# Lecture Notes on Forcing at Logic Seminar(Fall 2006)

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**TALK SLOWLY AND WRITE NEATLY!!**

## 0.1 Introduction

### 0.1.1 Statement of CH

Today we are going to begin the proof of the independence of the Continuum Hypothesis form the other axioms of ZFC. Before we begin the proof, we will want to review the statement of $CH$. However in order to do this we first need some definitions and theorems.

**Size of Sets**

**Definition 0.1.1.1.** Let $(V, \epsilon) \models ZFC$ and let $A, B \in V$. Then $V \models |A| \leq |B|$ if

$$V \models (\exists f) f : A \rightarrow B \text{ and } f \text{ is injective}$$

**Cantor-Bernstein Theorem**

**Theorem 0.1.1.2** (CantorBernsteinSchroeder)**.** *Let* $V \models$

*ZFC and let* $V \models f : A \to B$ *and* $g : B \to A$ *are injective. Then* $V \models (\exists h) h : A \to B$ *and* $h$ *is bijective.*

*Proof.* Basic result □

We can now state the Continuum Hypothesis say

**Continuum Hypothesis**

**Definition 0.1.1.3.** The Continuum Hypothesis(CH) says

$$(\forall x)|\mathbb{N}| \leq |x| \leq |\mathbb{R}| \to |x| = |\mathbb{N}| \vee |x| = |\mathbb{R}|$$

### 0.1.2 Statement of Theorem

Now we are ready to begin the proof of the independence of $CH$ from $ZFC$. In this lecture we will use Forcing to deal with the case of $\neg CH$. However, we are even going to show a little bit more than just that $CH$ can't be proved from the axioms of $ZFC$. We are going to show that

$$Con(ZFC) \Leftrightarrow Con(ZFC + \neg CH)$$

So we are going to show that $ZFC$ and $ZFC + \neg CH$ are equiconsistent.

## Equiconsistent

**Definition 0.1.2.1.** We say that two theories $S, T$ are equiconsistent if $S \vdash Con(T)$ and $T \vdash Con(S)$.

Now to understand why this is saying more than just $ZFC \nvdash CH$ consider the statement

$$\Theta = Con(ZFC)$$

## ZFC + Con(ZFC)

**Theorem 0.1.2.2.** *$ZFC$ and $ZFC+\Theta$ are not equiconsistent.*

*Proof.* Now we know $ZFC+\Theta \vdash ZFC$ and so $Con(ZFC) \Rightarrow Con(ZFC+\Theta)$. Hence, if $ZFC \vdash Con(ZFC+\Theta)$ then $ZFC \vdash Con(ZFC)$. However, by Godel's incompleteness theorem we also know that $ZFC \nvdash Con(ZFC)$ and so we must conclude that $ZFC \nvdash Con(ZFC + \Theta)$ and so $ZFC$ and $ZFC + \Theta$ are not equiconsistent. $\square$

What this shows is that saying $ZFC + \neg CH$ is equiconsistent is stronger than just saying $ZFC \nvdash CH$. However, if this is your first time seeing these concepts don't worry to much about them for this lecture, just put them in the back of your mind and think about them when you get home.

So for this lecture we will use the following (slightly imprecise) definition of consistent.

## Completeness Theorem

**Definition 0.1.2.3.** We say that a theory $T$ is consistent if $T$ has a model.

This is just Godel's completeness theorem.

So what we will show is that

**CH is Consistent With ZFC**

**Theorem 0.1.2.4.** *$ZFC$ has a model if and only if $ZFC + \neg CH$ has a model.*

Now it is obvious that if $ZFC + \neg CH$ has a model that $ZFC$ has a model (just use the same model) so what we really have to show is

**Theorem 0.1.2.5.** *If $ZFC$ has a model then $ZFC + \neg CH$ has a model.*

Let $V \models ZFC$. Well if $V \models \neg CH$ we are done. So what we really need to show is

**Theorem 0.1.2.6.** *If there is a model of $ZFC + CH$ then there is a model of $ZFC + \neg CH$.*

## 0.2 Generic Extensions

### 0.2.1 Forcing Conditions

So given a model $M \models ZFC + CH$ how are we going to come up with a model of $ZFC + \neg CH$. Intuitively we are going to add $\omega_2$ many real numbers to our universe.

However, on first glance it is not clear how we can do this. How are we to find a set which isn't in our universe? And if we add such a set, how do we know that we will still have a model of $ZFC$? It was in order to answer these question that forcing was developed.

**Notion of Forcing**

**Definition 0.2.1.1.** Let $M \models ZFC$. We call a partial order $\langle \mathbb{P}, \leq \rangle \in M$ a Notion of Forcing and the elements of $P$ Forcing Conditions.

**Compatible Elements**

**Definition 0.2.1.2.** If $p, q \in \mathbb{P}$ and $p < q$ then we say that $p$ is stronger than $q$. If $(\exists r) r \leq p \wedge r \leq q$ then we say that $p$ and $q$ are Compatible; Otherwise they are Incompatible $(p \perp q)$

## Anti-Chain

**Definition 0.2.1.3.** A set $A \subseteq \mathbb{P}$ is an anti-chain if

$$(\forall p, q \in A)(p \neq q) \rightarrow p \perp q$$

## Dense Subset

**Definition 0.2.1.4.** A set $D \subseteq \mathbb{P}$ is dense if

$$(\forall p \in \mathbb{P})(\exists q \in D) q \leq p$$

## Filter

**Definition 0.2.1.5.** A set $F \subseteq \mathbb{P}$ is a filter on $\mathbb{P}$ if

(i) $F$ is non-empty

(ii) If $p \leq q$ and $p \in F$ then $q \in F$

(iii) If $p, q \in F$ then there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

We are eventually going to use a filter to construct our extension of the universe. So in particular, given two different filters we very well may get two different universes. However, once we have finished the construction we will see that we can use our filter as an oracle. Any question we want to know about the model we made we can answer once we are given the filter used to construct the model.

In this sense we want to think of a forcing condition as an "approximation" to a filter. The forcing condition can't tell us everything we need to know about the model, but it can answer certain questions. And, for any question we will see that there will be a forcing condition which can answer it.

However, there are certain questions for which you will get different answers depending on which forcing condition you look at. And so we can think of a forcing condition as "forcing" to be true those statements it knows are true.

Under this interpretation we see that a forcing condition $q$ is stronger than a forcing condition $p$ if it is a better approximation to the filter and hence can tell us more information.

Similarly two forcing conditions $p, q$ are compatible if there is some way to extend both of them and hence if there is some universe in which both everything which $p$ forces to be true is, and also everything which $q$ forces to be true is.

Similarly an Anit-chain is a collection of forcing conditions which, pairwise, force contradictory statements to be true.

And a dense set can be thought of as a statement which, doesn't contradict any other statement (i.e. given any approximation to the universe you can still find a universe in which the statement corresponding to the dense set is true).

#### 0.2.2 Generic Extensions

**Generic**

**Definition 0.2.2.1.** A set $G \subseteq \mathbb{P}$ is generic over $M$ if

(i) $G$ is a filter

(ii) If $D \subseteq \mathbb{P}$ is dense and $D \in M$ then $G \cap D \neq \emptyset$

So a filter is generic over $M$ if it intersects every dense set. Now if you recall we can think of dense sets as statements which can't be refuted by any approximation to the filter. So intuitively, a filter is generic if any property which can't be refuted by an approximation to the filter, must in fact be true.

Similarly if we have the following definition.

