Lecture Notes on Forcing at Logic Seminar(Fall 2006)

by Nathanael Leedom Ackerman

February 19, 2007

TALK SLOWLY AND WRITE NEATLY!!

0.1 Introduction

0.1.1 Statement of CH

Today we are going to prove the independence of the Continuum Hypothesis as well as the Axiom of Choice from the other axioms of ZF. First though we need to review some of the concepts of last semester.

0.2 Generic Extensions

0.2.1 Forcing Conditions

Notion of Forcing

Definition 0.2.1.1. Let $M \models ZFC$. We call a partial order $\langle \mathbb{P}, \leq \rangle \in M$ a Notion of Forcing and the elements of P Forcing Conditions.

Compatible Elements

Definition 0.2.1.2. If $p, q \in \mathbb{P}$ and $p < q$ then we say that p is stronger than q. If $(\exists r) r \leq p \wedge r \leq q$ then we say that p and q are Compatible; Otherwise they are Incompatible $(p \perp q)$

Anti-Chain

Definition 0.2.1.3. A set $A \subseteq \mathbb{P}$ is an anti-chain if

$$(\forall p, q \in A)(p \neq q) \rightarrow p \perp q$$

Dense Subset

Definition 0.2.1.4. A set $D \subseteq \mathbb{P}$ is dense if

$$(\forall p \in \mathbb{P})(\exists q \in D) q \leq p$$

Filter

Definition 0.2.1.5. A set $F \subseteq \mathbb{P}$ is a filter on $\mathbb{P}$ if

(i) F is non-empty

(ii) If $p \leq q$ and $p \in F$ then $q \in F$

(iii) If $p, q \in F$ then there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

0.2.2 Generic Extensions

Generic

Definition 0.2.2.1. A set $G \subseteq \mathbb{P}$ is generic over M if

(i) G is a filter

(ii) If $D \subseteq \mathbb{P}$ is dense and $D \in M$ then $G \cap D \neq \emptyset$

Similarly if we have the following definition.

D-Generic

Definition 0.2.2.2. Let $\mathcal{D} \subseteq \{D \subseteq \mathbb{P} : D \text{ is dense}\}$. Then we say $G \subseteq \mathbb{P}$ is $\mathcal{D}$-Generic if

(i) G is a filter

(ii) $(\forall D \in \mathcal{D}) G \cap D \neq \emptyset$

This we have the following theorem

Generics for Countable Collections of Dense Set

Theorem 0.2.2.3. *Let $V \models ZFC$ and let $\langle P, \leq \rangle \in V$ be a partial order and let $\mathcal{D}$ be a countable collection of dense sets with $\mathcal{D} \subseteq V$. Then there is a $\mathcal{D}$-generic ultrafilter G in V.*

Proof. Let $D_1, D_2, \ldots$ be the sets in $\mathcal{D}$. Let $p_0 = p$, and for each n, let p_n be such that $p_n \leq p_{n-1}$ and $p_n \in D_n$. The set

$$G = \{q \in P : q \geq p_n \text{ for some } n \in \mathbb{N}\}$$

is a $\mathcal{D}$ generic filter on P and $p \in G$. □

However, in ZFC this is the best we can do. And in fact we even have the following.

Generics Aren't in the Ground Model

Theorem 0.2.2.4. *Let $M \models ZFC$ and let $\langle \mathbb{P}, \leq \rangle \in M$ be a partial order such that*

$$(\forall p \in \mathcal{P})(\exists q, r \in \mathcal{P})(q \leq p \wedge r \leq p \wedge r \perp q)$$

If $G \subseteq \mathbb{P}$ is a generic filter over M then $G \notin M$.

Proof. If $G \in M$ then $D = \mathcal{P} - G \in M$. Note though that if $p \in \mathcal{P}$ and q, r are as in the condition on the partial ordering then we can't have $q \in G$ and $r \in G$. Hence one of $q \in D$ or $r \in D$. So in particular we have that D is dense.

$\Rightarrow\Leftarrow G \cap D = \emptyset$. □

Forcing Over Finite Axioms

Before we continue it is worth explaining how we will prove the independence of our axioms.

- What we would like is to have a countable model of ZFC because then we know there would be a generic extension of ZFC.

- Explain how instead we look at a large finite fragment containing any contradiction and work there.
- Say that is enough

We are now ready to state one of the main theorems about forcing which we will prove.

The Generic Model Theorem

Theorem 0.2.2.5 (The Generic Model Theorem)**.** *Let $M \models ZFC$ be a transitive model and let $(\mathbb{P}, \leq) \in M$ be a notion of forcing in M. If $G \subseteq \mathbb{P}$ is generic over M then there exists a transitive model $M[G]$ such that*

(i) $M[G] \models ZFC$

(ii) $M \subseteq M[G]$ *and* $G \in M[G]$

(iii) $Ord^{M[G]} = Ord^M$

(iv) If N is a transitive model of ZFC such that $M \subseteq N$ and $G \in N$

We call such a model a Generic Extension and it will be definable from M and the filter G.

