts. So we need some other way to limit the number of models of our theory.

Specifically we have

Theorem 3.3.2. *Let* S_∞ *be the group automorphism group of* ω*. Then there is a group action of* S_∞ *on Models which preserves isomorphisms classes.*

We then also have a theorem

Theorem 3.3.3. *A set A $\subseteq$Models is Borel if and only if there is a $\sigma \in \mathcal{L}_{\omega_1,\omega}$ such that $A = \{M \in$Models: $M \models \sigma\}$.*

Hence we have another variant of Vaught's conjecture which says

Theorem 3.3.4 (Vaught's Conjecure)**.** *Every Borel subset of Models has either countably many or continuum many orbits under the action of S_∞.*

In this way Vaught's conjecture is also related to group actions and Polish spaces.

3.4 Types

Definition 3.4.1. Let A be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\sigma)$. We say that $p(\mathbf{x}) \subseteq A$ is an n-type

- There is a model M and an n-tuple $a \in M$ such that $(\forall \varphi \in p) M \models \varphi(\mathbf{a})$

- For all $\varphi(\mathbf{x}) \in A$ either $\varphi(\mathbf{x}) \in p(\mathbf{x})$ or $\neg\varphi(\mathbf{x}) \in p(\mathbf{x})$.

Theorem 3.4.2. *For every $p \subseteq A$ of sentences there are countably many or continuum many n-types consistent with it.*

Proof. Let $f : \omega \to A$ be a bijection. Then define $R(P, M, \mathbf{x}) \subseteq 2^\omega \times$ Models $\times \omega^n$ as follows

$R(P, M, \mathbf{x}) \Rightarrow$

- $P(n) = 1 \to M \models f(n)[\mathbf{x}]$
- $P(n) = 0 \to M \models \neg f(n)[\mathbf{x}]$

Then P is an n-type if and only if $(\exists M)(\exists \mathbf{x})R(P, M, \mathbf{x})$ which is a Σ^1_1 subset of 2^ω. $\square$

4 Vaught Tree

4.1 Weakly Scattered

Theorem 4.1.1. *Let* $\varphi \in \mathcal{L}_{\omega_1,\omega}(\sigma)$*. If there exists a fragment* A *such that*

- $\varphi \in A$
- *There are* 2^ω *many types over* A *contianing* φ

then φ *has a perfect kernel of models.*

Definition 4.1.2. Let $\varphi \in \mathcal{L}_{\omega_1,\omega}(\sigma)$. We say that φ is *Weakly Scattered* if for all countable fragments A such that $\varphi \in A$, there are only countably many types over A containing φ

Lemma 4.1.3. *If* φ *is not weakly scattered then* φ *has a perfect kernel of models.*

4.2 Vaught Tree

Definition 4.2.1. Let A be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and let M be a model. Define $A[M] = A \cup \bigwedge\{\varphi(\mathbf{x}) \in p(\mathbf{x}) : (\exists \mathbf{a} \in M) M \models p(\mathbf{a})\}$

Definition 4.2.2. Let $\varphi \in \mathcal{L}_{\omega_1,\omega}$ and let A be the smallest fragment containing φ. We then define the Vaught tree for φ as follows. Nodes are pairs (B, T) such that B is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and T is a complete theory in B

- $(B, T) \in$ Level 0 if and only if $B = A$ and $\vdash T \to \varphi$.
- $(B, T) \in$ Level $\beta + 1$ if there is some node $(C, T') \in$ Level β and some model M such that $B = C[M]$
- $(B, T) \in$ Level $\omega * \alpha$ if $B = \bigcup_{i<\omega*\alpha} C_i$ and there are nodes $(C_i, T_i) \in$ Level i where $C_i \subseteq C_j$ if $i \leq j$.

Given two nodes (B, T) and (C, S) we say $(B, T) \preceq (C, S)$ if and only if $B \subseteq C$ and $\vdash S \to T$.