**D-Generic**

**Definition 0.2.2.2.** Let $\mathcal{D} \subseteq \{D \subseteq \mathbb{P} : D \text{ is dense}\}$. Then we say $G \subseteq \mathbb{P}$ is $\mathcal{D}$-Generic if

(i) $G$ is a filter

(ii) $(\forall D \in \mathcal{D}) G \cap D \neq \emptyset$

This we have the following theorem

## Generics for Countable Collections of Dense Set

**Theorem 0.2.2.3.** *Let $V \models ZFC$ and let $\langle P, \leq \rangle \in V$ be a partial order and let $\mathcal{D}$ be a countable collection of dense sets with $\mathcal{D} \subseteq V$. Then there is a $\mathcal{D}$-generic ultrafilter $G$ in $V$.*

*Proof.* Let $D_1, D_2, \ldots$ be the sets in $\mathcal{D}$. Let $p_0 = p$, and for each $n$, let $p_n$ be such that $p_n \leq p_{n-1}$ and $p_n \in D_n$. The set

$$G = \{q \in P : q \geq p_n \text{ for some } n \in \mathbb{N}\}$$

is a $\mathcal{D}$ generic filter on $P$ and $p \in G$. □

However, in $ZFC$ this is the best we can do. And in fact we even have the following.

## Generics Aren't in the Ground Model

**Theorem 0.2.2.4.** *Let $M \models ZFC$ and let $\langle \mathbb{P}, \leq \rangle \in M$ be a partial order such that*

$$(\forall p \in \mathcal{P})(\exists q, r \in \mathcal{P})(q \leq p \wedge r \leq p \wedge r \perp q)$$

*If $G \subseteq \mathbb{P}$ is a generic filter over $M$ then $G \notin M$.*

*Proof.* If $G \in M$ then $D = \mathcal{P} - G \in M$. Note though that if $p \in \mathcal{P}$ and $q, r$ are as in the condition on the partial ordering then we can't have $q \in G$ and $r \in G$. Hence one of $q \in D$ or $r \in D$. So in particular we have that $D$ is dense.

$\Rightarrow\Leftarrow G \cap D = \emptyset$. □

We are now ready to state one of the main theorems about forcing which we will prove.

## The Generic Model Theorem

**Theorem 0.2.2.5** (The Generic Model Theorem)**.** *Let $M \models ZFC$ be a transitive model and let $(\mathbb{P}, \leq) \in M$*

*be a notion of forcing in $M$. If $G \subseteq \mathbb{P}$ is generic over $M$ then there exists a transitive model $M[G]$ such that*

*(i)* $M[G] \models ZFC$

*(ii)* $M \subseteq M[G]$ *and* $G \in M[G]$

*(iii)* $Ord^{M[G]} = Ord^M$

*(iv) If $N$ is a transitive model of $ZFC$ such that $M \subseteq N$ and $G \in N$*

We call such a model a Generic Extension and it will be definable from $M$ and the filter $G$. Specifically each element of $M[G]$ will have a name in $M$ describing how it is constructed. So the model $M[G]$ can be described, modulo the oracle $G$, inside the ground model $M$.

### 0.2.3 Forcing Relation

Specifically we will show that corresponding to each forcing notion $(P, \leq)$ there is a forcing language and a forcing relation $\Vdash_{\mathbb{P}}$ both defined in ground model $M$. And it is about these that we will now state the next two main theorems about forcing.

## Forcing Theorem

**Theorem 0.2.3.1** (The Forcing Theorem). *Let $(P, \leq)$ be a notion of forcing in a ground model $M$. If $\sigma$ is a sentence of the forcing language then fore every $G \subseteq P$ generic over $M$*

$$M[G] \models \sigma \text{ if and only if } (\exists p \in G) p \Vdash \sigma$$

*Where the $\sigma$ on the left interprets constants according to $G$.*

And the last of our main theorems about forcing is

## Properties of Forcing

**Theorem 0.2.3.2** (Properties of Forcing). *Let $(\mathbb{P}, \leq)$ be a notion of forcing in the ground model $M$, and let $M^{\mathbb{P}}$ be the class (in $M$) of all names. Further let $\Vdash$*

*be the corresponding forcing relation ($p \Vdash \sigma$ is read $p$ forces $\sigma$).*

*Then for all formulas $\varphi, \psi$ in the forcing language*

*(i) (a) If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$*

*(b) There is no $p$ such that $p \Vdash \varphi$ and $p \Vdash \neg\varphi$*

*(c) For every $p$ there is a $q \leq p$ such that either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.*

*(ii) (a) $p \Vdash \neg\varphi$ if and only if there is no $q \leq p$ such that $q \Vdash \varphi$.*

*(b) $p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$*

*(c) $p \Vdash (\forall x)\varphi$ if and only if $p \Vdash \varphi(\dot{a})$ for all $a \in M^{\mathbb{P}}$*

*(d) $p \Vdash \varphi \vee \psi$ if and only if $(\forall q \leq p)(\exists r \leq q) r \Vdash \varphi$ or $r \Vdash \psi$*

*(e)* $p \Vdash (\exists x)\varphi$ *if and only if* $(\forall q \leq p)(\exists r \leq q)(\exists \dot{a} \in M^{\mathbb{P}})r \Vdash \varphi(\dot{a})$

*(iii) If* $p \Vdash \exists x\varphi$ *then for some* $\dot{a} \in M^{\mathbb{P}}$*,* $p \Vdash \varphi(\dot{a})$

## 0.3 Relationship Between Boolean Algebras and Partial Orders

While the forcing relation can be defined directly in terms of the partial order, it turns out that its properties and the properties of the generic extension are determined by a boolean algebra associated to the partial order. And it is over this boolean algebra which we will construct our boolean valued model of $ZFC$.

### Separative Partial Ordering

**Definition 0.3.0.3.** A partially ordered set $(\mathbb{P}, \leq)$ is Separative if for all $p, q \in \mathbb{P}$

$$(\forall p, q \in \mathbb{P})(p \not\leq q) \rightarrow (\exists r \in \mathbb{P})(r \leq p) \wedge (r \perp q)$$

What this says is that if it isn't the case that $q$ forces everything that $p$ does, then there is some extension of $p$ which forces something incompatible with $q$. This is there to ensure that there is enough branching.

Examples of non-separative partial orderings are

A linear ordering (with more than one element)

The set of all infinite subsets of $\omega$ ordered by inclusion.

The reason why separative partial orderings are important is that they are "essentially" boolean algebras in the following sense.

To review

## Boolean Algebra

**Definition 0.3.0.4.** We say $\langle B, \vee, \cdot, -, 0, 1, \leq \rangle$ is a boolean algebra if

$\langle B, \leq \rangle$ is a lattice

$lub(a, b) = a \vee b$ for all $a, b \in B$

$glb(a, b) = a \cdot b$ for all $a, b \in B$

$a \vee (b \cdot c) = (a \cdot b) \vee (a \cdot c)$ for all $a, b, c \in B$

$a \cdot (b \vee c) = (a \vee b) \cdot (a \vee c)$ for all $a, b, c \in B$

$-a$ is the unique element such that $a \vee (-a) = 1$ and $a \cdot (-a) = 0$

## Complete Boolean Algebra

**Definition 0.3.0.5.** We say a boolean algebra $B$ is complete if for all $A \subseteq B$

The least upper bound of $A$ exists in $B$ (denoted $\Sigma A$)

The greatest lower bound of $A$ exists in $B$ (denoted $\Pi A$)

We then have

## Partial Order Associated to A Boolean Algebra

**Definition 0.3.0.6.** Let $B$ be a boolean algebra. We then define $(B^+, \leq)$ to be the linear order

$$B^+ = B - \{0\}$$

$$(\forall a, b \in B^+) a \leq b \leftrightarrow a \cdot b = a$$

We say that $D \subseteq B$ is dense if it is dense in the partial ordering $(B^+, \leq)$.

