Lecture Notes on Barwise Compactness at Logic Seminar (Fall 2007)

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TALK SLOWLY AND WRITE NEATLY!!

1 Admissible Sets

Definition 1.0.1. Recall that in the language of set theory we have

- $(\forall x \in y)\varphi \Leftrightarrow (\forall x)(x \in y \rightarrow \varphi)$
- $(\exists x \in y)\varphi \Leftrightarrow (\exists x)(x \in y \wedge \varphi)$

A Δ_0-formula is one which is built up from atomic formulas and their negations using only

$$\wedge, \vee, (\forall x \in y), (\exists x \in y)$$

A Σ-formula is one which is built up from atomic formulas and their negations using only

$$(\exists x), \wedge, \vee, (\forall x \in y), (\exists x \in y)$$

A Π-formula is one which is built up from atomic formulas and their negations using only

$$(\forall x), \wedge, \vee, (\forall x \in y), (\exists x \in y)$$

(here $\wedge, \vee$ are finite).

Easy lemmas

Lemma 1.0.2.

- *The negation of a Δ_0 formula is logically equivalent to a Δ_0 formula.*
- *The negation of a Π formula is logically equivalent to a Σ formula*
- *The negation of a Σ formula is logically equivalent to a Π formula.*

Proof. Immediate □

Lemma 1.0.3. *Let a, b be transitive sets with $a \subset b$ and let $c_1, \ldots, c_n \in a$.*

- *If* $\varphi(x_1, \ldots, x_n$ *is a* Δ_0*-formula then*

$$\langle a, \in \rangle \models \varphi(c_1, \ldots, c_n) \Leftrightarrow \langle b, \in \rangle \models \varphi(c_1, \ldots, c_n)$$

- *If* $\varphi(x_1, \ldots, x_n$ *is a* Σ*-formula then*

$$\langle a, \in \rangle \models \varphi(c_1, \ldots, c_n) \Rightarrow \langle b, \in \rangle \models \varphi(c_1, \ldots, c_n)$$

- *If* $\varphi(x_1, \ldots, x_n$ *is a* Σ*-formula then*

$$\langle b, \in \rangle \models \varphi(c_1, \ldots, c_n) \Rightarrow \langle a, \in \rangle \models \varphi(c_1, \ldots, c_n)$$

Proof. Immediate □

Definition 1.0.4. $\mathcal{A}$ is an *admissible set* if

- $\mathcal{A}$ is non-empty
- $\mathcal{A}$ is transitive
- If $x \in \mathcal{A}$ then the transitive closure of x, $TC(x)$, is in $\mathcal{A}$.

- (Δ_0-Separation) If $\varphi(x, y_1, \ldots, y_n)$ is a Δ_0 formula and $b_1, \ldots, b_n, c \in \mathcal{A}$ then

$$\{a \in c : \langle \mathcal{A}, \in \rangle \models \varphi(a, b_1, \ldots, b_n)\} \in \mathcal{A}$$

- (Σ-Reflection) If $\varphi(y_1, \ldots, y_n)$ is a Σ-formula, $b_1, \ldots, b_n \in \mathcal{A}$ and

$$\langle \mathcal{A}, \in \rangle \models \varphi(b_1, \ldots, b_n)$$

then there is a transitive set $a \in \mathcal{A}$ such that

$$b_1, \ldots, b_n \in a \text{ and } \langle a, \in \rangle \models \varphi(b_1, \ldots, b_n)$$

Definition 1.0.5. A set $X \subseteq \mathcal{A}$ is Σ on $\mathcal{A}$ if and only if there is a Σ-formula $\varphi(x, y_1, \ldots, y_n)$ and elements $b_1, \ldots, b_n \in \mathcal{A}$ such that

$$X = \{a \in \mathcal{A} : \langle A, \in \rangle \models \varphi(a, b_1, \ldots, b_n)\}$$

We call $\varphi(x, b_1, \ldots, b_n)$ is a Σ-definition of X.