4.3 Atomic Formulas

Definition 4.3.1. We say that a type $p(\mathbf{x})$ over a countable fragment A is *Atomic* if there is a formula $\varphi(\mathbf{x}) \in A$ such that

- $\varphi(\mathbf{x}) \in p(\mathbf{x})$
- $(\forall \psi(\mathbf{x}) \in p(\mathbf{x})) \vdash \varphi(\mathbf{x}) \to \psi(\mathbf{x})$.

Definition 4.3.2. We say a model M is an atomic model for a fragment A if for all $\mathbf{a} \in M$ type$(\mathbf{a})$ is an atomic type.

Lemma 4.3.3. *If M is an atomic model for a fragement A then $A[M] = A$.*

Theorem 4.3.4. *If $M \models \varphi$ then there is some node (A, T) at a countable level in the Vaught tree such that*

- *$M \models T$*
- *M is atomic for the fragment A.*

Proof. Assume $(A[M], T)$ is a node in the Vaught tree, $M \models T$ and M is not atomic for $A[M]$. Then there is some tuple $\mathbf{a}$ such that type($\mathbf{a}$) is not generated by a single formula.

Hence, if we let $S(A, \mathbf{a}) = \{\mathbf{b} \in M : (\forall \varphi(\mathbf{x}) \in A) M \models \varphi(\mathbf{a}) \leftrightarrow \varphi(\mathbf{b})\}$ then we must have $S(A[M], \mathbf{a}) \subsetneq S(A, \mathbf{a})$ because type($\mathbf{a}$) is non-principle.

In paricular, because M is countable, there must be some node (B, T') of the Vaught tree (at a countable level) such that $M \models T'$ and $S(B, \mathbf{a}) = S(B[M], \mathbf{a})$ for all $\mathbf{a} \in M$.

But then we must have that every formula of $B[M]$ is implied by one of B. And in particular we have that M is an atomic model for B. □

Definition 4.3.5. We say the Vaught rank of a sentence

φ is the smallest α such that every countable model φ is atomic for some node in the Vaught tree at level $\leq \alpha$.

4.4 Level's of Vaught Tree

Theorem 4.4.1. *Let $\varphi \in \mathcal{L}_{\omega_1,\omega}$. Then for each countable α there is a bijection between the number of nodes at level α and a Σ^1_1 set.*

Proof. Let $f : 2^\omega \to (2^\omega)^\omega$ be a bijection.

Let $<_\alpha \subseteq \omega \times \omega$ be a linear order of order type $\alpha + 1$.

Let $T(X, <_\alpha, A, T) \Leftrightarrow$

- $f(X)_i$ encodes a fragment $F[f(X)_i]$.
- $F[f(X)_i] \subseteq F[f(X)_j]$ if and only if $<_\alpha (i, j)$
- Let $a \in \omega$ be the element such that $\{n \in \omega : n <_\alpha\}$ has order type α in $<_\alpha$. Then $f(X)_a = A$

- $T \subseteq A$ and there is a model M such that $M \models T$

Then (A, T) is a node of the Vaught tree on level α if and only if $(\exists X)T(X, <_\alpha, A, T)$. $\square$

Corollary 4.4.2. *Either the nodes of the Vaught tree at level α contain a perfect kernel or there are countably many of them.*

Now we know that for every model there is some node for which it is atomic. However we don't know that every node has a model which is atomic for that node. And in general this isn't the case. But if our theory is weakly scattered then it is. And that is what we prove next.

5 Model Existence Theorem

Now we will prove the Model Existence Theorem. The Model Existence Theorem captures the essence of a Henkin construction of a countable model in the context of $\mathcal{L}_{\omega_1,\omega}$.

5.1 Model Existence Theorem

The theorem is incredibly useful and with it we can prove many different results. First we need some convenient notation.