**Lemma 0.3.0.7.** *Let $B$ be a boolean algebra and let $D \subseteq B$ be a dense set. Then $(D, \leq)$ is a separative partial order.*

*Proof.* If $r$ is incompatible with $q$ then any $r' \leq r$ is incompatible with $q$. So if $p \not\leq q$ are elements of our dense set, let $a = p \cdot q$. S $p \cdot \neg a) \cdot q = 0$. Hence if

$r = p \cdot (\neg a)$ then $r \leq p$ and is incompatible with $q$. Let $r'$ then be an element of $D$ which is less than $r$. We then have $r' \leq p$ and $r'$ is incompatible with $q$. □

## Embedding Partial Orders Into Boolean Algebr

**Theorem 0.3.0.8.** *Let* $(\mathbb{P}, \leq)$ *be a separative partial order. Then there is a complete boolean algebra* $B$ *such that*

*(i)* $P \subseteq B^+$ *and* $\leq$ *agrees with the partial ordering on* $B$.

*(ii)* $P$ *is dense in* $B$

*The algebra* $B$ *is unique up to isomorphism.*

*Proof.* We say $A \subseteq P$ is a cut if $(\forall a \in A), b \leq a \rightarrow b \in A$. For each $p \in \mathbb{P}$ let $U_p = \{q : q \leq p\}$ be the cut which we want to correspond to $p$.

**Definition 0.3.0.9.** We say a cut $U \subseteq P$ is regular if

$$(\forall p \in \mathbb{P}) p \notin U \rightarrow (\exists q \leq p) U_q \cap U = \emptyset$$

Now notice that $U_p$ is a regular cut and that every cut $U$ there is a $p$ such that $U$ contains $U_p$

Notice that the intersection of regular cuts is a regular cut. So every cut $U$ is contained in a least regular cut $\overline{U}$. Then $\langle B, \vee, \times, \neg \rangle$ is the complete Boolean algebra we want where

$$B = \{U : U \text{ is a regular cup of } \mathbb{P}\}$$

$$(\forall u, v \in B) u \times v = u \cap v$$

$$(\forall u, v \in B) u \vee v = \overline{u \cup v}$$

$$\neg U = \{p : U_p \cap U = \emptyset\}$$

□

But what is more, when the partial order is not separative, we can replace it by one that is.

# Embedding Partial Orders Into Separative Boole

**Theorem 0.3.0.10.** *Let* $(P, \leq)$ *be a partial ordering. Then there is a separative partial ordering* $(Q, \preceq)$ *and a surjective mapping* $h : P \twoheadrightarrow Q$ *such that*

*(i)* $x \leq y$ *implies* $h(x) \preceq h(y)$

*(ii)* $x$ *and* $y$ *are compatible in* $P$ *if and only if* $h(x)$ *and* $h(y)$ *are compatible in* $Q$*.*

*We call* $Q$ *the* *Separative Quotient of P* *and it is unique up to isomorphism.*

*Proof.* Lets define the following equivalence relation on $P$:

$x \sim y$ if and only if $(\forall z)(z$ is compatible with $x \leftrightarrow z$ is compatible with $y)$

Let $Q = P/\sim$ and let us define

$$[x] \preceq [y] \leftrightarrow (\forall z \leq x)[z \text{ and } y \text{ are compatible}]$$

It isn't hard to check that $(Q \ \preceq)$ is a separative partial ordering and that $h(x) = [x]$ satisfies the conditions of the lemma. $\square$

**Corollary 0.3.0.11.** *For every partial ordering* $(P, \leq)$ *there is a complete boolean algebra* $B = B(P)$ *and a mapping* $e : P \to B^+$ *such that*

*(i)* $(\forall p, q \in P)$ *if* $p \leq q$ *then* $e(p) \leq e(q)$

*(ii)* $p$ *and* $q$ *are compatible if and only if* $e(p) \cdot e(q) \neq 0$

*(iii)* $\{e(p) : p \in P\}$ *is dense in* $B$

$B$ *is unique up to isomorphism.*

*Proof.* This is gotten by simply combining the previous results. $\square$

Now that we understand the relationships between boolean algebras and partial orders we can examine what we mean when we say that the information contained in a generic filter on $P$ is contained in an ultrafilter on $B(P)$. Specifically we mean.

## Relationship Between BA Generics and PO Gen

**Theorem 0.3.0.12.** *(i) In the ground model $M$, let $Q$ be the separative quotient of $P$ and let $h$ map $P$ onto $Q$ such that the conditions of the previous theorem hold.*

*(a) If $G \subseteq P$ is generic over $M$ then $h(G) \subseteq Q$ is also generic over $M$.*

*(b) If $H \subseteq Q$ is generic over $M$ then $h^{-1}(H) \subseteq P$ is generic over $M$.*

*(ii) In the ground model $M$ let $P$ be a dense subset of*

*a partial ordered set* $Q$*.*

*(a) If* $G \subseteq Q$ *is generic over* $M$ *then* $H = G \cap P \subseteq P$ *is generic over* $M$*.*

*(b) If* $H \subseteq P$ *is generic over* $M$ *then* $G = \{q \in Q : (\exists p \in H) p \leq q\}$ *is generic over* $M$

*Proof.* This simply requires checking definitions. □

**Corollary 0.3.0.13.** *Let* $e : P \to B(P)$ *be as in the previous theorems, let* $G \subseteq P$*, and let* $H = \{u \in B : (\exists p \in G) e(p) \leq u\}$*. Then*

- $G$*,* $H$ *are both definable from each other.*
- $G$ *is generic if and only if* $H$ *is.*
- $M[G] = M[H]$

*Proof.* Immediate from the definitions. □

So $P$, $B(P)$ produce the same models.

**Definition 0.3.0.14.** Let $B$ be a boolean algebra. $U \subseteq B$ is an Ultrafilter if

- If $a \in U$ and $a \leq b$ then $b \in U$.
- If $a \in U$ and $b \in U$ then $a \cdot b \in U$
- $(\forall u \in U)(u \in U) \vee (-u \in U)$

## Generic Ultrafilter in a Boolean Algebra

**Definition 0.3.0.15.** If $(P, \leq) \in M$ then $(B(P)$ is a complete boolean algebra$)^M$. But, outside of $M$ $B(P)$ is still a boolean algebra even if it isn't complete. So we say an ultrafilter $G$ is Generic over $M$ if

$$\bigwedge X \in G \text{ whenever } X \in M \wedge X \subseteq G$$

**Lemma 0.3.0.16.** *$G$ is a generic ultrafilter on a boolean algebra $B$ if and only if $G$ is a generic filter on $B^+$.*

*Proof.* Immediate from the definitions. □

## 0.4 Boolean Valued Models

Before we can prove these important theorems about forcing (and to explain why the connection between forcing notions and boolean algebras are important) we will have to introduce the general idea of a Boolean Valued Model. A Boolean Valued Model is just a model, in the sense of model theory, where the truth values of statements are elements of a fixed Boolean algebra as opposed to just elements of {True, False}.