0.2.3 Forcing Relation

Forcing Theorem

Theorem 0.2.3.1 (The Forcing Theorem)**.** *Let* $(P, \leq)$ *be a notion of forcing in a ground model* M*. If* σ *is a sentence of the forcing language then fore every* $G \subseteq P$ *generic over* M

$$M[G] \models \sigma \text{ if and only if } (\exists p \in G) p \Vdash \sigma$$

Where the σ *on the left interprets constants according to* G*.*

And the last of our main theorems about forcing is

Properties of Forcing

Theorem 0.2.3.2 (Properties of Forcing)**.** *Let* $(\mathbb{P}, \leq)$ *be a notion of forcing in the ground model* M*, and let*

$M^{\mathbb{P}}$ be the class (in M) of all names. Further let $\Vdash$ be the corresponding forcing relation ($p \Vdash \sigma$ is read p forces σ).

Then for all formulas φ, ψ in the forcing language

(i) (a) If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$

(b) There is no p such that $p \Vdash \varphi$ and $p \Vdash \neg\varphi$

(c) For every p there is a $q \leq p$ such that either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.

(ii) (a) $p \Vdash \neg\varphi$ if and only if there is no $q \leq p$ such that $q \Vdash \varphi$.

(b) $p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$

(c) $p \Vdash (\forall x)\varphi$ if and only if $p \Vdash \varphi(\dot{a})$ for all $a \in M^{\mathbb{P}}$

(d) $p \Vdash \varphi \vee \psi$ if and only if $(\forall q \leq p)(\exists r \leq q) r \Vdash \varphi$ or $r \Vdash \psi$

(e) $p \Vdash (\exists x)\varphi$ *if and only if* $(\forall q \leq p)(\exists r \leq q)(\exists \dot{a} \in M^{\mathbb{P}})r \Vdash \varphi(\dot{a})$

(iii) If $p \Vdash \exists x\varphi$ *then for some* $\dot{a} \in M^{\mathbb{P}}$, $p \Vdash \varphi(\dot{a})$

Boolean Algebra

Definition 0.2.3.3. We say $\langle B, \vee, \cdot, -, 0, 1, \leq\rangle$ is a boolean algebra if

$\langle B, \leq\rangle$ is a lattice

$lub(a, b) = a \vee b$ for all $a, b \in B$

$glb(a, b) = a \cdot b$ for all $a, b \in B$

$a \vee (b \cdot c) = (a \cdot b) \vee (a \cdot c)$ for all $a, b, c \in B$

$a \cdot (b \vee c) = (a \vee b) \cdot (a \vee c)$ for all $a, b, c \in B$

$-a$ is the unique element such that $a \vee (-a) = 1$ and $a \cdot (-a) = 0$

Complete Boolean Algebra

Definition 0.2.3.4. We say a boolean algebra B is complete if for all $A \subseteq B$

The least upper bound of A exists in B (denoted ΣA)

The greatest lower bound of A exists in B (denoted ΠA)

0.3 Boolean Valued Models

0.3.1 Boolean Valued Models

Definition of Boolean Valued Model on Atomic

Definition 0.3.1.1. Let B be a complete boolean algebra and let $L = \{R_i : i \in I\}$ be a relational language. We say that $\Updownarrow = \langle M, \| = \|, \langle \|R_i\| : i \in I\rangle\rangle$ is a B (or boolean valued) model of L if we have (with slight abuse of notation)

(a) For each $R_i \in L$,

$$\|R\| : M^n \to B$$

where n is the arity of R

(b) $\| = \| : M^2 \to B$ such that

(i) $(\forall x \in M)\|x = x\| = 1$

(ii) $(\forall x, y \in M)\|x = y\| = \|y = x\|$

(iii) $(\forall x, y, z \in M)\|x = y\| \cdot \|y = z\| \leq \|x = z\|$

(iv) $(\forall x_1, \ldots, x_n, y_1, \ldots, y_n \in M)(\forall R_i \in L)\|R_i(x_1, \ldots, x_n)\| \cdot \Pi_{i \in i}\|x_i = y_i\| \leq \|R_i(y_1, \ldots, y_n)\|$

Once we've defined a boolean valued model on atomic formulas we can extend the definition in a natural way to arbitrary formula of first order logic.

Definition of Boolean Valued Model on 1st Orde

Definition 0.3.1.2. Let $\Updownarrow = \langle M, \| = \|, \langle \|R_i\| : i \in$

$I\rangle\rangle$ be a B-valued model of a language L for some complete boolean algebra B. We extend $\|\cdot\|$ to be a function from 1st order formulas of L to B as follows:

(a) If $\varphi(\mathbf{x})$ is atomic then use the $\|\cdot\|$ from the definition of B-valued model.

