1 Background

One of the most fruitful approaches to countable model theory in recent years is the Descriptive set theory approach. The idea is that there is a natural topology on the collection of all countable models of a countable 1st order theory.

To see this we first have to be very precise on what exactly we mean by a countable model. In this case we mean a model on ω.

Definition 1.1. If $L = \langle R_i : i \in \omega \rangle$ such that arity$(R_i) = n_i$ we can define $X_L = \Pi_{i \in \omega} 2^{\omega^{n_i}}$ as the space of all models of L. If $M \in X_L$ we say that $M \models R_i(m_1, m_2, \cdots, m_{n_i})$ iff $\pi_i(M)(m_1, m_2, \cdots, m_{n_i}) = 1$.

Now that we have our space we need to put a topology on it. The topology we we will choose will be the obvious one.

Definition 1.2. The basic open sets of X_L are of the form $\{M : M \models R_i(m_0, \cdots, m_{n_i})\}$ for natural numbers some natural numbers $m_0, \cdots, m_{n_i}$.

Notice that this corresponds to the standard topology on X_L considered as cantor space.

We then have also have that if $S_\infty = \{$automorphisms of $\omega\}$ then there is a natural group action of S_∞ on X_L.

Now that we have our topology, we can ask what properties does it have. As it turns out, the properties which are most important are encapsulated in the fact that X_L is a Polish space.

SEPARATE BLACKBOARD

Definition 1.3. To review a space X with distinguished "Open" sets $O(X)$ is Polish iff:

Topological Space
$O(X)$ is closed under finite intersections.
$O(X)$ is closed under ω unions.
X is Hausdorff

Second Countable
There is a countable set $B \subseteq O(X)$ such that every element of $O(X)$ is a countable union of elements of B.
Metric Space
There is a function $d : X \times X \to R$ such that $\forall x, y, z \in X$

$d(x, x) = 0$
$d(x, y) = d(y, x)$
$d(x, y) + d(y, z) \geq d(x, z)$

Complete
Every Cauchy Sequence Converges.

Now because Polish spaces are metric spaces, they look (in some strong sense) very much like the reals. In particular we have a well known result

Theorem 1.4. *Any Polish space is homeomorphic to a* G_δ *subspace of* $[0, 1]^\omega$

So, if we are going to study uncountable model theory we have to come up with different conditions which generalize a metric space enough to allow us to prove the results we need, which doesn't make explicit use of R and which is satisfied by 2^κ.

This is where L-Polish spaces come in. Now one of the crucial properties of metric spaces which we will try and generalize is that there is a basis of sets which in some sense have a "size" in R (i.e. the basis of all open balls).

Definition 1.5. Let $L = \langle \kappa, < \rangle$ be a linear order on κ. A space X with distinguished "Open" sets $O(X)$ is L-Polish iff:

$O(X)$ is closed under $< \kappa$ intersections.
$O(X)$ is closed under κ unions.
X is Hausdorff

For each $i \in L$ let $\mathcal{C}_i \subseteq O(X)$ be "the open sets of size at least i".
Then

(1)$|\mathcal{C}_i| \leq \kappa$ for each $i \in L$.
(2)$\mathcal{C}_i$ is a cover of X for each $i \in L$.
(3)$\bigcup_{i\in\kappa} \mathcal{C}_i$ is a basis for X
(4)$(\forall e \in \kappa, U \text{ open }, x \in U)(\exists e' \in \kappa)(e' \preceq e) \wedge (\{x\}_{e'} \subseteq U)$
(5)$(\forall e \in \kappa, U \text{ open }, x \in U)(\exists e' \in \kappa, V \in \mathcal{C}_{e'})[(e' \preceq e) \wedge (x \in V) \wedge (\overline{V} \subseteq U)]$
(6)$(\forall e' \prec e' \in L)\mathcal{C}_{e'} \subseteq \mathcal{C}_e$

Complete
Every Cauchy Sequence Converges.