### 0.4.1 Boolean Valued Models

**Definition of Boolean Valued Model on Atomic**

**Definition 0.4.1.1.** Let $B$ be a complete boolean algebra and let $L = \{R_i : i \in I\}$ be a relational language. We say that $\Updownarrow = \langle M, \| = \|, \langle \|R_i\| : i \in I\rangle\rangle$ is a $B$ (or boolean valued) model of $L$ if we have (with slight abuse of notation)

(a) For each $R_i \in L$,

$$\|R\| : M^n \to B$$

where $n$ is the arity of $R$

(b) $\| = \| : M^2 \to B$ such that

(i) $(\forall x \in M)\|x = x\| = 1$

(ii) $(\forall x, y \in M)\|x = y\| = \|y = x\|$

(iii) $(\forall x, y, z \in M)\|x = y\| \cdot \|y = z\| \leq \|x = z\|$

(iv) $(\forall x_1, \ldots, x_n, y_1, \ldots, y_n \in M)(\forall R_i \in L)\|R_i(x_1, \ldots, x_n)\| \cdot \Pi_{i \in i}\|x_i = y_i\| \leq \|R_i(y_1, \ldots, y_n)\|$

Once we've defined a boolean valued model on atomic formulas we can extend the definition in a natural way to arbitrary formula of first order logic.

## Definition of Boolean Valued Model on 1st Orde

**Definition 0.4.1.2.** Let $\Updownarrow = \langle M, \| = \|, \langle \|R_i\| : i \in$

$I\rangle\rangle$ be a $B$-valued model of a language $L$ for some complete boolean algebra $B$. We extend $\|\cdot\|$ to be a function from 1st order formulas of $L$ to $B$ as follows:

(a) If $\varphi(\mathbf{x})$ is atomic then use the $\|\cdot\|$ from the definition of $B$-valued model.

(b) For negation, conjunction, ect

$$\begin{aligned}
\|\neg\varphi(a_1,\ldots,a_n)\| &= -\|\varphi(a_1,\ldots,a_n)\| \\
\|(\varphi\wedge\psi)(a_1,\ldots,a_n)\| &= \|\varphi(a_1,\ldots,a_n)\| \vee \|\varphi(a_1,\ldots,a_n)\| \\
\|(\varphi\vee\psi)(a_1,\ldots,a_n)\| &= \|\varphi(a_1,\ldots,a_n)\| \cdot \|\varphi(a_1,\ldots,a_n)\|
\end{aligned}$$

(c) For quantifiers

$$\begin{aligned}
\|\exists x\varphi(x,a_1,\ldots,a_n)\| &= \Sigma_{a\in M}\|\varphi(a,a_1,\ldots,a_n)\| \\
\|\forall x\varphi(x,a_1,\ldots,a_n)\| &= \Pi_{a\in M}\|\varphi(a,a_1,\ldots,a_n)\|
\end{aligned}$$

We will use the standard short hands of $\varphi(\mathbf{x}) \rightarrow \psi(\mathbf{x}) :=$

$\neg\varphi(\mathbf{x}) \vee \psi(\mathbf{x})$ and $\varphi(\mathbf{x}) \leftrightarrow \psi(x) := (\varphi(\mathbf{x}) \rightarrow \psi(\mathbf{x})) \wedge (\psi(\mathbf{x}) \rightarrow \varphi(\mathbf{x}))$

## Definition of Valid

**Definition 0.4.1.3.** We say that $\varphi(\overline{a})$ is valid in $M$ if $\|\varphi(\overline{a})\| = 1$. We say that an implication $\varphi(\overline{a}) \rightarrow \psi(\overline{a})$ is valid if $\|\varphi(\overline{a})\| \leq \|\psi(\overline{a})\|$ (it is not hard to check that these definitions agree).

**Lemma 0.4.1.4.** *We have that for each first order formula* $\varphi$

$$\|\overline{a} = \overline{b}\| \cdot \|\varphi(\overline{a})\| \leq \|\varphi(\overline{b})\|$$

*(where* $\|\overline{a} = \overline{b}\| = \Pi_{i\in n}\|a_i = b_i\|$*). And so we have*

$$(\overline{a} = \overline{b}) \wedge \varphi(\overline{a}) \rightarrow \varphi(\overline{b})$$

*is valid (as we would hope if our definition of* $=$ *is correct).*

*Proof.* Immediate from the definitions. □

## Definition Full Models

**Definition 0.4.1.5.** We say a that a boolean valued model $M$ is full if for all $\overline{a} \in M$ and all 1st order formulas $\exists x \varphi(x, \mathbf{y})$ there exists a $a \in M$ such that

$$\|\varphi(a, \overline{a})\| = \|(\exists x)\varphi(x, \overline{a})\|$$

Full models are important because of the following theorem.

**Definition 0.4.1.6.** Let $F$ be an ultrafilter on $B$. Let $\mathcal{M}$ be a full $B$-valued model. For each 1st order formula $\varphi(\mathbf{x})$ we say that

$\mathcal{M}/F \models \varphi(a_1, \ldots, a_n)$ if and only if $\|\varphi(a_1, \ldots, a_n)\| \in F$

## Moding Out By An Ultrafilter

**Theorem 0.4.1.7.** *If $F$ is an ultrafilter on $B$ and $\mathcal{M}$ is a $B$-valued model then then $\mathcal{M}/F$ is well define and is a classical (or 2-valued) model*

*Proof.* Case 1: $\varphi$ is atomic.

Then the theorem is true by definition.

Case 2: $\varphi = \psi \wedge \eta$

In this case we know $\|\psi \wedge \eta\| \in F$ if and only if $\|\psi\| \in F$ and $\|\eta\| \in F$ because $F$ is an ultrafilter. So we are done by induction.

Case 3: $\varphi = \neg\psi$

In this case we know $\|\neg\psi\| \in F$ if and only if $\|\psi\| \notin F$ because $F$ is an ultrafilter. So we are done by induction.

Case 4: $\varphi = \exists x \psi(x, \dots)$

In this case we will need fullness to choose an $a \in \mathcal{M}$ such that $M \models \|\exists x \psi(x, \dots)\| = \|\psi(a, \dots)\|$. So in particular we have $\|\exists x \psi(x, \dots)\| \in F$ if and only if $(\exists a \in \mathcal{M})\|\psi(a, \dots)\| \in F$. And so by induction we are done. □

An easy way to think of this is we can view an ultrafilter as a boolean algebra homomorphism from $B$ to the boolean algebra 2. Then $\mathcal{M}/F$ is just the induced 2-valued model gotten from applying that homomorphism to $\|\varphi(\mathbf{x})\|$ for each 1st order formula $\varphi(\mathbf{x})$.

## 0.5 $V^B$

Now that we know what boolean valued models are, and we know how to get a boolean algebra from a notion of forcing, it is time to consider the boolean valued model we are interested in.

### 0.5.1 Definition

We want $V^B$, our boolean valued model, to be a generalization of a model of $ZFC$. And as such we will define it as a generalization of the Von Von Neumann Hierarchy.

**Definition of $V^B$**

**Definition 0.5.1.1.** Let $L = \{\in\}$ be the language of set theory and let $B$ be a boolean algebra. We then define our boolean valued model $(V^B, \| \in \|, \| = \|)$ as follows

- – $V_0^B = \emptyset$

  – $V_{\alpha+1}^B = \{f \text{ s.t. } f : V_\alpha^B \to B\}$

  – $V_{\lambda * \omega}^B = \bigcup_{i \leq \lambda * \omega} V_i^B$

  – $V^B = \bigcup_{i \in ORD} V_i^B$

- – $\|x \in y\| = \Sigma_{t \in \text{dom}(y)}(\|x = t\| \times y(t))$

  – $\|x \subset y\| = \Pi_{t \in \text{dom}x}(x(t) \Rightarrow \|t \in y\|)$

  – $\|x = y\| = \|x \subset y\| \cdot \|y \subset x\|$

where we define

$$(\forall b, c \in B) b \to c = -b \vee c$$

So what is going on here. We are saying that in order to calculate the truth value of the statement $\|x \in y\|$ all we need to do is go through all of the elements of $y$ and add up the value that $x$ is one of those elements. Similarly, the value that $x$ is a subset of $y$ is just the product over elements of $x$ of the value of the statement "$z \in x$ implies $z \in y$". And lastly, the truth value of $x = y$ is just the truth value of $x$ is a subset of $y$ and $y$ is a subset of $x$.

