Lecture Notes on Aczel's Antifoundation Axiom at Logic Seminar (Fall 2007)

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TALK SLOWLY AND WRITE NEATLY!!

1 An Introduction to the Axiom

1.1 Well-Founded Case

One of the most basic results in ZFC is Mostrowski's Collapsing lemma:

Definition 1.1.1. Let E be a binary relation on P. Fore each $x \in P$ we define

$$ext_E(x) = \{z \in P : zEx\}$$

A relation E is extensional if

$$ext_E(x) = ext_E(y) \Leftrightarrow x = y$$

Theorem 1.1.2 (Mostrowski's Collapsing Lemma)**.** *If E is a well-founded extensional relation on a class P then there is a unique transitive class M and isomorphism π between (P, E) and $(M, \in)$.*

Proof. We let $\pi(x) = \{\pi(z) : zEx\}$. Because E is well-founded this map is well-defined.

We then let $M = \pi"[P]$. It is clear that π a surjective homomorphism, and because E is extensional we see it is also injective (and hence an isomorphism). □

Intuitively this lemma says that every well-founded directed graph with a root represents a unique set.

This is a very nice property and one which we would like to make precise.

Definition 1.1.3. An Accessible Pointed Graph (APG) is a triple (G, E, p) where

- (G, E) is a directed graph
- $(\forall x \in G) \bigvee_{n\in\omega}(\exists y_0, y_1, \cdots, y_n) \bigwedge_{i<n} E(y_i, y_{i+1}) \wedge y_0 = p \wedge y_n = x$

Definition 1.1.4. A decoration of an APG (G, E, p) is a map $\pi : G \to V$ such that $\pi(x) = \{\pi(y) : E(y, x)\}$. We say that (G, E, p) is a picture of $\pi(p)$.

So Mostrowski's collapsing lemma says that for every wellfounded tree there is a decoration.

Examples

Draw trees 0, 1, 2, 3.

However, it isn't hard to show by induction that

Theorem 1.1.5. *If* (G, E, p) *is a well-founded APG and* π, π' *decorations of* (G, E, p) *then* $\pi = \pi'$.

Proof. We prove this by induction on the height of (G, E, p).

If height$(()G) = 0$ then $G = \{p\}$ and so $\pi(p) = \{\pi(y) : E(y, p)\} = \emptyset = \pi'(p)$

Assume the theorem is true for all graphs of height $< \alpha$ and let height$(()G) = \alpha$.

For all $x \in G$ there is an APG (Gx, E, x) where

$$Gx = \{y \in G : \bigvee_{n \in \omega} (\exists y_0, y_1, \cdots, y_n) \bigwedge_{i<n} E(y_i, y_{i+1}) \wedge y_0 = x \wedge y_n = y\}$$

Further $\pi|_{Gx}, \pi'|_{Gx}$ are decorations of (Gx, E, p). So, because height$(()Gx) <$ height$(()G) = \alpha$ if $x \neq p$ we know that $\pi|_{Gx} = \pi'|_{Gx}$ (if $x \neq p$)

Now $\pi(x) = \pi|_{Gx}(Gx) = \pi'|_{Gx}(Gx) = \pi'(x)$ for all $x \in G - \{p\}$ $\pi(x) = \pi'(x)$.

So all that is left is to show that $\pi(p) = \pi'(p)$

But $\pi(p) = \{\pi(x) : xEp\} = \{\pi'(x) : xEp\} = \pi'(p)$. And we are done □

So between these two theorems we have that every well-founded tree has a unique decoration or equivalently that every wellfounded tree is the picture of a unique set.

However, once we leave the realm of wellfounded sets, there is no reason why we should restrict ourselves to graphs which are trees.

1.2 Ill-Founded Case

Now that we have considered the well-founded case lets consider the ill-founded case. The first thing to notice is that every set has a picture which represents it.

Theorem 1.2.1. *Let A be a set. Then there is an APG $(T_A, \prec, p_A)$ and a decoration π of T_A such that $\pi(p_A) = A$*

Proof. Let $T_A = \{\langle a_i : i \leq n\rangle \subseteq TC(A \cup \{A\}) : a_{i+1} \in a_i \text{ and } a_0 = A\}$

Let $\langle a_i : i \leq n+1\rangle \prec \langle b_i : i \leq n\rangle$ if and only if $a_i = b_i$ for all $i \leq n$.