Definition 5.1.1. Let $\varphi \in \mathcal{L}_{\omega_1,\omega}(\sigma)$. Then we define $(\sigma\neg)$ as follows

- $\varphi\neg = \neg\varphi$ if φ is atomic.
- $\neg\varphi\neg = \varphi$.
- $(\bigvee_{i\in I} \varphi_i)\neg = \bigwedge_{i\in I}(\varphi_i\neg)$
- $(\bigwedge_{i\in I} \varphi_i)\neg = \bigvee_{i\in I}(\varphi_i\neg)$
- $(\exists x\varphi)\neg = (\forall x)(\varphi\neg)$
- $(\forall x\varphi)\neg = (\exists x)(\varphi\neg)$

So $\neg\varphi$ is always logically equivalent to $\varphi \neq$

Definition 5.1.2. Let σ be a countable signature and let C be a countable set of new constant symbols with

$\tau = \sigma \cup C$. A set of countable collections of sentences S is a Consistence Property if and only if for all $s \in S$ the following holds

- Either $\varphi \notin s$ or $(\varphi\neg) \notin s$

- If $(\neg\varphi) \in s$ then $s \cup \{\varphi\neg\} \in S$
- If $(\bigwedge_i \varphi_i) \in s$ then for each i $s \cup \{\varphi_i\} \in S$.
- If $(\forall x \varphi(x)) \in s$ then for all $c \in C$, $s \cup \{\varphi(c)\} \in S$
- If $\bigvee_i \varphi_i \in s$ then for some i $s \cup \{\varphi_i\} \in S$
- If $(\exists x)\varphi(x) \in s$ then for some $c \in C$ $s \cup \{\varphi(c)\} \in S$
- If $(c = d) \in s$ then $s \cup \{d = c\} \in S$.

  If $c = d, \varphi(c) \in s$ then $s \cup \{\varphi(d)\} \in s$

Theorem 5.1.3 (Model Existence Theorem). *If S is a consistency property and $s_0 \in S$ then there is a* *countable model of* s_0*.*

Proof. We can assume without loss of generality that every subset of a member of S is a member of S. Let Y be the least set of sentences such that

- $s_0 \subset Y$
- If $c \in C$ and $\varphi(t) \in Y$ then $\varphi(c) \in Y$
- If $(\neg\varphi) \in Y$ then $(\varphi\neg) \in Y$.
- If $c, d \in C$ then $c = d \in Y$

Y is clearly countable and let $X = \{\varphi_i : i \in \omega\} = \{\varphi \in Y : \varphi \text{ is a sentence}\}$ and let $C = \{c_i : i \in \omega\}$.

We want to construct $s_0 \subset s_1 \subset_2 \subset \dots$

Assume we have chosen s_n. We then choose s_{n+1} such that

- $s_n \subset s_{n+1} \in S$
- If $s_n \cup \{\varphi_n\} \in S$ then $\varphi_n \in s_{n+1}$
- If $s_n \cup \{\varphi_n\} \in S$ and $\varphi_n = \bigvee \Psi$ then there is a $\psi \in \Psi$ such that $\psi \in s_{n+1}$
- If $s_n \cup \{\varphi_n\} \in S$ and $\varphi_n = (\exists x)\psi$ then there is a $c \in C$ such that $\psi(c) \in s_{n+1}$

Let $s_\omega = \bigcup_{n\in\omega} s_n$.

We then let $c \sim d$ if $c = d \in s_\omega$ and we define $A = C/\sim$

It then say $A \models R(c_1, \ldots, c_n)$ if and only if $R(c_1, \ldots, c_n) \in s_\omega$. It isn't hard to show by induction that this is well defined and that for each $\varphi \in s_\omega$ $A \models \varphi$. $\square$

5.2 Omitting Types Theorem

Theorem 5.2.1 (Omitting Types). *Let L_A be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and let $T \subset L_A$ be a set of sentences and for each $n < \omega$ let $\Phi_n(x_1, \ldots, x_{p_n})$ be a set of formulas with at most the free variables $x_1, \ldots, x_{p_n}$. Suppose that*

(i) T has a model

(ii) For all $n < \omega$ and for all $\psi(x_1, \ldots, x_{p_n}) \in L_A$, if $T \cup \{(\exists x_1, \ldots x_{p_n})\psi\}$ has a model then there is a $\varphi \in \Phi_n$ such that $T \cup \{(\exists x_1, \ldots x_{p_n})\psi \wedge \varphi\}$ has a model.