The notion of X being Π on $\mathcal{A}$ is defined analogously.

Example of Admissible Set

As an example we have that V_ω = all hereditarily finite sets is an admissible set and for any $X \subseteq V_\omega$

- $X \in V_\omega$ if and only if X is finite
- X is Δ on V_ω if and only if X is computable.
- X is Σ on V_ω if and only if X is computably enumerable.

This (along with a lot of much deeper work) leads to the following definitions.

Definition 1.0.6. Let $\mathcal{A}$ be an admissible set and let $X \subseteq \mathcal{A}$.

- X is $\mathcal{A}$-finite if and only if $X \in \mathcal{A}$
- X is $\mathcal{A}$-computable if and only if X is Δ on $\mathcal{A}$.

- X is $\mathcal{A}$-computably enumerable if and only if X is Σ on $\mathcal{A}$.

Example of Admissible Set

If $\mathcal{A}$ is a transitive set such that $\mathcal{A} \models ZF - Powerset$ then $\mathcal{A}$ is an admissible set.

In particular $H_\kappa = \{x : |TC(x)| < \kappa\}$ is admissible.

Definition 1.0.7. Suppose $\varphi(x_1, \ldots, x_n)$ is a formula where y does not occur. The *relativization of* φ *to* y $(\varphi^y(x_1, \ldots, x_n))$ is defined to be the formula obtained by replacing each quantifier $(\forall z), (\exists z)$ in φ by the bounded quantifier $(\forall z \in y), (\exists z \in y)$.

So for any formula of set theory φ, φ^y is a Δ_0 formula which has as free variables those free variables of φ with the extra variable y. We write

$$\langle \mathcal{A}, \in \rangle \models \varphi^a(b_1, \ldots, b_n) \text{ for } \langle \mathcal{A}, \in \rangle \models \varphi^y(a, b_1, \ldots, b_n)$$

Notice that for $a \in \mathcal{A}$ with $b_1, \ldots, b_n \in a$ we have

$$\langle a, \in \rangle \models \varphi(b_1, \ldots, b_n) \text{ if and only if } \langle \mathcal{A}, \in \rangle \models \varphi^a(b_1, \ldots, b_n)$$

Lemma 1.0.8. *Let $\mathcal{A}$ be an admissible set.*

- *If $a, b \in \mathcal{A}$ then $a \times b, a \cap b, \langle a, b \rangle, a - b$ and $\{a, b\} \in \mathcal{A}$. And, if $f \in \mathcal{A}$ then $dom(f), range(f) \in \mathcal{A}$.*
- *(Δ-Separation) If $a \in \mathcal{A}$, $X \subseteq \boldsymbol{a}$ is Δ on $\mathcal{A}$ then $X \in \mathcal{A}$.*
- *(Σ-Replacement) If*
  - *G is a function*
  - *$graph(G) \subseteq \mathcal{A} \times \mathcal{A}$*
  - *G is Σ on $\mathcal{A}$*
  - *$a \in \mathcal{A}$ and $a \subseteq dom(G)$*

then $graph(G|a) \in \mathcal{A}$*.*

- *(Definition by* Σ*-Recursion) Suppose*
  - $G(x, y)$ *be a function*
  - $(\forall a, b \in \mathcal{A}) G(a, b) \in \mathcal{A}$
  - $graph(G) \cap \mathcal{A}$ *is* Σ *on* $\mathcal{A}$

  and suppose
  - $F(x)$ *is a function with domain* $\mathcal{A}$
  - $(\forall a \in \mathcal{A}) F(a) = G(a, graph(F|TC(a)))$ *(we say* F *is defined recursively by* G*)*

  Then $graph(F)$ *is* Δ *on* $\mathcal{A}$ *and* F *maps* $\mathcal{A}$ *into* $\mathcal{A}$

Proof. Part (i):
Let $a, b \in \mathcal{A}$. By Σ-reflection there is a $c \in \mathcal{A}$ such that $a, b \in c$. By Δ_0-separation $\{a, b\} \in \mathcal{A}$ and hence $\{\{a\}, \{a, b\}\} = \langle a, b \rangle \in \mathcal{A}$.