We then have a decoration $\pi : T_A \to TC(A)$ given by $\pi(\langle a_i : i \leq n \rangle) = a_n$ $\square$

So this tells us we are on the right track. Because even if we allow ill-founded sets we know that every set at least has some picture.

We are now ready to state the anti-foundation axiom **Axiom (AFA):** Every accessible pointed graph has a unique decoration. Examples: Lets give some examples of ill-founded sets

Define Ω

Draw picture. This is the simplest example of an ill-founded set.

$$\Omega = \{\Omega\}$$

Notice that even though this may look like an infinite object it is in fact hereditarily finite.

Now lets suppose a is a set and we want to find a set which satisfies

$$a^* = \{a, a*\}$$

Draw Graph, then draw labeled graph

There are two things to notice about this. First if we allow ourselves to use labeled graphs then we can express this graph in a much more concise way.

Second, this looks a little bit like an algebraic equation. We will get to this a little later.

Definition 1.2.2. A labeled graph is a quadruple (G, E, p, l) such that

- (G, E, p) is an APG
- $l : G \to SETs$

Definition 1.2.3. A labeled decoration of a labeled graph (G, E, p, l) is a function $\pi : G \to SETs$ such that

- $\pi(a) = \{\pi(b) : bEa\} \bigcup l(a)$

Definition 1.2.4 (Labeled Anti-Foundation Axiom)**.** Every labeled graph has a unique labeled decoration.

It is clear that the Labeled Anti-Foundation Axiom implies the AFA (with just the empty set of labels). We will show the other direction.

Class APG's

First though we need to widen our definition of APG.

Definition 1.2.5. A system is a triple (M, E, p) such that

- M is a class
- $E \subseteq M \times M$
- $p \in M$
- $(\forall a \in M)\ a_{child} = \{b \in M : E(b, a)\}$ is a set.

Intuitively it is just a class sized APG. We then have

Theorem 1.2.6. *Assuming AFA, every system has a unique decoration.*

Proof. Let M be a system. To each $a \in M$ we can associate an apg Ma in a way identical to how we formed Gx for a set APG G.

To see that Ma is a set let $X_0 = \{a\}$ and let $X_{n+1} = \bigcup_{y \in X_n} y_{child}$. Then $Ma = \bigcup_{n \in \omega} X_n$

So for each $a \in M$, there is a unique decoration d_a of Ma. Now if we let $d(a) = d_a(a)$ we want to show that d is a unique decoration of Ma.

First notice that d is a decoration because if $x \in a_{child}$ then $Mx \subseteq Ma$ and the restriction of d_a to Mx will be the unique decoration of Mx (i.e. d_x).

Hence if $x \in a_{child}$ then $d_a(x) = d_x(x) = d(x)$. So for each $a \in M$

$$d(a) = d_a(a) = \{d_a(x) : x \in a_{child}\} = \{d(x) : x \in a_{child}\}$$

So d is a decoration of M.

To see uniqueness it suffices to observe that any decoration must restrict to a decoration of Ma for all $a \in M$.

Hence because each such restriction is unique, the whole structure must be unique. □

We then also have

Theorem 1.2.7. *Every Labeled System (which is defined in the obvious way) has a unique decoration.*

Proof. Let M be a labeled system with labeling function l_M and let M' be the system having as nodes all ordered pairs (i, a) such that either

- $i = 1$ and $a \in M$
- $i = 2$ and $a \in V$

and having edges

- $(1, a) \to (1, b)$ whenever $E(a, b) \in M$.
- $(1, a) \to (2, b)$ whenever $a \in M$ and $b \in l_M(b)$
- $(2, a) \to (2, b)$ whenever $b \in a$

By AFA M' has a unique decoration π so

- For each $a \in M$

  $$\pi(1,a) = \{\pi(1,b) : E(a,b)\} \cup \{\pi(2,b) : b \in l_M(a)\}$$

- For each $a \in V$

$$\pi(2,a) = \{\pi(2,b) : b \in a\}$$

Now the assignment of $\pi(2,a)$ to a is a decoration of V and so by AFA we must have $\pi(2,a) = a$ for all $a \in V$.