Now on first glance this definition seems very circular. After all the truth value of $=$ is defined in terms of the truth value for $\in$ which is in turn defined in terms of the truth value of $=$, ect. However, this infinite cycle terminates because we see that each element of the definition is defined in terms of elements of lower rank.

## Definition of Rank

**Definition 0.5.1.2.** Let $x \in V^B$. Define the rank of $x$ ($\rho(x)$) to be the least $\alpha$ such that $x \in V^B_\alpha$.

We then have

## Properties of Rank

**Theorem 0.5.1.3.** *If rank of* $(x, y)$ *is* $(\rho(x), \rho(y))$ *and*

*$(a, b)$ is used in the definition of $\|x = y\|, \|x \in y\|$ or $\|x \subset y\|$ then $\rho(a) \leq \rho(x), \rho(b) \leq \rho(y)$ and $(\rho(x), \rho(y)) \neq (\rho(a), \rho(b))$.*

*And hence in particular, this definition is well defined.*

*Proof.* This is immediate from looking at the definition.

□

This definition of $\| \in \|, \| = \|$ is one of the most complicated parts of the forcing machinery. But fortunately, once you know it works, you never have to think about it again.

### 0.5.2 Boolean Valued Model

Now that we have defined $V^B$ we need to show that in fact $(V^B, \| = \|, \| \in \|)$ is a full boolean valued model of the language $L = \{\in\}$ of set theory.

## $V^B$ Is B-valued Model of =

**Theorem 0.5.2.1.** *(a)* $(\forall x \in V^B)\|x = x\| = 1$

*(b)* $(\forall x, y \in V^B)\|x = y\| = \|y = x\|$

*Proof.* It suffices to show that for all $x \in V^B$ $\|x \subset x\| = 1$, i.e. that $x(t) \Rightarrow \|t \in x\| = 1$, or equivalently that $x(t) \leq \|t \in x\|$. We prove this by induction on the rank of $x$.

If $t \in \mathrm{dom}(x)$ then we have $\|t = t\| = 1$ by induction and hence by the definition of $\|t \in x\|$ we have $x(t) = \|t = t\| \cdot x(t) \leq \|t \in x\|$ □

## $V^B$ is B-valued Model

**Theorem 0.5.2.2.** *(a)* $(\forall x, y, z \in V^B)\|x = y\| \cdot \|y = z\| \leq \|x = z\|$

*(b)* $(\forall x, y, z \in V^B)\|x \in y\| \cdot \|y = z\| \leq \|x \in z\|$

*(c)* $(\forall x, y, z \in V^B)\|y \in x\| \cdot \|y = z\| \leq \|z \in x\|$

*Proof.* We will prove this by induction on triples $\langle \rho(x), \rho(y), \rho(z) \rangle$.

Part (i):

It suffices to prove $\|x \subset y\| \cdot \|y = z\| \leq \|x \subset z\|$. Let $t \in \text{dom}(x)$ be arbitrary. We wish to show

$$(*)\ \|y = z\| \cdot (x(t) \Rightarrow \|t \in y\|) \leq x(t) \rightarrow \|t \in z\|$$

(using the definition of $\|x \subset z\|$). By the inductive hypothesis we have $\|t \in y\| \cdot \|y = z\| \leq \|t \in z\|$. Thus $\|y = z\| \cdot (-x(t) \vee \|t \in y\|) = (\|y = z\| - x(t)) \vee (\|y = z\| \cdot \|t \in y\|) \leq -x(t) \vee \|t \in z\|$ and $(*)$ follows.

Part (ii):

Let $t \in \text{dom}(y)$ be arbitrary. by the induction hypothesis we have $\|x = z\| \cdot \|x = t\| \leq \|z = t\|$ and so

$$\|x = z\| \cdot \|x = t\| \cdot y(t) \leq \|z = t\| \cdot t(t)$$

Taking the sum over all $t \in \mathrm{dom}(y)$ we get

$$\|x = z\| \cdot \sum_{t \in \mathrm{dom}(y)} (\|x = t\| \cdot y(t)) \leq \sum_{t \in \mathrm{dom}(y)} (\|z = t\| \cdot y(t))$$

that is $\|x = z\| \cdot \|x \in y\| \leq \|z \in y\|$.

Part (iii):

Let $t \in \mathrm{dom}(x)$. By the definition of $\|x = z\|$ we have $x(t) \cdot \|x = z\| \leq \|t \in z\|$ and so

$$\|y = t\| \cdot x(t) \cdot \|x = z\| \leq \|y = t\| \cdot \|t \in z\|$$

By the inductive hypothesis we have $\|y = t \cdot \|t \in z\| \leq \|y \in z\|$ and therefore

$$\|y = t\| \cdot x(t) \cdot \|x = z\| \leq \|y \in z\|$$

Taking the sum of the left-hand side over all $t \in \mathrm{dom}(x)$ we get

$$\sum_{t \in \mathrm{dom}(x)} (\|y = t\| \cdot x(t)) \cdot \|x = z\| \leq \|y \in z\|$$

that is $\|y \in x\| \cdot \|x = z\| \leq \|y \in z\|$. $\square$

**Lemma 0.5.2.3.** *If $W$ s a set of pairwise disjoint elements of $B$ and if $a_u, u \in W$ are elements of $V^B$ then there exists some $a \in V^B$ such that $u \leq \|a = a_u\|$ for all $u \in W$.*

*Proof.* Let $D = \bigcup_{n \in W} \operatorname{dom}(a_u)$ and for every $t \in D$ let $a(t) = \sum\{u \cdot a_u(t) : u \in W\}$. Since the $u$'s are pairwise disjoint we have $u \cdot a(t) = u \cdot a_u(t)$ for each $u \in W$ and each $t \in D$. In other words $u \leq (a(t) \Rightarrow a_u(t))$ and $u \leq (a_u(t) \Rightarrow a(t))$ and so $u \leq \|a = a_u\|$. □

## $V^B$ is Full

**Theorem 0.5.2.4.** *$V^B$ is full. I.e. given a formula $\varphi(x, \dots)$ there exists some $a \in V^B$ such that*

$$\|\varphi(a, \dots)\| = \|\exists x \varphi(x, \dots)\|$$

*Proof.* We have already that

$$\|\varphi(a, \dots)\| \leq \|\exists x \varphi(x, \dots)\|$$

for all $a$ so we just need to find an $a \in V^B$ such that

$$\|\varphi(a, \dots)\| \geq \|\exists x \varphi(x, \dots)\|$$

let $u_0 = \|\exists x \varphi(x, \dots)\|$. Let

$$D = \{u \in B : \text{ there is some } a_u \text{ such that } u \leq \|\varphi(a_u, \dots)\|\}$$

It is clear that $D$ is open and dense below $u_0$. Let $W$ be a maximal set of pairwise disjoint elements of $D$. Clearly $sum\{u : u \in W\} \geq u_0$. So there exists an $a \in V^B$ such that $u \leq \|a = a_u\|$ for all $u \in W$. Thus for each $u \in W$ we have $u \leq \|\varphi(a, \dots)\|$ and hence $u_0 \leq \|\varphi(a, \dots)\|$. □

So we have shown that in fact $(V^B, \| \in \|, \| = \|)$ is a full boolean valued model of $L$. It is also worth mentioning that in the proof of fullness we used the axiom of choice. And, in that proof is the only place where we use it in the entire forcing construction.