Where

Definition 1.6. Let $H \subseteq X$. $\{H\}_\epsilon = \bigcup\{U \in \mathcal{C}_\epsilon : U \cap H \neq \emptyset\}$

The idea behind the "sizes" of open sets is that we want to generalize the open balls in the case of a metric space. But, we don't want to require that our linear order has an addition on it (and so the triangle inequality won't make sense). Now there is one more thing which needs to be defined, and that is what it means to be a Cauchy Sequence.

Definition 1.7. Let $\langle K, \leq \rangle$ be a linear order. $\{x_k : k \in K\} \subseteq X$ is a Cauchy sequence if $(\forall l \in \kappa)(\exists k \in K)(\forall k', k" > k)(\exists A \in \mathcal{C}_l)$ s.t. $(x_{k'}, x_{k"} \in A)$ and

(*) $(\forall U)[(\exists i \in \kappa)U \supseteq \langle x_j : i < j < \kappa \rangle] \to [(\exists \epsilon \in \kappa, i' \geq i)\{\langle x_j : i < j < \kappa \rangle\}_\epsilon \subseteq U]$

The idea is that a sequence is Cauchy if for all sizes there is a a ball of that size which contains the tail of the sequence. And, if an open set contains the tail of a sequence then there is a size such that the set contains all balls of that size which intersect the sequence.

Definition 1.8. Let $\langle K, \leq \rangle$ be a linear order. $\langle x_k : k \in K \rangle \subseteq X$ converges to $x \in X$ iff $(\forall$ open $A)x \in A \leftrightarrow (\exists k \in K)$ s.t. $(\forall k' > k)x_{k'} \in A$.

We say $\langle x_k : k \in K \rangle \to x$.

We will not prove this here (as we have limited time) but we have the following theorem very useful theorem.

Theorem 1.9. *Let X be L-Polish and $A \subseteq X$. Then $\{x \in X : (\exists \langle x_i : i \in \kappa\rangle \subseteq A)\langle x_i : i \in \kappa\rangle \to x\} = \{x \in X : (\exists \langle x_i : i \in \kappa\rangle \subseteq A)\langle x_i : i \in \kappa\rangle \to x$ and $\langle x_i : i \in \kappa\rangle$ doesn't have to satisfy $(*)\} = \overline{A}$*

Now there is one one more feature of R which is important with regards to the metric spaces. That is that there is a map from ω^* into $(0, \infty)$ which is unbounded towards 0. For example the map $n \to 2^{-n}$.

Theorem 1.10. *Let $L = \langle \kappa, < \rangle$ be a linear order such that there is an injective map $f : \kappa* \rightarrowtail L$ such that $(\forall a \in L)(\exists m \in \kappa*)$ such that $f(m) \preceq a$*
Let $(X, \{\mathcal{C}_i : i \in L\})$ be a $L-$Polish Space and let $G = \bigcap_{i \in \kappa} U_i$ be a G_δ set.

Then, there is a sequence of covers $\{\mathcal{C'}_i : in \in L\}$ of G such that $(G, \mathcal{C'}_i)$ is a Complete $L-$Polish with the topology induced by the topology on X.

Proof. **DRAW A PICTURE OF WHAT IS GOING ON AND HOW IT WORKS IN THE POLISH CASE**
Observe that we can assume that if $m < m'$ $U_m \subseteq U_{m'}$.
Let $\overline{\mathcal{C}_{f(0)}} = \{U : (U \cap U_0 \neq \emptyset) \wedge (U \in \mathcal{C}_\epsilon, \epsilon \preceq f(0_m))\}$
If $\epsilon \notin$range(f) let $\overline{\mathcal{C}_\epsilon} = \{U : \overline{U} \subseteq G, U \in \mathcal{C}_{\epsilon'}, \epsilon' \preceq \epsilon\}$
If $\epsilon = f(m)(m \in M)$ let $\overline{\mathcal{C}_\epsilon} = \{U : \overline{U} \subseteq G, U \in \mathcal{C}_{\epsilon'}, \epsilon' \preceq \epsilon\} \cup \{U : \overline{U} \subseteq U_m, U \in \mathcal{C}_{\epsilon'}, \epsilon' \preceq \epsilon\}$

Let $\overline{\mathcal{C'}_\epsilon} = \bigcup_{\epsilon' \preceq \epsilon} \overline{\mathcal{C}_\epsilon}$