So if we let $ra = \pi(1,a)$ for all $a \in M$ then for $a \in M$

$$ra = \{rb : E(b,a) \in M\} \bigcup l_M(a)$$

So r is a labeled decoration of the labeled system M.

Uniqueness

Suppose r, r' are labeled decorations of the labeled system M. Then π' is a decoration of the system M' where

$$\pi'(1, a) = r'a \text{ if } a \in M$$

$$\pi'(2, a) = a \text{ if } a \in V$$

It follows from AFA that $\pi' = \pi$ and so for all $a \in M$

$$r'a = \pi'(1, a) = \pi(1, a) = ra$$

so $r = r'$ □

We have one more important theorem before we can start talking about systems of equations.

Theorem 1.2.8. *Assuming AFA. Let M be a labeled system with labels l_M whose set of labels are subsets of a class X.*

- *If $\pi : X \to V$ then there is a unique map $\pi^* : M \to V$ such that for each $a \in M$*

  $$\pi^*(a) = \{\pi^*(b) : E(b, a) \textit{ in } M\} \cup \{\pi(x) : x \in l_M(a)\}$$

- *Given $a_x \in M$ for $x \in X$ there is a unique $\pi : X \to V$ such that for all $x \in X$*

$$\pi(x) = \pi^*(a_x)$$

Proof. (1) For each $\pi : X \to V$ let M_π be the labeled system obtained from M by redefining the set of labels so that for each node a

$$l_{M_\pi}(a) = \{\pi(x) : x \in l_M(a)\}$$

(2) Let M' be the system having the same nodes as M and all the edges of M together with edges $a \to a_x$ whenever $a \in M$ and $x \in l_M(a)$.

We know that M' has a unique decoration ϕ. So for each $a \in M$

$$\phi(a) = \{\phi(b) : a \to b \text{ in } M\} \cup \{\phi(a_x) : x \in l_M(a)\}$$

Let $\pi(x) = \phi(a_x)$ for $x \in X$. Then ϕ is a labeled decoration of the labeled systems M_π so that $\phi = \pi^*$ and hence $\pi(x) = \phi(a_x)$ for $x \in X$.

For uniqueness let $\pi' : X \to V$ such that $\pi'(X) = (\pi^*)'(a_x)$ for $x \in X$. Then observe that $(\pi^*)'$ is a decoration of M' so that $(\pi^*)' = \phi$ and hence $\pi'(x) = (\pi^*)'(a_x) = \phi(a_x) = \phi(x)$ for $x \in X$. So $\pi = \pi'$ □

1.3 Solution Lemma

One thing we noticed before was that one way to express a set had a very algebraic feel. It looked like our sets were unique solutions to systems of equations. Now we will make this precise.

One of the consequences of AFA (just like existensionality) is that there is only one set which has no elements.

While atoms don't add anything to the complexity of our models they do make expressing certain facts much more convenient. As such we will want to allow a class of atoms for the following discussion.

Definition 1.3.1. Suppose that X is a class of atoms. We say a set A is an X-set if $(\forall y \in TC(A))(\forall z, z \notin y) \rightarrow y = \emptyset \vee y \in X$

Definition 1.3.2. A set A is pure if it is a $\emptyset$-Set. We will denote the class of pure sets V_{pure}

We will now work in the theory

- ZFC + Proper class of atoms
- AFA is satisfied for V_{pure}

We now have the following two theorems

Theorem 1.3.3 (Substitution Lemma)**.** *For each family of pure sets $P = (b_x)_{x \in X}$ there is a unique function*

$\pi_P : V \to V_{pure}$ *such that*

- $\pi_P(x) = b_x$ *if* $x \in X$
- $\pi_P(a) = \{\pi_P(b) : b \in a\}$

Proof. This follows immediatly from Theorem 1.2.8 part (1). □

Theorem 1.3.4 (Solution Lemma)**.** *Let a_x be an X-set for every $x \in X$. Then there is a unique solution to the set of equations $x = a_x$.*

That is there is a unique family of pure sets $(b_x)_{x \in X}$ such that $b_x = \pi a_x$

Proof. This follows immediatly from Theorem 1.2.8 part (2). □

Lets go through an example Let

- $w = (4, 3, z, \{y\})$

- $y = \{z, w, \{y\}\}$
- $z = \{2^{\omega}, \{z\}, \{y\}\}$

Then we know by the solution lemma that (under AFA) there is a unique triple of sets which satisfy these equations.