In addition to $V^B$ being a boolean valued model of $L$ we also have that each element of $V$ has a name inside $V^B$

## Canonical Name for a Set

**Definition 0.5.2.5.** Define $\hat{x}$ by $\in$ induction on $x \in V$.

(i) $\hat{\emptyset} = \emptyset$

(ii) For all $x \in V$, $\mathrm{dom}(\hat{x}) = \{\hat{y} : y \in x\}$ and $(\forall \hat{y} \in \mathrm{dom}(\hat{x}))\hat{x}(\hat{y}) = 1$

## Canonical Name for an Ultrafilter

**Definition 0.5.2.6.** We also see that there is a canonical name, $\dot{G}$ for our generic ultrafilter $G$.

$$\mathrm{dom}(\dot{G}) = \{\check{u} : u \in B\}$$

and

$$\dot{G}(u) = u \text{ for every } u \in B$$

# Canonical Name for an Ultrafilter is Correct

**Theorem 0.5.2.7.** *Assume $V^B \models ZFC$ and $G$ is an ultrafilter on $B$. Then*

$$V^B/\dot{G} \models \check{u} \in \dot{G} \text{ if and only if } u \in G$$

First we need a lemma

## On Pure Sets $V^B$ is 2 Valued

**Lemma 0.5.2.8.** *Let $x, y \in M$.*

- *If $x \neq y$ then $\|\check{x} = \check{y}\| = 0$.*
- *If $x \notin y$ then $\|\check{x} \in \check{y}\| = 0$*
- *If $x \in y$ then $\|\check{x} \in \check{y}\| = 1$*

*Proof.* We will prove this by induction on pairs $(\rho(x), \rho(y))$.

Base Case:

Notice that if $x = \emptyset$ then

$$\|\check{x} = \check{y}\| \leq \|\check{y} \subseteq \check{x}\| = \prod_{t \in \operatorname{dom} \check{y}} (\check{y}(t) \Rightarrow \|t \in \check{x}\|) = 0$$

<u>Case $x \notin y$:</u>

$$\|\check{x} \in \check{y}\| = \Sigma_{t \in \text{dom}(\check{y})}(\|\check{x} = t\| \cdot \check{y}(t)) = \Sigma_{t \in y} 0 \cdot \check{y}(\check{t}) = 0$$

The last step is gotten because by induction we know $\|\check{x} = \check{t}\| = 0$ for all $t \in y$.

<u>Case $x \in y$:</u>

$$\|\check{x} \in \check{y}\| = \Sigma_{t \in \text{dom}(\check{y})}(\|\check{x} = t\| \cdot \check{y}(t)) \geq \|\check{x} = \check{x}\| \cdot \check{y}(\check{x}) = 1$$

<u>Case $x \neq y$:</u>

Without loss of generality we can assume that there is a $s \in x$ such that $s \in y$. We then have

$$\begin{aligned} \|x = y\| &\leq \|\check{x} \subset \check{y}\| \\ &= \prod t \in \text{dom}(\check{x})(\check{x}(t) \Rightarrow \|t \in \check{y}\|) \\ &= \prod t \in x(1 \Rightarrow \|\check{t} \in \check{y}\|) \\ &\leq (1 \Rightarrow \|\check{s} \in \check{y}\|) \\ &= 0 \end{aligned}$$

□

## Proof That Canonical Name For a Ultrafilter is

*Theorem.* $V^B/\dot{G} \models \check{u} \in \dot{G}$ if and only if $\|\check{u} \in \dot{G}\| \in G$.

So it suffices to show that $\|\check{u} \in \dot{G}\| = u$

But we have

$$\begin{aligned} \|\check{u} \in \dot{G}\| &= \Sigma_{t \in \mathrm{dom}(\dot{G})}(\|\check{y} = t\| \times \dot{G}(t)) \\ &= \Sigma_{t \in B}\|\check{u} = \check{t}\| \times \dot{G}(\check{t}) \\ &= \dot{G}(\check{u}) \\ &= u \end{aligned}$$

□

We then have the following Boolean valued version of absoluteness.

## $\Delta_0$ Formulas

**Lemma 0.5.2.9.** *If* $\varphi(x_1, \ldots, x_n)$ *is* $\Delta_0$ *formula with*

$x_1, \ldots, x_n \in V$ *then* $V \models \varphi(x_1, \ldots, x_n)$ *if and only if* $V^B \models \|\varphi(\check{x}_1, \ldots, \check{x}_n)$

*Proof.* By induction on the complexity of the formula.

□

**$\Sigma_1$ Formulas**

**Lemma 0.5.2.10.** *If* $\varphi(x_1, \ldots, x_n)$ *is* $\Sigma_1$ *formula with* $x_1, \ldots, x_n \in V$ *then* $V \models \varphi(x_1, \ldots, x_n)$ *implies* $V^B \models \|\varphi(\check{x}_1, \ldots, \check{x}_n)$

*Proof.* By induction on the complexity of the formula.

□

### 0.5.3 $V^B$ Satisfies ZFC

Now we are ready to prove that $V^B$ makes all the axioms of $ZFC$ valid (as well as several other results concerning $V^B$).

**Ordinals**

**Theorem 0.5.3.1.** *For every $x \in V^B$*

$$\|x \textit{ is an ordinal }\| = \sum_{\alpha \in Ord} \|x = \check{\alpha}\|$$

*Proof.* For every $x \in V^B$

$$\|x \text{ is an ordinal}\| = \sum_{\alpha \in Ord} |x = \check{\alpha}\|$$

Since "$x$ is an ordinal" is $\Delta_0$ we have

$$\sum_{\alpha \in Ord} |x = \check{\alpha}\| \leq \|x \text{ is an ordinal}\|$$

On the other hand if $\gamma$ is an ordinal then

$$\|x \text{ is an ordinal and } x \in \check{\gamma}\| \leq \sum_{\alpha \in \gamma} \|x = \check{\alpha}\|$$

But we also have

$$\|x \text{ is an ordinal}\| \leq \|x \in \check{\alpha}\| + \|x = \check{\alpha}\| + \|\check{\alpha} \in x\|$$

However there is only a set of $\alpha$'s such that $\|\alpha \in x\| \neq 0$ because

$$\|\check{\alpha} \in x\| = \sum_{t \in \mathrm{dom}(x)} \|\check{\alpha} \in t\| \cdot x(t)$$

Hence there is a $\gamma$ such that $\|x \text{ is an ordinal}\| \leq \|x \subset \check{\gamma}\|$ and we have $\|x \text{ is an ordinal}\| \leq \sum_{\alpha \leq \gamma} \|x = \alpha\|$ $\square$

## Extensionality

**Theorem 0.5.3.2.** *$V^B$ makes extensionality valid. I.e.*

$$(\forall X, Y \in V^B)\|(\forall u)(u \in X \leftrightarrow u \in Y)\| \leq \|X = Y\|$$

*Proof.* Let $X, Y \in V^B$. By the definition of $a \Rightarrow b$ we observe that if $a \leq a'$ then $(a' \Rightarrow b) \leq (a \Rightarrow b)$. Thus for any $u \in V^B$ we have $(\|u \in X\| \Rightarrow \|u \in Y\|)$ and therefore

$$\prod_{u \in V^B} (\|u \in X\| \Rightarrow \|u \in Y\|) \leq \prod_{u \in V^B} (X(u) \Rightarrow \|u \in Y\|)$$