Let $\mathcal{C'}_\epsilon = \{U \cap G : U \in \overline{\mathcal{C'}_\epsilon}\}$

Claim 1.11. *If $U \in \mathcal{C}'_e$ and $e \preceq e'$ then $U \in \mathcal{C}'_{e'}$*

Proof. Immediate from the definition of $\mathcal{C}'_e$. □

Claim 1.12. *$\mathcal{C}'_\epsilon$ is a cover of G.*

Proof. First note that if $x \in U \in \mathcal{C}_\epsilon, \overline{U} \subseteq G$ then $U \in \mathcal{C}'_\epsilon$ by construction.
So lets assume $x \in G - \bigcup\{U : \overline{U} \subseteq G, U \in \mathcal{C}_\epsilon\}$

Let $f(m) \preceq \epsilon$ which we know exists by the definition of f.

Let $x \in A \in \mathcal{C}_{f(m)}$ which must exist as $\mathcal{C}_{f(m)}$ is a cover of X.

So in particular $x \in A \cap U_m$ which is open, and so there is an $\epsilon' \preceq f(m)$ such that $\{x\}_{\epsilon'} \subseteq U_m \cap A$.

Let $x \in B \in \mathcal{C}_{\epsilon'}$

Let $\epsilon" \preceq \epsilon'$ such that $(\exists D \in \mathcal{C}_{\epsilon"})x \in D, \overline{D} \subseteq B$

So, $\overline{D} \subseteq U_m, D \in \mathcal{C}_{\epsilon"}, \epsilon" \preceq f(m)$ so $D \in \overline{\mathcal{C}_{f(m)}} \subseteq \overline{\mathcal{C}'_{f(m)}}$

Hence $x \in D \cap G \in \mathcal{C}'_{f(m)} \subseteq \mathcal{C}'_\epsilon$

So, because x was arbitrary we have that $\mathcal{C}'_\epsilon$ is a cover of G. □

Claim 1.13. *$\bigcup_{e \in L} \mathcal{C}'_e$ is a basis for G.*

Proof. It suffices to show that if $U \in \mathcal{C}_e$ then $U \cap G$ is the union of elements of $\bigcup_{e \in L} \mathcal{C}'_e$. But, if $U \cap G \neq \emptyset$ then by the definition of $\mathcal{C}'_e$ $U \cap G \in \mathcal{C}'_e$ for some e (and in fact for some $e = f(m)$ for some $m \in M$) □

This therefore proves that $(G, \mathcal{C}'_e)$ is a κ-Presize Space.

Claim 1.14. $(\forall e \in \kappa, U \textit{ open in } X, x \in U \cap G)(\exists e' \in \kappa)(e' \preceq e) \wedge ((\{x\}_{e'})^G \subseteq U \cap G)$

Proof. Let e' be such that $(\{x\}_{e'})^X \subseteq U$.
Now notice by condition (3) of being an L-Size Space that $\overline{\mathcal{C}'_{e'}} \subseteq \mathcal{C}_{e'}$ and so $(\{x\}_{e'})^G \subseteq (\{x\}_{e'})^X \cap G \subseteq U \cap G$. □

Claim 1.15. $(\forall e \in \kappa, U \textit{ open }, x \in U \cap G)(\exists e' \in \kappa, V \in \mathcal{C}'_{e'})[(e' \preceq e) \wedge (x \in V) \wedge (\overline{V} \subseteq U \cap G)]$

Proof. First note it suffices to assume that that $U \in \mathcal{C}_e$ for some e (as the covers $\mathcal{C}_i$ for a basis for X)
If $\overline{U} \subseteq G$ then $(\forall B \in \mathcal{C}_\epsilon) B \subseteq U \to B \in \mathcal{C}'_\epsilon$ and so this property is inherited from the same property for $\{\mathcal{C}_i : i \in L\}$
So, assume $\overline{U} \not\subseteq G$