2 The Axiom in More Detail

Before we continue we want to break AFA into two parts.

Definition 2.0.5. Let AFA_1 be "every graph has at least one decoration"

Let AFA_2 be "every graph has at most one decoration"

Now we see that the axiom of foundaiton (that every set is well founded) implies AFA_2 and the negatin of AFA_1.

2.1 Bisimulations

Now we want to ask the question "When do two graphs represent the same set?"

Definition 2.1.1. We say that $a \equiv b$ if there is a graph G which is a picture of both a and b.

Definition 2.1.2. Let M be a system. A binary relation $R \subseteq M \times M$ is a <u>Bisimulation</u> on M if $R \subseteq R^+$ where for $a, b \in M$

$$aR^+b \Leftrightarrow (\forall x \in a_{child})(\exists y \in b_{child})xRy \wedge (\forall y \in b_{child})(\exists x \in a_{child})xRy$$

Notice that if $R_0 \subseteq R$ then $R_0^+ \subseteq R^+$. So $()^+$ is monotone.

Theorem 2.1.3. $\equiv$ *is a bisimulation on* V

Proof. Let $(G, \to, *)$ be an APG which is a picture of a, b (so $a \equiv b$). Let d_a, d_b be decorations of G by a, b respectively.

Now for each $\alpha \in a$ there is an $x \in *_{child}$ such that $d_a(\alpha) = x$.

There is then a $\beta \in b$ such that $d_b(x) = \beta$. So, Gx is a picture of α and β and hence $\alpha \equiv \beta$. We have therefore shown

$$a \equiv b \to (\forall \alpha \in a)(\exists \beta \in b)\alpha \equiv \beta$$

The other direction is identical.

So $a \equiv^+ b$ and $\equiv \subseteq \equiv^+$ and $\equiv$ is a bisimulation. □

In general any system M will have many bisimulations.

Theorem 2.1.4. *Every system M has a unique max-*

ium bisimulation $\equiv_M$ *i.e.*

(1) $\equiv_M$ *is a bisimulaiton on* M

(2) If R *is a bisimulation on* M *then for all* $a, b \in M$

$$aRb \Rightarrow a \equiv_M b$$

In fact $a \equiv_M b \Leftrightarrow (\exists R)aRb$ *and* R *is a small bisimulation.*

Proof. Let $\equiv_M$ be defined as above.

For (1) let $a \equiv_M b$. Then aRb for some small bisimulation R on M. So trivially

$$xRy \Rightarrow x \equiv_M y \text{ for all } x, y \in M$$

So by monoticity of $()^+$ we have

$$xR^+y \Rightarrow x \equiv_M^+ y \text{ for all } x, y \in M$$

But as aRb and $R \subseteq R^+$ it follows that $a \equiv^+_M b$

For (2) let R be a bisimulation on M and let aRb. It isn't hard to see that if $R_0 = R \cap (Ma) \times (Mb)$ then aR_0b and R_0 is a bisimulation. Further, as Ma, Mb are small so is R_0 and so $a \equiv_M b$. □

Theorem 2.1.5. *For all sets a, b*

$$a \equiv b \Leftrightarrow a \equiv_V b$$

Proof. As $\equiv$ is a bisimulation on V and $\equiv_V$ is the maximum bisimulation the implication from left to right holds

For the converse it suffices to show that if R is a bisimulation on V then for all sets a, b

$$aRb \to a \equiv b$$

Let R be such a bisimulation on V. Define the sytstem M_0 as follows.

The nodes of M_0 are the elements of R, i.e. ordered pairs (a, b) such that aRb. The edges of $M_0(a, b)$ are defined so that

$$(a, b) \to (x, y) \text{ in } M_0(a, b) \Leftrightarrow x \in a \wedge y \in b$$

Now observe that d_1, d_2 are both decorations of M_0 where

$$d_1(a, b) = a$$

$$d_2(a, b) = b$$

So if aRb then the APG $M_0(a, b)$ is a picture of both a and b using the restrictions of d_1, d_2. So if aRb then $a \equiv b$. □

In general we have

Theorem 2.1.6. *For each system M $\equiv_M$ is an equivalence relation such that*

$$(\forall a, b \in M) a \equiv_M b \Leftrightarrow a \equiv^+_M b$$

Proof. ****

This is 2.7

Maybe waive my hands at it

**** □

Definition 2.1.7. A system M is Extensional if for all $a, b \in M$

$$a_{child} = b_{child} \rightarrow a = b$$

It is Strongly Extensional if for all $a, b \in M$

$$a \equiv_M b \rightarrow a = b$$

The connection between AFA and the axiom of existensionality is

Theorem 2.1.8. *$AFA_2 \Leftrightarrow V$ is strongly extensional.*

Proof. ⇒Assume AFA_2.