To see $a \times b \in \mathcal{A}$ notice

$$\langle \mathcal{A}, \in \rangle \models (\forall x \in a)(\forall y \in b)(\exists z)(z = \langle x, y \rangle)$$

Now this is a Σ formula so by Σ reflection there is a $c \in \mathcal{A}$ such that $a, b \in c$ and

$$\langle c, \in \rangle \models (\forall x \in a)(\forall y \in b)(\exists z)(z = \langle x, y \rangle)$$

So $a \times b \subset c$. But

$$a \times b = \{u \in c : (\exists x \in a)(\exists y \in b)(u = \langle x, y \rangle)\}$$

The rest follow by Δ_0 separation.

Part (ii):

Let $\varphi(x, b_1, \ldots, b_n)$ and $\psi(x, b_1, \ldots, b_n)$ be Σ and Π definitions of X. So for all $x \in X$

$$\langle \mathcal{A}, \in \rangle \models \varphi(x, b_1, \ldots, b_n)$$

hence by Σ-reflection $x \in X$ implies

$$\langle \mathcal{A}, \in \rangle \models (\exists y)(x, b_1, \ldots, b_n \in y \wedge y \text{ is transitive } \wedge \varphi^y(x, b_1, \ldots, b_n))$$

Call this formula $\overline{\varphi}(x)$

Similarly $x \notin X$ implies

$\langle \mathcal{A}, \in \rangle \models (\exists y)(x, b_1, \ldots, b_n \in y \wedge y$ is transitive $\wedge \neg \psi^y(x, b_1, \ldots, b_n))$

Call this formula $\overline{\psi}(x)$

Now, since $X \subseteq a$ we have

$$\langle \mathcal{A}, \in \rangle \models (\forall x \in a)\overline{\varphi} \vee \overline{\psi}(x)$$

So in particular there is a $d \in \mathcal{A}$ such that $a, b_1, \ldots, b_n \in d$, d is transitive and

$$\langle d, \in \rangle \models (\forall x \in a)\overline{\varphi} \vee \overline{\psi}(x)$$

We claim that

$$x \in X \Leftrightarrow \langle \mathcal{A}, \in \rangle \models x \in a \wedge \overline{\varphi}^d(x)$$

This is because if $\langle \mathcal{A}, \in \rangle \models x \in a \wedge \overline{\varphi}^d(x)$ then because

$\overline{\varphi}$ is Σ, $\langle \mathcal{A}, \in \rangle \models x \in a \wedge \overline{\varphi}(x)$ and hence

$$\langle \mathcal{A}, \in \rangle \models \varphi(x, b_1, \ldots, b_n)$$

and so $x \in X$

Now if $\langle \mathcal{A}, \in \rangle \not\models x \in a \wedge \overline{\varphi}^d(x)$ then either

- $x \notin a$ and hence $x \notin X$
- $\langle \mathcal{A}, \in \rangle \not\models \overline{\varphi}^d(x)$. In this case $\langle \mathcal{A}, \in \rangle \models \overline{\psi}^d(x)$. But $\overline{\psi}$ is also a Σ formula so $\langle \mathcal{A}, \in \rangle \models \neg\varphi(x)$ and hence $x \notin X$.

Part (iii):

First notice that every $R \subset \mathcal{A} \times \mathcal{A}$ is also a subset of $\mathcal{A}$. Further it isn't hard to see that R is Σ if and only if there is a formula $\varphi(x, y, b_1, \ldots, b_n)$ and $b_1, \cdots, b_n \in \mathcal{A}$ such that $R = \{\langle a, b \rangle : \langle \mathcal{A}, \in \rangle \models \varphi(a, b, b_1, \ldots, b_n)\}$. We call such a φ a Σ definition of R.