Then there is a model M of T such that for every Φ_n $M \models (\forall x_1, \ldots x_{p_n}) \bigvee \Phi_n(x_1, \ldots x_{p_n})$

Proof. Let L_A^* be the set of formulas obtained by replacing all occurrences of finitely many free variables with constants $c \in C$.

Let S be the set of all s of the form

$$s = s_0 \cup T \cup \{\bigvee \Phi_n(c_1, \dots c_{p_n})\}$$

where s_0 is a finite set of sentences of L_A^* and only finitely many $c \in C$ occur in s_0 and $s_0 \cup T$ has a model.

Lemma 5.2.2. *S is a consistency property.*

Proof. The only nontrivial step is the disjunction one.

<u>Case 1</u>: $\bigvee \Theta \in s_0 \cup T$

Let A be a model of $s_0 \cup T$. Then for some $\theta \in \Theta$ $A \models s_0 \cup \{\theta\}$ and hence $s \cup \{\theta\} \in S$.

<u>Case 2</u>: $\bigvee \Theta = \bigvee \Phi_n(c_1, \dots c_{p_n})$ for some $n < \omega$ and

$c_1, \ldots, c_{p_n} \in C$. So

$$\Theta = \{\varphi(c_1, \ldots c_{p_n}) : \varphi \in \Phi_n\}$$

Let $d_1, \ldots, d_m$ be all the constants in $C - \{c_0, \ldots, c_{p_n}\}$ which occur in s_0. So

$$s_0 = s_0(c_1, \ldots, c_{p_n}, d_1, \ldots, d_m)$$

Since $s \in S$ the set

$$T \cup \{(\exists x_1, \ldots, x_{p_n})(\exists y_1, \ldots, y_m) \bigwedge s_0(x_1, \ldots, x_{p_n}, y_1, \ldots, y_m)\}$$

has a model. So, by the condition (ii) there is a $\varphi \in \Phi_n$ such that

$$T \cup \{(\exists x_1, \ldots, x_{p_n})(\exists y_1, \ldots, y_m) \bigwedge s_0(x_1, \ldots, x_{p_n}, y_1, \ldots, y_m) \wedge \varphi(x_1, \ldots$$

It follows that $s \cup \{\varphi(c_1, \ldots c_{p_n})\} \in S$ because $\varphi(c_1, \ldots c_{p_n}) \in \Theta$ and

$$T \cup s_0 \cup \{\varphi(c_1, \ldots c_{p_n})\}$$

has a model.

□

Now by (i) the set

$$T \cup \{\bigvee \Phi_n(c_1, \dots c_{p_n}) : n < \omega, c_i \in C\}$$

belongs to S. So in particular it has a model where every element is one of the c_i. So T has model

$$M \models T \wedge \bigwedge_{n \in \omega} (\forall x_1, \dots x_{p_n}) \bigvee \Phi_n(x_1, \dots x_{p_n})$$

□

We have shown that for each model there is a minimal node in the Vaught tree such that the node is atomic.

Theorem 5.2.3. *If T is a weakly scattered theory and (B, T_B) is a node of the theory, then there is an atomic model of T.*

Proof. Because the theory T is weakly scattered, there are only ω many types and so in particular there are at most ω many non-principle types. So we can find a model which omits all of them. $\square$

So when counting models of weakly scattered theories there is a bijection between isomorphism classes of modes and nodes in the Vaught tree. So when studying weakly scattered theories it suffices to study their Vaught trees.