We know that if $a \equiv_V b$ then by a previous theorem we have $a \equiv b$. So there is an APG G and decorations d_1 of G by a and d_2 of G by b. But then by AFA_2 $d_1 = d_2$ and so $a = b$. Hence V is strongly extensional.

$\underline{\Leftarrow}$Assume V is strongly extensional.

Let d_1, d_2 be decorations of a graph G. If $x \in G$ then $d_1(x) \equiv d_2(x)$ as Gx is a picture of both $d_1(x)$ and $d_2(x)$. Hence $d_1(x) \equiv_V d_2(x)$ and so $d_1(x) = d_2(x)$ as V is strongly extensional. So $d_1 = d_2$ and we have AFA_2. $\square$

2.2 System Maps

Definition 2.2.1. A System Map $\pi : M \to M'$ is a map such that

$$(\forall a \in M)(\pi a)_{child} = \{\pi b : b \in a_{child}\}$$

For example, a system map $G \to V$ is simply a decoration of G.

Definition 2.2.2. Let $\pi : M \to M'$ be the quotient of M with respect to an equivalence relation. i.e.

$$aRb \Leftrightarrow \pi a = \pi b$$

where π is surjective.

Then π can be made into a system map if we put a graph structure on M' given by $(\pi a, \pi b)$ is an edge in M' if and only if (a, b) is an edge in M.

We say that π is the Quotient of the system M with respect to R.

Theorem 2.2.3. *Let $\pi_1, \pi_2 : M \to M'$ be system maps.*

(1) If R is a bisimulation on M then

$$(\pi_1 \times \pi_2)R = \{(\pi_1 a_1, \pi_2 a_2) : a_1 R a_2\}$$

Is a bisimulation on M'.

(2) If S is a bisimulation on M' then

$$(\pi_1 \times \pi_2)^{-1} S = \{(a_1, a_2) \in M \times M : \pi_1 a_1 S \pi_2 a_2\}$$

is a bisimulation on M

Proof. <u>(1)</u>

Let $S = (\pi_1 \times \pi_2)R$ and let $b_1 S b_2$ and $b_1' \in (b_1)_{child}$.

Then there are a_1, a_2 such that

- $a_1 R a_2$
- $b_1 = \pi_1 a_1$ and $b_2 = \pi_2 a_2$

As $b_1' \in (\pi_1 a_1)_{child}$ there is an $a_1' \in (a_1)_{child}$ such that $b_1' = \pi a_1'$

Now as R is a bisimulation there is a $a_2' \in (a_2)_{child}$ such that $a_1' R a_2'$. Now if we let $b_2' = \pi_2 a_2'$ then we have

- $b'_1 S b'_2$
- $b'_2 \in (b_2)_{child}$

So we have proved

$$(\forall b'_1 \in (b_1)_{child})(\exists b'_2 \in (b_2)_{child}) b'_1 S b'_2$$

We have the other direction is similar and so S is a bisimulation.

<u>(2)</u>

Let $R = (\pi_1 \times \pi_2)^{-1} S$ and let $a_1 R a_2$ and $a'_1 \in a_{child}$. Then $(\pi_1 a_1) S (\pi_2 a_2)$ and $\pi_1 a'_1 \in (\pi_1 a_1)_{child}$. As S is a bisimulation on M' there is $b'_2 \in (\pi_2 a_2)_{child}$ such that $(\pi_1 a'_1) S b'_2$. So $b'_2 = \pi_2 a'_2$ for some $a'_2 \in (a_2)_{child}$. So $a'_1 R a'_2$ for some $a'_2 \in (a_2)_{child}$. Thus if $a_1 R a_2$ then

$$(\forall a'_1 \in (a_1)_{child})(\exists a'_2 \in (a_2)_{child}) a'_1 R a'_2$$

and similarly

$$(\forall a'_2 \in (a_2)_{child})(\exists a'_1 \in (a_1)_{child}) a'_1 R a'_2$$

□

Definition 2.2.4. If $\pi : M \to M'$ is the quotient of the system M with respect to $\equiv_M$ then we say it is the Strongly extensional quotient of M