While the left-hand side is equal to $\|(\forall u)(u \in X \rightarrow u \in Y)\|$, the right-hand side is easily seen to equal $\|X \subset Y\|$. Consequently

$$\|(\forall u)(u \in X \leftrightarrow u \in Y)\| \leq \|X = Y\|$$

□

## Pairing

**Theorem 0.5.3.3.** *$V^B$ makes Pairing valid.*

*Proof.* Given $a, b \in V^B$ let $c = \{a, b\}^B \in V^B$ be such that $\mathrm{dom}(c) = \{a, b\}$ and $c(a) = c(b) = 1$. Then $\|a \in c \wedge b \in c\| = 1$. This with separation is enough for the pairing axioms. □

## Separation

**Theorem 0.5.3.4.** *$V^B$ makes Separation valid.*

*Proof.* We prove that for every $X \in V^B$ there is a $Y \in V^B$ such that

$$\|Y \subset X\| = 1 \text{ and } \|(\forall x \in X)(\varphi(z)) \leftrightarrow z \in Y\| = 1$$

Let $Y \in V^B$ be as follows

$$\mathrm{dom}(Y) = \mathrm{dom}(X),\ Y(t) = X(t) \cdot \|\varphi(t)\|$$

For every $x \in V^B$ we have $\|x \in Y\| = \|x \in X\| \cdot \|\varphi(x)\|$ and this gives us separation. $\square$

## Union

**Theorem 0.5.3.5.** *$V^B$ makes Union valid.*

*Proof.* We prove that for every $X \in V^B$ there is a $Y \in V^B$ such that

$$\|(\forall u \in X)(\forall v \in u)(u \in Y)\| = 1$$

If $X \in V^B$ then letting $Y \in V^B$ be defined as follows verifies the above condition.

$$\text{dom}(Y) = \bigcup\{\text{dom}(u) : u \in \text{dom}(X)\},\ Y(t) = 1 \text{ for all } t \in \text{dom}(Y)$$

$\square$

## Powerset

**Theorem 0.5.3.6.** *$V^B$ makes Powerset valid.*

*Proof.* We prove that for every $X \in V^B$ there is a $Y \in V^B$ such that

$$\|\forall u(u \subset X \to u \in Y)\| = 1$$

Let $Y$ be such that

$$\text{dom}(Y) = \{u \in V^B : \text{dom}(u) = \text{dom}(X) \text{ and } u(t) \leq X(t) \text{ for all } t\}$$

$$Y(u) = 1 \text{ for all } u \in \text{dom}(Y)$$

To verify that $Y$ satisfies the condition we need we make the following observation. If $u \in V^B$ is arbitrary let $u' \in V^B$ be such that $\text{dom}(u') = \text{dom}(X)$ and $u'(t) = X(t) \cdot \|t \in u\|$ for all $t \in \text{dom}(X)$. Then

$$\|u \subset X\| \leq \|u = u'\|$$

Hence we can include in $Y$ only the "Representative" $u$'s □

## Infinity

**Theorem 0.5.3.7.** *$V^B$ makes Infinity valid.*

*Proof.* Let $\varphi(x) = x$ is an ordinal and $(\forall\alpha \in x)\alpha{+}1 \in x$. This is $\Delta_0$ and $V \models \varphi(\omega)$ so $V^B \models \|\varphi(\check{\omega})\| = 1$ □

## Replacement

**Theorem 0.5.3.8.** *$V^B$ makes Replacement valid.*

*Proof.* It suffices to verify the collection principle. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|(\forall u \in X)(\exists v \varphi(u,v)) \to (\exists v \in Y)\varphi(u,v)\| = 1$$

Here we let

$\mathrm{dom}(Y) = \bigcup\{S_u : u \in \mathrm{dom}(X)\}$ $Y(t) = 1$ for all $t \in \mathrm{dom}(Y)$

where $S_u \subset V^B$ is some set such that

$$\sum_{v \in V^B} \|\varphi(u,v)\| = \sum_{v \in S_u} \|\varphi(u,v)\|$$

□

## Regularity

**Theorem 0.5.3.9.** *$V^B$ makes Regularity valid.*

*Proof.* We prove that for every $X \in V^B$

$$\|X \text{ is non-empty } \rightarrow (\exists y \in X)(\forall z \in y) z \notin X\| = 1$$

If this isn't true then

$$\|\exists u(u \in X) \wedge (\forall y \in X)(\exists z \in y) z \in X\| = b \neq 0$$

Let $y \in V^B$ be the least $\rho(y)$ such that $\|y \in X\|\dot{b} \neq \emptyset$. Then $\|y \in X\|\dot{b} \leq \|(\exists z \in y) z \in X\|$ so there exists a $z \in \text{dom}(y)$ such that $\|z \in X\| \cdot \|y \in X\| \cdot b \neq 0$. Since $\rho(z) < \rho(y)$ this is a contradiction. □

## Choice

**Theorem 0.5.3.10.** *$V^B$ makes Choice valid.*

*Proof.* For every set $S \in V$ we have $\|\check{S}$ can be well-ordered$\| = 1$ because that is a $\Sigma_1$ statement in $V$.

Now we prove that for every $X \in V^B$ three exists some

$S$ and $f \in V^B$ such that

$$\|f \text{ is a function on } \check{S} \text{ and range}(f) \supset X\| = 1$$

This will show that $\}X$ can be well ordered$\} = 1$.

Let $S = \text{dom}(x)$ and let $f \in V^B$ be as follows

$\text{dom}(f) = \{(\check{x}, x)^B : x \in S\}$ $f(t) = 1$ for all $t \in \text{dom}(f)$

(where $(a, b)^B = \{\{a\}^B, \{a, b\}^B\}^B$)

These $S$ and $f$ satisfy the condition. $\square$

## 0.6 Forcing Relation

### 0.6.1 Forcing Relation

Let $M$ be a transitive model of ZFC and let $(P, <) \in M$ be a notion of forcing. We will now introduce a forcing language by specifying names and by defining a forcing relation $\Vdash$. We will then prove the fundamental properties of forcing with regard to these names and this relation.

If $(P, <)$ is a notion of forcing we have sen that there is a complete boolean algebra $B(P)$ such that $P$ embeds in $B$ by a mapping $e : P \to B$ such that the image of $P$ is dense and $e$ preserves compatibility and incompatibility.

## Names for a partial order

**Definition 0.6.1.1.** We let $M^P = M^{B(P)}$ where $M^{B(P)}$ is the boolean valued model we defined previously with respect to the boolean algebra $B(P)$. We say the elements of $M^P$ are $\underline{P\text{-names}}$ (we will usually express names with dots over them).

The forcing langauge is the language of set theory with the names as constants. The forcing relation $\Vdash_P$ (or just

$\Vdash$) is defined by

$$p \Vdash \varphi(\dot{a}_1, \ldots, \dot{a}_n) \text{ if and only if } e(p) \leq \|\varphi(\dot{a}_1, \ldots, \dot{a}_n)\|$$

where $\varphi$ is a formula of set theory and $\dot{a}_1, \ldots, \dot{a}_n$ are names.

We will mention that both the names and the forcing relation an be defined directly from $P$ with out reference to $B(P)$.

Recall the properties we wan the forcing relation to satisfy.