Let $f(m) \preceq e$.
Let $B \in \mathcal{C}_{f(m)}$ such that $x \in B$.
So, $x \in B \cap U_m$ which is open.
And hence, $\exists e' \preceq f(m)$ such that $\{x\}_{e'} \subseteq B \cap U_m$.
Let $D \in \mathcal{C}_{e'}$ such that $x \in D$.
Let $e" \preceq e'$ such that $(\exists E \in \mathcal{C}_{e"}) x \in E, \overline{E} \subseteq D$
So, $\overline{E} \subseteq U_{f(m)}$ and hence $E \in \overline{\mathcal{C}'_{f(m)}}$ and $E \cap G \in \mathcal{C}'_{f(m)}$
And, we have $x \in E \cap G$, $(\overline{E \cap G})^G = \overline{E} \cap G \subseteq U \cap G$ and $E \cap G \in \mathcal{C}'_{e"}$ with $e" \preceq e$.

So we are done. □

We therefore have $(G, \mathcal{C}'_i)$ satisfies everything but completeness.

Claim 1.16. *Let $\{x_k : k \in K\}$ be a Cauchy sequence in $(G, \mathcal{C}'_i)$. Then $\{x_k : k \in K\}$ is a Cauchy sequence in*

$(X, \mathcal{C}_i)$

Proof. This is because $(\forall e \in L)\overline{\mathcal{C}'_e} \subseteq \mathcal{C}_e$. □

Claim 1.17. *If $\{x_k : k \in K\}$ is a Cauchy sequence in $(G, \mathcal{C}'_i)$ then it converges to a point $x \in G$*

Proof. First notice that $\{x_k : k \in K\}$ is a Cauchy sequence in $(X, \mathcal{C}_i)$ and hence converges to a point $x \in X$ (as $(X, \mathcal{C}_i)$ is $L-$Complete).
So, it suffices to show that $x \in U_m$ for all $m \in M$.
5 Notice that $x \in \overline{\{x_k : k \in K\}}$.
Now fix $m \in M$

$(\exists U \in \mathcal{C}'_{f(m)})$ s.t. $(\exists n)(\forall n' > n)x_{n'} \in U$.
But then $\overline{U} \subseteq U_m$ (by the construction of $\mathcal{C}'_{f(m)}$).
But $x \in lim(U) = \overline{U} \subseteq U_m$

So $x \in G$ as m was arbitrary.
But as the definition of convergence only makes reference to the topology, $\{x_k : k \in K\}$ converges to y in $(G, \mathcal{C}'_i)$.
So $(G, \mathcal{C}'_i)$ is $L-$Complete as $\{x_k : k \in K\}$ was an arbitrary Cauchy sequence in G. □

But, we then have that $(G, \mathcal{C}'_i)$ is a Complete $L-$Size Space and we are done. □

2 More Results

Theorem 2.1. *$\overline{H} \subseteq \{H\}_\epsilon$ for $\epsilon \in L$.*

Proof. Let $x \in \overline{H}$.
Let $x \in U, U \in \mathcal{C}_\epsilon$ (which we know exists because $\mathcal{C}_\epsilon$ is a cover of the space.)

$x \in U \to H \cap U \neq \emptyset$.
So, $x \in H_\epsilon$. $\square$

Theorem 2.2. $\bigcap_{i<\kappa} H_i \subseteq \overline{H}$

Proof. Let $x \in \bigcap_{i<\kappa} H_i$
Fix V such that $x \in V$
Let $\epsilon < \kappa$ be such that $\{x\}_\epsilon \subseteq V$.
So $(\exists U \text{ open}) x \in U \in \mathcal{C}_\epsilon$ and $U \cap H \neq \emptyset$ (This must exists because $x \in H_\epsilon$)
So $U \subseteq V$ and hence $V \cap H \neq \emptyset$
Hence $x \in lim(H) = \overline{H}$ as V was arbitrary.
Therefore $\overline{H} \supseteq \bigcap_{i<\kappa} H_i$ (because x, i were arbitrary) $\square$

2.1 L-Polish subspaces are G-delta

Definition 2.3. If $B \subseteq X$ define $diam(B) = \{\epsilon : (\exists U \in \mathcal{C}_\epsilon) U \supseteq B\}$
If Y is a topological space, $x \in Y$ and $f : Y \to X$ is a function then define $osc_f(x) = \bigcup\{diam(f(U)) : x \in U, U$ open$\}$
Let $A^f_\epsilon = \{x \in Y : \epsilon \in osc_f(x)\}$