Theorem 2.2.5. *Every system M has a strongly extensional quotient.*

Proof. This is strait forward. □

We then have

Theorem 2.2.6. *The following are equivalent*

(1) M is strongly extensional

(2) For each (small) system M_0 there is at most one system map $M_0 \to M$

(3) For each system M' every system map $M \to M'$ is injective.

Proof. First we show (1) $\Leftrightarrow$ (2).

Assume (1)

Suppose $\pi_1, \pi_2 : M_0 \to M$ are system maps. By a previous theorem we have $(\pi_1 \times \pi_2)(=_{M_0})$ is a bisimulation R on M. If $m \in M_0$ then $(\pi_1 m)R(\pi_2 m)$ and so $\pi_1 m \equiv_M \pi_2 m$. Hence $\pi_1 m = \pi_2 m$ by strong extensionality. Thus $\pi_1 = \pi_2$. So we have now proved (2)

Assume (2)

Let M_0 be the system whose nodes are ordered pairs (a, b) such that $a \equiv_M b$ and whose edges are

$$(a, b) \to (a', b') \Leftrightarrow a \to a' \wedge b \to b'$$

Now let $\pi_1, \pi_2 : M_0 \to M$ are defined by

$$\pi_1(a, b) = a$$

$$\pi_2(a, b) = b$$

It isn't hard to show that π_1, π_2 are system maps.

Clear enough?

We therefore have by (2) that $\pi_1 = \pi_2$. Now whenever $a \equiv_M b$ then $(a, b) \in M_0$ and so

$$a = \pi_1(a, b) = \pi_2(a, b) = b$$

So M is strongly extensional.

We now show (1) $\Leftrightarrow$ (3)

Assume (1).
Let $\pi : M \longrightarrow M'$ be a system map. By a previous theorem we have $(\pi \times \pi)^{-1}(=_{M'})$ is a bisimulation R on M.

Hence if $\pi(a) = \pi(b)$ then aRb and hence also $a \equiv_M b$.

But then $a = b$ as M is strongly extensional and so π is injective. And we have (3)

Assume (3).
Let $\pi : M \to M'$ be a strongly extensional quotient of M.

We know that π is a surjective system map by definition, and by (3) we know that it is also injective. Hence π is an isomorphism $M \cong M'$.

But we know that M' is strongly extensional (by the definition of strongly extensional) and so M must be as well. So we have proved (1). □

Only do if time

Theorem 2.2.7. *Let M be a system such that any* *two nodes of M lie in a common apg of the form Mc. Then M is strongly extensional iff Mc is strongly extensional for ever node c.*

Proof. **** □

2.3 Exact Pictures

ONLY DO IF TIME

3 A Model of the Axiom

3.1 Complete Systems

Definition 3.1.1. Let M be a system. An *M-Decoration* of a graph G is a system map $G \to M$

So for example V-decoration is just a decoration.

Definition 3.1.2. A system M is *Complete* if every graph has a unique M-Decoration.

Lemma 3.1.3. *Every complete system is strongly extensional.*

Proof. By previous results. □

Notice that AFA holds if and only if V is complete.

Constructing a model of ZFC + AFA

Every APG is of the form Ga where a is some node in G.

Let V_0 be the system whose nodes are APG's with an edge between Ga, Gb if and only if there is an edge between a, b in G.

Let $\pi_c : V_0 \to V_c$ be a strongly extensional quotient of V_0.

Theorem 3.1.4. *For each system M there is a unique system map $M \to V_c$*

Proof. If $a \in M$ then $Ma \in V_0$. Further the map

$M \to V_0$ which takes $a \to Ma$ is clearly a system map. Composing with π_c gets us a system map $M \to V_c$.