Now let $\varphi(x, y, b_1, \ldots, b_n)$ be a Σ definition of G. Then $G|a$ has the Σ definition

$$x \in a \wedge \varphi(x, y, b_1, \ldots, x_n)$$

So $G|a$ is Σ on A. We have, since $a \subset \mathrm{dom}(G)$

$$\langle \mathcal{A}, \in \rangle \models (\forall x \in a)(\exists y)\varphi(x, y, b_1, \ldots, b_n)$$

So by Σ-reflection there exists $c \in \mathcal{A}$ such that c is transitive $a, b_1, \ldots, b_n \in c$ and

$$\langle c, \in \rangle \models (\forall x \in a)(\exists y)\varphi(x, y, b_1, \ldots, b_n)$$

So $graph(G|a) \subseteq a \times c \in \mathcal{A}$. Finally $G|a$ has the Π-definition

$$\psi(x, y, b_1, \ldots, b_n) \Leftrightarrow x \in a \wedge (\forall z \in c)(\varphi(x, y, b_1, \ldots, b_n) \rightarrow z = y)$$

Hence by Δ separation $G|a \in \mathcal{A}$

Part (iv):

First notice the relation $u \in TC(x)$ has the Σ definition

$$u \in TC(x) \Leftrightarrow x \neq u \wedge (\exists v)[x \in v \wedge (\forall y \in v)(y \neq u \rightarrow (\exists z)(z \in y \cap v))$$

Hence $w = TC(x)$ has the Σ definition

$$w = TC(x) \Leftrightarrow x \subset w \wedge (w \text{ is transitive }) \wedge (\forall u \in w) u \in TC(x)$$

Now let $\varphi(x, y, z, b_1, \ldots, b_n)$ be a Σ definition of the binary function $G|\mathcal{A}$. Let F' be the subset of $\mathcal{A}$ which has the Σ-definition

$$\psi(x, z, b_1, \ldots, b_n) \Leftrightarrow (\exists w)(\exists f)[w = TC(x) \wedge f \text{ is a function } \wedge$$

$$\text{dom}(f) = w \wedge \varphi(x, f, z, b_1, \ldots, b_n) \wedge (\forall r \in w)(\exists s)(s = TC(r) \wedge \varphi(r, f|s,$$

Here $\text{dom}(f)$, $f|s$, $f(r)$ can be eliminated by introducing the appropriate bounded quantifiers.

We show that for all ordinals α and all $a \in \mathcal{A} \cap V_\alpha$ $F'|a = F|a$ and $F'|a \in \mathcal{A}$.

Case $\alpha = 0$:

This is trivial

Case α is a limit:

This is also immediate as any set in $\mathcal{A} \cap V_\alpha$ must already have been dealt with.

Case $\alpha = \beta + 1$ where the conditions hold for β:

Let $a \in V_{\beta+1} \cap \mathcal{A}$, $x \in a$. Let $w = TC(x)$. Then $w \in \mathcal{A} \cap V_\beta$ and by hypothesis $F|w = F'|w \in \mathcal{A}$.

Let $f = F|w$. Then $F(x) = G(x, f) \in \mathcal{A}$. It is then clear from the definition that $\langle x, F(x) \rangle$ belongs to the relation F'. Moreover, w, f are the only sets which satisfy the part of ψ within the square brackets. Hence there is no $\langle x, z \rangle \in F'$ with $z \neq F(x)$. Therefore $F'|a = F|a$.

But $F'|a$ has the Σ definition $\psi \wedge x \in a$. Further, since F is a total function and $F'|a = F|a$ we have $a \subseteq \mathrm{dom}(F'|a)$. Hence, by Σ replacement we have $F|a \in \mathcal{A}$.