Lemma 2.4. *For any function $f : X \to Y, \epsilon \in \kappa$ A^f_ϵ is open.*

Proof. $x \in A^f_\epsilon \Leftrightarrow \epsilon \in osc_f(x) \Leftrightarrow \epsilon \in \bigcup\{diam(f(U)) : x \in U, U \text{ open}\} \Leftrightarrow (\exists U_x \ni x)\epsilon \in diam(f(U_x)) \Leftrightarrow (\exists U_x \ni x)(\exists V \in mcC_\epsilon)V \supseteq diam(f(U_x))$
Hence, if $y \in U_x$ we also have $(\exists U_x \ni y)(\exists V \in mcC_\epsilon)V \supseteq diam(f(U_x))$ and so $\epsilon \in osc_f(y)$.
So A^f_ϵ contains an open neighborhood of ever one of it's elements and hence is open. $\square$

Corollary 2.5. *$\{x : osc_f(x) = \kappa\}$ is G_δ*

Proof. $A^f = \{x : osc_f(x) = \kappa\} = \bigcap_{i\in\kappa} A_i^f$ □

Lemma 2.6. *Let $A \subseteq X$. If $f : A \to Y$ is continuous and $a \in A$ then $a \in A^f$*

Proof. Let $V \in \mathcal{C}_e, f(a) \in V$. Then $f^{-1}(V)$ is open and $a \in f^{-1}(V)$ and so $e \in osc_f(a)$ and $a \in A_e^f$.
So because e was arbitrary we have $a \in A^f$. □

Lemma 2.7. *$\langle x_k : k \in K\rangle$ is a Weak Cauchy sequence iff $\bigcup_{i\in K} diam(\langle x_k : i < k \in K\rangle) = \kappa$.*

Proof. $\langle x_k : k \in K\rangle$ is a Weak Cauchy sequence iff $(\forall\epsilon \in \kappa)(\exists i \in K)(\exists U \in \mathcal{C}_\epsilon)(U \supseteq \langle x_k : i < k \in K\rangle)$
iff $(\forall\epsilon \in \kappa)(\exists i \in K)\epsilon \in diam(\langle x_k : i < k \in K\rangle)$
iff $\bigcup_{i\in K} diam(\langle x_k : i < k \in K\rangle) = \kappa$. □

Corollary 2.8. *Let $f : X \to Y$, and let $\langle x_k : k \in K\rangle$ be a Weak Cauchy sequence such that $\langle x_k : k \in K\rangle \to x$ and $x \in A^f$. Then $\langle f(x_k) : k \in K\rangle$ is a weak Cauchy sequence.*

Proof. $e \in osc_f(x) \Rightarrow (\exists U \ni x)$ s.t. $(\exists V \in \mathcal{C}_e)V \supseteq f(U)$
$\Rightarrow (\exists\alpha)(\langle x_k : \alpha < k \in K\rangle \subseteq U \to \langle f(x_k) : \alpha < k \in K\rangle \subseteq V)$ $\Rightarrow (\exists\alpha)e \in diam(\langle f(x_k) : \alpha < k \in K\rangle)$
$\Rightarrow e \in \bigcup_{i\in K} diam(\langle f(x_k) : i < k \in K\rangle)$
So, $\kappa = osc_f(x) \subseteq \bigcup_{i\in K} diam(\langle f(x_k) : i < k \in K\rangle)$
Hence by the previous lemma we have $\langle f(x_k) : k \in K\rangle$ is Weak Cauchy. □

Lemma 2.9. *Let $(Y, \{\mathcal{C}_i : i \in \kappa\})$ be a Hausdorff κ-Presize space. Let $A \subseteq X, f : A \to Y$ be continuous.
If $\langle x_i : i \in K\rangle, \langle z_i : i \in K'\rangle \subseteq A$, $\langle x_i : i \in K\rangle \to x, \langle z_i :$*

$i \in K'\rangle \to x$ and $\langle f(x_i) : i \in K\rangle \to y, \langle f(z_i) : i \in K'\rangle \to y'$ then $y = y'$.