Theorem 5.2.4. *For every weakly scattered theory T one of the following happens*

- *There is an α such that there is a perfect kernel of nodes of the Vaught tree and so there is a perfect kernel of models of T.*

- *There is a β such that the Vaught tree of T doesn't split at any level above β and each level has only countably many nodes. In this case T has ω many* *models.*

- *Every level of the Vaught tree has ω many nodes and for all countable ordinals α there is a new node of the Vaught tree at level α. In this case T has ω many nodes.*

Definition 5.2.5. If T is weakly scattered and has no perfect kernel of models then we say T is scattered.

6 Absoluteness of Vaught's Conjecture

All that is left to prove that which of the previous three is independent of the model of set theory we are working in.

It is clear that if T is not scattered in $L[T]$ then it isn't scattered in any larger model of set theory.

So it suffices to show that if T is scattered in $L[T]$ then T is scattered in any larger model of set theory, and that if the Vaught tree is unbounded in $L[T]$ then it is unbounded in every model of set theory.

6.1 Absoluteness of Scattered

First we need a couple of useful results which we won't prove here.

Theorem 6.1.1 (Shoenfield Absoluteness). *Let* $\varphi(x)$ *be* $\mathbf{\Sigma}^1_2$ *and* $V \models ZFC$*. Then* $V \models \varphi(a) \Leftrightarrow L(a) \models \varphi(a)$

Definition 6.1.2. We say that a set $(A, \in)$ is Admissible if it satisfies Kripke-Plank Set Theory (KP)

- (Extensionality) $(\forall x) x \in y \leftrightarrow x \in z \Rightarrow y = z$
- (Induction) For all $\varphi(x)$, $(\forall x)[(\forall y \in x \varphi(y)) \rightarrow \varphi(x)] \Rightarrow (\forall x)\varphi(x)$
- (Empty Set) $(\exists x)(\forall y) y \notin x$
- (Pairing) $(\forall x, y)(\exists z) a \in z \Leftrightarrow a = x \vee a = y$
- (Union) $(\forall x)(\exists y) z \in y \Leftrightarrow (\exists a \in x) z \in a$
- (Δ_0-Separation) For all Δ_0 formulas $\varphi(x)$, $(\forall a)(\exists b) z \in b \leftrightarrow z \in a \wedge \varphi(z)$

- (Δ_0-collection) For all Δ_0 formulas $\varphi(x, y)$, such that $A \models (\forall x)(\exists y)\varphi(x, y)$ then $A \models (\forall a)(\exists b)(\forall x \in a)(\exists y \in b)\varphi(x, y)$

Admissible sets are INCREDIBLY useful and interesting objects. And at some point next year I will be happy to talk about them in more detail.

Theorem 6.1.3. *Let $D(x, y)$ be a Δ_0 formula. Let $p, b \in A$ where A is a countable admissible set. And let*

$$S^D_{p,b} = \{x : x \in V \wedge x \subseteq b \wedge D(x, p)\}$$

Then if $S^D_{p,b} \notin A$ then $|S^D_{p,b}| = 2^\omega$

Theorem 6.1.4. *Suppose T is scattered in $L[T]$ and $V \models ZFC$. Then T is scattered in V.*

Proof. Let $D(x, y, T, \mathbf{z}) \Leftrightarrow$

- $y = (B_0, T_0)$ where B_0 is a fragment of $\mathcal{L}_{\omega_1,\omega}$ and T_0 is an ω-consistent, complete subset of B_0 such that

$T_0 \to T$.

- $x = (B_1, T_1)$ where B_1 is a fragment of $\mathcal{L}_{\omega_1,\omega}$ and T_1 is an ω-consistent, complete subset of B_1 such that $T_1 \to T_0$.

- $\mathbf{z}$ are the parameters needed to define these things (e.g. ω, ect.)

We have seen that $D(x, y, T, \mathbf{z})$ can be expressed as a Δ_0 formula where $\mathbf{z}$ is independent of x, y, T.

Let $SC(T) \Leftrightarrow$

- $(\exists \in_A \subseteq \omega \times \omega)$ such that $(\omega, \in_A)$ is a countable well-founded admissible set.

- $(\exists n \in \omega)(\{m : m \in_A n\}, \in_A)$ is a countable well-founded admissible set.