So F maps $\mathcal{A}$ into $\mathcal{A}$.

So $\psi(x, z, b_1, \ldots, b_n)$ is a Σ definition of F. But we also have

$$\eta(x, z, b_1, \ldots, b_n) \Leftrightarrow (\exists y)\psi(x, y, b_1, \ldots, b_n) \wedge y \neq z$$

is a Σ-definition of $\mathcal{A} - F$. Hence F is Δ on $\mathcal{A}$. □

Definition 1.0.9. We say a set x is *hereditarily countable* $x \in HC$ if $TC(x)$ is countable.

Lemma 1.0.10. *Let $\mathcal{A}$ be an admissible set such that $\mathcal{A} \subseteq HC$. Then the following are Δ on $\mathcal{A}$*

- $ORD \cap \mathcal{A}$
- $L_{\mathcal{A}} = \mathcal{L}_{\omega_1,\omega} \cap \mathcal{A}$

- $Ax(L_{\mathcal{A}}) = \{\varphi \in L_{\mathcal{A}} : \varphi$ *is an axiom of* $\mathcal{L}_{\omega_1,\omega}\}$
- *The set of atomic formulas of* $L_{\mathcal{A}}$
- *The relation* $\varphi \in L_{\mathcal{A}}$ *and* v_γ *is a free variable in* φ.

Proof. ****IF TIME**** □

2 Model Existence Theorem

Now we will recall the Model Existence Theorem. The Model Existence Theorem captures the essence of a Henkin construction of a countable model in the context of $\mathcal{L}_{\omega_1,\omega}$.

2.1 Model Existence Theorem

The theorem is incredibly useful and with it we can prove many different results. First we need some convenient notation.

Definition 2.1.1. Let $\varphi \in \mathcal{L}_{\omega_1,\omega}(\sigma)$. Then we define

$(\sigma\neg)$ as follows

- $\varphi\neg = \neg\varphi$ if φ is atomic.
- $\neg\varphi\neg = \varphi$.
- $(\bigvee_{i\in I} \varphi_i)\neg = \bigwedge_{i\in I}(\varphi_i\neg)$
- $(\bigwedge_{i\in I} \varphi_i)\neg = \bigvee_{i\in I}(\varphi_i\neg)$
- $(\exists x\varphi)\neg = (\forall x)(\varphi\neg)$
- $(\forall x\varphi)\neg = (\exists x)(\varphi\neg)$

So $\neg\varphi$ is always logically equivalent to $\varphi \neq$

Definition 2.1.2. Let σ be a countable signature and let C be a countable set of new constant symbols with $\tau = \sigma \cup C$. A set of countable collections of sentences S is a <u>Consistence Property</u> if and only if for all $s \in S$ the following holds

- Either $\varphi \notin s$ or $(\varphi\neg) \notin s$

- If $(\neg\varphi) \in s$ then $s \cup \{\varphi\neg\} \in S$
- If $(\bigwedge_i \varphi_i) \in s$ then for each i $s \cup \{\varphi_i\} \in S$.
- If $(\forall x \varphi(x)) \in s$ then for all $c \in C$, $s \cup \{\varphi(c)\} \in S$
- If $\bigvee_i \varphi_i \in s$ then for some i $s \cup \{\varphi_i\} \in S$
- If $(\exists x)\varphi(x) \in s$ then for some $c \in C$ $s \cup \{\varphi(c)\} \in S$
- If $(c = d) \in s$ then $s \cup \{d = c\} \in S$.

  If $c = d, \varphi(c) \in s$ then $s \cup \{\varphi(d)\} \in s$

Theorem 2.1.3 (Model Existence Theorem)**.** *If S is a consistency property and $s_0 \in S$ then there is a countable model of s_0.*

Proof. We proved this in the talks on Vaught's Conjecture last semester. □

3 Barwise Compactness

Before we prove Barwise's compactness theorem we will need one result.