The uniqueness of the system map follows from the strong extensionality of V_c. □

Corollary 3.1.5. *V_c is complete.*

Proof. Apply the theorem to all small systems. □

Theorem 3.1.6. *The following are equivalent for a system M*

(1) For each system M' there is a unique system map $M' \to M$

(2) M is complete

(3) $M \cong V_c$.

Proof. That (3) implies (1) is a consequence of the previous theorem.

That (1) implies (2) is trivial. So it suffices to show (2) implies (3).

Let M be a complete system. Let $\pi : M \to V_c$ be the unique system map which exists by the previous theorem. We therefore have π is injective as M is strongly extensional.

If $a \in V_c$ then $V_c a$ is an APG with a unique M-decoration d say. Then $\pi \circ d : V_c a \to V_c$ is a system map.

As V_c is strongly extensional $\pi \circ d$ must be the identity map on $V_c a$. In particular $a = \pi(da)$. Thus π is surjective as well as injective.

So $\pi : M \cong V_c$. □

3.2 Full Systems

Definition 3.2.1. A system M is full if for every $x \subseteq M$ there is a unique $a \in M$ such that $x = a_{child}$

Examples:

(1) V is a full system. Or when M is a class such that $M = pow(M)$. For example $M = V_{well-founded}$

(2) If $\pi : M \rightarrow M$ is any bijection on a full sstem then we get a new full system M_π having the same nodes as M but where

$$a \rightarrow b \in M_\pi \Leftrightarrow \pi a \rightarrow b \in M$$

Theorem 3.2.2. *Each complete system is full*

Proof. Let $x \subseteq M$ be a set where M is a complete system. Let G_0 be the graph such that

- Nodes are any node that is connected to an element of G_0.

- Edges are any edge in M between nodes in G_0.

Let G be obtained by adding a new node $*$ and a new edge $(*, y)$ for each $y \in x$.

Draw Picture

As M is complete, G has a unique decoration d. Restricting d to G_0 we obtain M decorations of every element of x.

But the identity map is clearly the unique M decoration of G_0 so $dx = x$ for all $x \in G_0$. Hence if $a = d*$ then $a \in M$ such that

$$a_{child} = \{dy : * \to y \text{ in } G\} = x$$

Uniqueness

Now suppose $a' \in M$ such that $a'_M = x$. Then we get an M-decoration d' of G with $d'* = a'$ and $d'y = y$ for all

$y \in G_0$. As d is the unique M-decoration of G, $d = d'$ so that

$$a' = d'* = d* = a$$

So we have shown that there is a unique $a \in M$ such that $a_{child} = x$ □

3.3 Interpreting AFA

Any system M determines an interpretation of the language of set theory in which variables range over nodes of M and the predicate $\in$ is interpreted $\in_M$ where for $a, b \in M$

$$a \in_M b \Leftrightarrow a \in b_{child}$$

Theorem 3.3.1. *If M is full then it satisfies all the axioms of ZFC^-*

Proof. **Do I want to show?** □

Theorem 3.3.2. *Each complete system is a model of AFA*

Proof. If x is a subset of M then let x^M be the unique $a \in M$ such that $x = a_M$. For $a, b \in M$ let

$$(a, b)^M = \{\{a\}^M, \{a, b\}^M\}^M$$

So $(a, b)^M$ is the element of M which is the standard set theoretic representation in M of the ordered pair of a and b.

Here we represent a graph as an ordered pair consisting of a set and a binary relation on it.

So for $c \in M$ $M \models$"c is a graph" if and only if there are $a, b \in M$ such that

$$c = (a, b)^M$$

$$M \models b_{child} \subseteq \{(x, y)^M : x, y \in a_{child}\}$$

i.e. $M \models$"b is a binary relation on a".

Hence for such a $c \in M$ we may define a graph G having as nodes the elements of a_{child} and having as edges the pairs (x, y) such that $(x, y)^M \in b_{child}$.

As M is complete G has a unique M-decoration. This must be the unique map $d : a_{child} \to M$ such that for all $x \in a_{child}$

$$dx = \{dy : (x, y)^M \in b_{child}\}$$

Now let $f = \{(x, dx)^M : x \in a_{child}\}^M$. Then $f \in M$ and it is easy to check

$$M \models \text{“f is the unique decoration of the graph c”}$$

Thus we have proved that in M every graph has a unique decoration. i.e. M is a model of AFA. $\square$