Proof. Assume $y \neq y'$

Let $y \in U, y' \in U', U \cap U' = \emptyset$ and U, U' open (we know this exists because the space is Hausdorff.

So $f^{-1}(U), f^{-1}(U')$ are both open and $f^{-1}(U) \cap f^{-1}(U') = \emptyset$

But, $(\exists e \in K, e' \in K')\langle f(x_i) : e < i \in K\rangle \in U$ and $\langle f(z_i) : e' < i \in K'\rangle \subseteq U'$ (by the definition of convergence)

So $(\exists e \in K, e' \in K')\langle x_i : e < i \in K\rangle \in f^{-1}(U)$ and $\langle z_i : e' < i \in K'\rangle \subseteq f^{-1}(U')$.

Hence $x \in f^{-1}(U)$ and $x \in f^{-1}(U')$ $\Longleftrightarrow$

So $y = y'$. □

Theorem 2.10. *Let $(Y, \{\mathcal{D}_i : i \in \kappa\})$ be a L-Polish space.*
Let $(X, \{\mathcal{C}_i : i \in \kappa\})$ be an L-Polish space with $A \subseteq X$.
If $f : A \to Y$ is continuous then there is a G_δ set G such that $A \subseteq G \subseteq \overline{A}$ and a continuous extension $g : G \to Y$.

Proof. Let $G = \overline{A} \cap \{x : osc_f(x) = \kappa\}$.
G is G_δ because both $\overline{A}$ and $\{x : osc_f(x) = \kappa\}$ are.
Further since f is continuous on a, $A \subseteq G$.
Now let $x \in G$. Since $x \in \overline{A}$ and $\overline{A} = lim(A) = c - lim(A)$ there is a Cauchy sequence $\langle x_i : i \in K\rangle$ such that $\langle x_i : i \in K\rangle \to x$.
And so $\langle f(x_i) : i \in K\rangle$ is a Cauchy sequence and must converge to a unique point y (because Y is complete).
Let $g(x) = y$
First notice that by the previous lemmas this definition

doesn't depend on the sequence.

Claim 2.11. *For all $B \subseteq G$, $g(\overline{B}) \subseteq \overline{g(B)}$.*

Proof. Let $B \subseteq G$. So in particular $\overline{B} \subseteq \overline{A}$ and so $\overline{B} = \overline{B \cap A}$.
Let $x \in \overline{B \cap A}$. So there is a Cauchy sequence $\langle x_i : i \in K\rangle \subseteq B \cap A$ such that $\langle x_i : i \in K\rangle \to x$ (as $lim(B \cap A) = c - lim(B \cap A)$).
Therefore $\langle g(x_i) : i \in K\rangle = \langle f(x_i) : i \in K\rangle \subseteq f(B \cap A) \subseteq g(B \cap A)$ is a Cauchy sequence (because f is continuous)
And, we have $g(x) \in c{-}lim(g(B{\cap}A)) = lim(g(B{\cap}A)) = \overline{g(B \cap A)} \subseteq \overline{g(B)}$
Hence $g(\overline{B}) \subseteq \overline{g(B)}$ because x was arbitrary. □

Claim 2.12. *g is continuous.*

Proof. Let B be closed in Y. It suffices to show that $A = g^{-1}(B)$ is closed in X (for an arbitrary B)
We have $g(A) \subseteq B$ and so $x \in \overline{A} \Rightarrow g(x) \in g(\overline{A}) \subseteq \overline{g(A)} \subseteq \overline{B} = B$.
So $x \in g^{-1}(B) = A$ and so $A = \overline{A}$ □

□

Theorem 2.13. *Let $(Y, \{\mathcal{D}_i : i \in \kappa\})$, $(X, \{\mathcal{C}_i : i \in \kappa\})$ be L-Polish Spaces. Further let $Y \subseteq X$. Then Y is a G_δ subset of X.*

Proof. $id_Y : Y \to Y$ satisfies the conditions of the previous theorem. So, there is a G_δ set G such that $Y \subseteq G \subseteq \overline{Y}$.
But, $Y = c - lim(Y)(= \overline{Y})$ as Y is complete.
So $G = Y$. □