## Properties of Forcing

**Theorem 0.6.1.2** (Properties of Forcing)**.** *Let* $(\mathbb{P}, \leq)$ *be a notion of forcing in the ground model* $M$*, and let* $M^{\mathbb{P}}$ *be the class (in* $M$*) of all names. Further let* $\Vdash$ *be the corresponding forcing relation (*$p \Vdash \sigma$ *is read* $p$ *forces* $\sigma$*).*

*Then for all formulas $\varphi, \psi$ in the forcing language*

*(i) (a) If* $p \Vdash \varphi$ *and* $q \leq p$ *then* $q \Vdash \varphi$

*(b) There is no* $p$ *such that* $p \Vdash \varphi$ *and* $p \Vdash \neg\varphi$

*(c) For every* $p$ *there is a* $q \leq p$ *such that either* $q \Vdash \varphi$ *or* $q \Vdash \neg\varphi$.

*(ii) (a)* $p \Vdash \neg\varphi$ *if and only if there is no* $q \leq p$ *such that* $q \Vdash \varphi$.

*(b)* $p \Vdash \varphi \wedge \psi$ *if and only if* $p \Vdash \varphi$ *and* $p \Vdash \psi$

*(c)* $p \Vdash (\forall x)\varphi$ *if and only if* $p \Vdash \varphi(\dot{a})$ *for all* $a \in M^{\mathbb{P}}$

*(d)* $p \Vdash \varphi \vee \psi$ *if and only if* $(\forall q \leq p)(\exists r \leq q) r \Vdash \varphi$ *or* $r \Vdash \psi$

*(e)* $p \Vdash (\exists x)\varphi$ *if and only if* $(\forall q \leq p)(\exists r \leq q)(\exists \dot{a} \in M^{\mathbb{P}}) r \Vdash \varphi(\dot{a})$

*(iii) If* $p \Vdash \exists x \varphi$ *then for some* $\dot{a} \in M^{\mathbb{P}}$, $p \Vdash \varphi(\dot{a})$

*Proof.* (i) (a) If $q \leq p$ then $e(q) \leq e(p)$

(b) $\|\varphi\| \cdot \|\neg\varphi\| = 0$

(c) If $e(p) \cdot \|\varphi\| \neq 0$ then there is a $q \leq p$ such that $e(q) \leq \|\varphi\|$ and similarly if $e(p) \cdot \|\neg\varphi\| \neq 0$

(ii)(a) Left to right: Use (i)(a) and (b). Right to left: If $p$ does not force $\neg\varphi$ then $e(p) \cdot \|\varphi\| \neq 0$ and proceed as in (i)(c).

(b) Immediate

(c) Because our model is full

(iii) Because our model is full. □

## Informal Name for Universe

**Definition 0.6.1.3.** We can also introduce an informal name for $M$ because we know

$$p \Vdash \dot{a} \in \check{M} \text{ if and only if } \forall q \leq p \exists r \leq q \exists x (r \Vdash \dot{a} = \check{x})$$

However notice that in $\check{M}$ isn't a set, but rather we simply use the above as a shorthand for $\dot{a} \in \check{M}$.

### 0.6.2 Forcing Theorem

Now that we know that we can find a boolean valued model which makes the axioms of ZFC valid, we will want an explicit way of representing the two valued analog of the model. Let $G$ be a generic ultrafilter.

**Interpretation by G**

**Definition 0.6.2.1** (Interpretation by $G$)**.** For each $x \in V^B$ we define $x^G$ by induction on $\rho(x)$.

(i) $\emptyset^G = \emptyset$

(ii) $x^G = \{y^G : x(y) \in G\}$

We then let $V[G] = \{x^G : x \in V^B\}$

**Interpretation agrees with Boolean Values on R**

**Lemma 0.6.2.2.** *Let $G$ be an $V$-generic ultrafilter on $B$. Then for all names $x, y \in V^B$*

*(i)* $x^G \in y^G$ *if and only if* $\|x \in y\| \in G$

*(ii)* $x^G = y^G$ *if and only if* $\|x = y\| \in G$

*Proof.* We prove (i), (ii) by induction on the pairs $(\rho(x), \rho(y))$.

(i)

$$\begin{aligned}
\|x \in y\| \in G &\leftrightarrow \exists t \in \text{dom}(y)(y(t) \in G \text{ and } \|x = t\| \in G) \\
&\leftrightarrow \exists t(y(t) \in G \text{ and } x^G = t^G) \\
&\leftrightarrow x^G \in \{t^G : y(t) \in G\} \\
&\leftrightarrow x^G \in y^G
\end{aligned}$$

(ii)

$$\begin{aligned}
\|x = y\| \in G &\leftrightarrow \textstyle\prod_{t \in \text{dom}(x)}(x(t) \Rightarrow \|t \in y\|) \in G \\
&\leftrightarrow (\forall t \in \text{dom}(x))(x(t) \in G \text{ implies } \|t \in y\| \in G) \\
&\leftrightarrow \forall t(x(t) \in G \text{ implies } t^G \in y^G) \\
&\leftrightarrow \{t^G : x(t) \in G\} \subset y^G \\
&\leftrightarrow x^G \subset y^G
\end{aligned}$$

□

## Interpretation agrees with Boolean Values on Fo

**Lemma 0.6.2.3.** *Let $G$ be an $V$-generic ultrafilter on $B$. Then for all $x_1, \ldots, x_n \in V^B$*

$M[G] \models \varphi(x_1^G, \ldots, x_n^G)$ *if and only if* $\|\varphi(x_1, \ldots, x_n)\| \in G$

*Proof.* We prove this by induction on the complexity of $\varphi$.

If $\varphi = \psi \wedge \xi$ or $\neg\psi$ then the result follows by properties of an ultrafilter.

If $\varphi = \exists x \psi(x, \dots)$ then we have

$$\begin{aligned} M[G] \models \exists x \psi(x, \dots) &\leftrightarrow (\exists x \in M[G]) M[G] \models \psi(x, \dots) \\ &\leftrightarrow (\exists x \in M^B) M[G] \models \psi(x^G, \dots) \\ &\leftrightarrow (\exists x \in M^B) \|\psi(x, \dots)\| \in G \\ &\leftrightarrow \textstyle\sum_{x \in M^B} \|\psi(x, \dots)\| \in G \\ &\leftrightarrow \|\exists x \psi(x, \dots)\| \in G \end{aligned}$$

The penultimate equivalence is because if $A = \{\|\psi(x, \dots)\| : x \in M^B\}$ then $A \in M$ and $A \subseteq B$. So since $G$ is generic we have

$$(\exists a \in A) a \in G \text{ if and only if } \sum A \in G$$

□

**Corollary 0.6.2.4.** $V[G] \models ZFC$

*Proof.* Immediate □

## $M[G]$ satisfies needed conditions

**Lemma 0.6.2.5.** *(i) $M \subset M[G]$ and both models have the same ordinals* *(ii)* $G \in M[G]$ *and if* $N \supset M$ *is a transitive model of* $ZFC$ *such that* $G \in N$ *then* $N \supset M[G]$

*Proof.* (i) For every $x \in M$ and generic $G$ we have $\check{x}^G = x$. That every ordinal in $M[G]$ is in $M$ follows from the fact that

$$\|x \text{ is an ordinal}\| = \sum_{\alpha \in Ord} \|x = \check{\alpha}\|$$

(ii) Let $\dot{G}$ be the canonical name for $G$. Then $\dot{G}^G = G$ and so $G \in M[G]$. And if $N \models ZFC$ is a transitive model such that $M \cup \{G\} \subset N$ then the construction can be carried out in $N$ and we get $M[G] \subset N$. □

## 0.7 Independence of CH

### 0.7.1 The Partial Order

### 0.7.2 The Model

### 0.7.3 Preservation of Cardinals

### 0.7.4 Countable Models