(b) For negation, conjunction, ect

$$\begin{aligned}
\|\neg\varphi(a_1,\ldots,a_n)\| &= -\|\varphi(a_1,\ldots,a_n)\| \\
\|(\varphi\wedge\psi)(a_1,\ldots,a_n)\| &= \|\varphi(a_1,\ldots,a_n)\| \vee \|\varphi(a_1,\ldots,a_n)\| \\
\|(\varphi\vee\psi)(a_1,\ldots,a_n)\| &= \|\varphi(a_1,\ldots,a_n)\| \cdot \|\varphi(a_1,\ldots,a_n)\|
\end{aligned}$$

(c) For quantifiers

$$\begin{aligned}
\|\exists x\varphi(x,a_1,\ldots,a_n)\| &= \Sigma_{a\in M}\|\varphi(a,a_1,\ldots,a_n)\| \\
\|\forall x\varphi(x,a_1,\ldots,a_n)\| &= \Pi_{a\in M}\|\varphi(a,a_1,\ldots,a_n)\|
\end{aligned}$$

We will use the standard short hands of $\varphi(\mathbf{x}) \rightarrow \psi(\mathbf{x}) :=$

$\neg\varphi(\mathbf{x}) \vee \psi(\mathbf{x})$ and $\varphi(\mathbf{x}) \leftrightarrow \psi(x) := (\varphi(\mathbf{x}) \to \psi(\mathbf{x})) \wedge (\psi(\mathbf{x}) \to \varphi(\mathbf{x}))$

Definition of Valid

Definition 0.3.1.3. We say that $\varphi(\overline{a})$ is valid in M if $\|\varphi(\overline{a})\| = 1$. We say that an implication $\varphi(\overline{a}) \to \psi(\overline{a})$ is valid if $\|\varphi(\overline{a})\| \leq \|\psi(\overline{a})\|$ (it is not hard to check that these definitions agree).

Lemma 0.3.1.4. *We have that for each first order formula* φ

$$\|\overline{a} = \overline{b}\| \cdot \|\varphi(\overline{a})\| \leq \|\varphi(\overline{b})\|$$

(where $\|\overline{a} = \overline{b}\| = \Pi_{i \in n} \|a_i = b_i\|$*). And so we have*

$$(\overline{a} = \overline{b}) \wedge \varphi(\overline{a}) \to \varphi(\overline{b})$$

is valid (as we would hope if our definition of $=$ *is correct).*

Proof. Immediate from the definitions. □

Definition Full Models

Definition 0.3.1.5. We say a that a boolean valued model M is full if for all $\overline{a} \in M$ and all 1st order formulas $\exists x \varphi(x, \mathbf{y})$ there exists a $a \in M$ such that

$$\|\varphi(a, \overline{a})\| = \|(\exists x)\varphi(x, \overline{a})\|$$

Full models are important because of the following theorem.

Definition 0.3.1.6. Let F be an ultrafilter on B. Let $\mathcal{M}$ be a full B-valued model. For each 1st order formula $\varphi(\mathbf{x})$ we say that

$\mathcal{M}/F \models \varphi(a_1, \ldots, a_n)$ if and only if $\|\varphi(a_1, \ldots, a_n)\| \in F$

Moding Out By An Ultrafilter

Theorem 0.3.1.7. *If F is an ultrafilter on B and $\mathcal{M}$ is a B-valued model then then $\mathcal{M}/F$ is well define and is a classical (or 2-valued) model*

Proof. Case 1: φ is atomic.

Then the theorem is true by definition.

Case 2: $\varphi = \psi \wedge \eta$

In this case we know $\|\psi \wedge \eta\| \in F$ if and only if $\|\psi\| \in F$ and $\|\eta\| \in F$ because F is an ultrafilter. So we are done by induction.

Case 3: $\varphi = \neg\psi$

In this case we know $\|\neg\psi\| \in F$ if and only if $\|\psi\| \notin F$ because F is an ultrafilter. So we are done by induction.

Case 4: $\varphi = \exists x \psi(x, \dots)$

In this case we will need fullness to choose an $a \in \mathcal{M}$ such that $M \models \|\exists x \psi(x, \dots)\| = \|\psi(a, \dots)\|$. So in particular we have $\|\exists x \psi(x, \dots)\| \in F$ if and only if $(\exists a \in \mathcal{M})\|\psi(a, \dots)\| \in F$. And so by induction we are done. □

An easy way to think of this is we can view an ultrafilter as a boolean algebra homomorphism from B to the boolean algebra 2. Then $\mathcal{M}/F$ is just the induced 2-valued model gotten from applying that homomorphism to $\|\varphi(\mathbf{x})\|$ for each 1st order formula $\varphi(\mathbf{x})$.

0.4 Independence of CH

0.4.1 The Partial Order

0.4.2 The Model

0.4.3 Preservation of Cardinals

0.4.4 Countable Models