- $(\{m : m \in_A n\}, \in_A) \models (\exists \mathbf{z}, T)$ where $\mathbf{z}$ is as above and T is our initial theory

- $(\exists a, b, c \in \omega) a \in_A n \wedge b \in_A n \wedge c \notin_A n$ where $(\omega, \in_A) \models S^D_{a,b}(c)$

First notice that $V \models SC(T)$ if and only if T is scattered.

Next notice that $SC(T)$ is a $\boldsymbol{\Sigma^1_2}$ sentence. The $(\exists x \in \omega^\omega)$ is obvious via the existence of the admissible set $(\omega, \in_A)$. However, the reason why it isn't a Σ^1_1 set is that we need to say that (ω, in_A) is well-founded.

I.e. we need to say that for all subsets there doesn't exist an infinite descending chain. This is a Π^1_1 statement (and in fact is a Π^1_1-complete statement) and hence the whole thing is Σ^1_2.

Therefore being scattered is absolute. $\square$

6.2 Absoluteness of Unbounded Trees

So all that is left is to show that if the Vaught tree is unbounded in $L[T]$ and $V \models ZFC$ then it is unbounded in V.

First we need the following theorem which we won't prove

Definition 6.2.1. We say a collection of sentences T of $\mathcal{L}_{\omega_1,\omega}$ is ω-complete if for all $\bigwedge \Phi \in T$ there is a $\phi \in \Phi$ such that $\phi \in T$.

Theorem 6.2.2. *Let T be a countable, finitely consistent, ω-complete collection of sentences of $\mathcal{L}_{\omega_1,\omega}$. Then T has a countable model.*

Definition 6.2.3. Let $T \in \mathcal{L}_{\omega_1,\omega}(\sigma)$ be a countable sentence in a countable language. We say the Vaught tree of T is unbounded if there exist new nodes at level α for each $\alpha \in \omega_1$.

Definition 6.2.4 (Unbounded Vaught Trees)**.** Suppose T is weakly scattered and has an unbounded Vaught tree in $L[T]$. Let $VT[T](X) \Leftrightarrow (\exists \mathcal{B}[T])(\exists \beta)$

- $\mathcal{B}[T] = \{\langle \alpha, B_\alpha, T_\alpha \rangle : \alpha \leq \beta\}$
- B_α is a fragment of $\mathcal{L}_{\infty,\omega}$ such that $B_\alpha \supset B_\gamma$ if $\alpha > \gamma$.
- $T_\alpha \subseteq B_\alpha$ is a κ-complete, finitely consistent theory such that $T_\alpha \rightarrow T_\gamma$ if $\alpha > \gamma$
- $T_0 \rightarrow T$
- $(B_\beta, T_\beta) = X$.

Theorem 6.2.5. *If T is a weakly scattered theory, T has an unbounded Vaught tree in V if and only if $(\forall \alpha \in \omega_1^V)$ there is a $X \in V$ of length α such that $VT[T](X)$*

Proof. By definition $\square$

Theorem 6.2.6. *If T is a weakly scattered theory, T has an unbounded Vaught tree in V if and only if $(\forall \alpha \in \omega_1^L[T])$ there is a $X \in L[T]$ of length α such that $VT[T](X)$*

Proof. Let $\kappa \in \omega_1^V$. Let $f : A \prec L[T]$ be the Σ_1 elementary substructure of $L[T]$ containing $\{T, \kappa\}$.

A is countable and $A = L_\alpha[T]$ for some α.

$(\exists^{L[T]} X) VT[T](X)$ where X has length κ if and only if $(\exists^A X) VT[T](X)$ where X has length $f^{-1}(\kappa)$ (because $VT[T]$ is Σ_1).

But this holds by assumption. So in $L[T]$ there are elements satisfying $VT[T](X)$ of arbitrary ordinal length.

Hence there must be elements of arbitrary countable (in

V) ordinal length. $\square$

7 EF Games