Theorem 3.0.4. *Let $\mathcal{A} \subset HC$ be a countable admissible set. Then there is a Σ formula $P(y)$ such that*

$$\varphi \in L_{\mathcal{A}} \wedge \models \varphi \Leftrightarrow (\exists y)\varphi \in P(y)$$

Recall $\models \varphi \Leftrightarrow (\forall M)M \models \varphi$

Proof. We will not prove this right now as there is not enough time. But we may prove this after the break. □

We now prove compactness of Barwise compactness

Theorem 3.0.5 (Barwise Compactness Theorem)**.** *Let $\mathcal{A} \subset HC$ be a countable admissible set. Suppose*

- *$X \subset L_{\mathcal{A}}$ is a set of sentences*

- *X is Σ on $\mathcal{A}$*

- *Every $x \subset X$ such that $x \in \mathcal{A}$ has a model.*

Then X has a model.

So Barwise compactness says "If X is an $\mathcal{A}$-c.e collection of sentences such that every $\mathcal{A}$-finite subset has a model then X has a model"

Proof. Let M be the language $L \cup C$ where $C = \{c_i : i \in \omega\}$ where each c_i is a constant.

Let S be the set of all $s \subset M_{\mathcal{A}}$ such that

- s is a set of sentences of $M_{\mathcal{A}}$

- s is Σ on $\mathcal{A}$

- Every $x \subset s$ such that $x \in \mathcal{A}$ has a model

- Only finitely many elements of C occur in s.

We claim s is a consistency property.

The only difficult part is the condition:

If $\bigwedge \Theta \in s \in S$ then for some $\theta \in \Theta, s \cup \{\theta\} \in S$

Suppose $\bigwedge \Theta \in s \in S$ but $s \cup \{\theta\} \notin S$ for all $\theta \in \Theta$. Then for all $\theta \in \Theta$ there is an $x \in \mathcal{A}$ such that $x \subseteq s$ and $x \cup \{\theta\}$ has no model. Therefore

$$\langle \mathcal{A}, \in \rangle \models (\forall \theta \in \Theta)(\exists x \subset s \wedge (\exists y) \text{``}(\bigwedge x \to \neg\theta)'' \in P(y))$$

This formula is a Σ-formula because $x \subset s$ and $P(y)$ are. So there is a transitive $a \in A$ such that all constants occurring in s belong to a and the above formula holds in $\langle a, \in \rangle$.

So

$$(1)\langle\mathcal{A},\in\rangle\models(\forall\theta\in\Theta)(\exists x\in a)[(x\subset s)^a\wedge(\exists y)``(\bigwedge x\to\neg\theta)''\in P(y))$$

Let

$$b=\{u\in a:\langle a,\in\rangle\models u\in s\}$$

Then $b\subset c$ and b is defined by the Δ_0-formula

$$u\in a\wedge(u\in s)^a$$

in $\langle\mathcal{A},\in\rangle$. Hence by Δ_0 separation $b\in\mathcal{A}$.

Also, since s is Σ on $\mathcal{A}$ we have $b\subset s$.

Finally if $\langle\mathcal{A},\in\rangle\models x\in a\wedge(x\subset s)^a$ then $x\subset b$. It follows from (1) that for all $\theta\in\Theta$ there is an $x\subset b$ with $(\exists y)``\bigwedge x\to\neg\theta''\in P(y)$ and so $\models\bigwedge x\to\neg\theta$ and $\models\bigwedge b\to\neg\theta$

Then $\models\bigwedge b\to\bigwedge_{\theta\in\Theta}\neg\theta$ and hence $b\cup\{\bigvee\Theta\}$ has no model.

But $b \cup \{\bigvee \Theta\} \subset s$ and hence $b \cup \{\bigvee \Theta\} \in S \cap \mathcal{A}$.

But this contradicts the definition of S. $\square$