Subject: Notes for my talk at Notre Dame

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1 Background

-So today we are going to construct theories a sequence of theories such the the Scott ranks of these theories are unbounded in ω_1.

-Now, the theories we will be constructing are modifications of Robin Knight's counter example to Vaughts conjecture. And, as the counterexample is incredibly complicated and as very few people have completely gone through it, in this talk I will black box many of the important theorems I need and just give an explanation for what they do and why they are important.

-Before I go on I want to define what it means for two things to be equal up to an ordinal. Suppose $f(x) = \langle \alpha_1(x), \alpha_2(x), \cdots \rangle$ for some function f. We say that $f(x)|\gamma = f(y)|\gamma$ if for all i $\alpha_i(x) = \alpha_i(y) < \gamma$ or $\alpha_i(x) \geq \gamma \wedge \alpha_i(y) \geq \gamma$. This is an informal idea and we will often use it as such.

1.1 Trees

-The basic structure of Knight's model is a tree on the collection finite tuples where extension preserves order.

-To be more precise we define a sequence of predicates P^n on the n-tuples such that if

$P^n(x_1, x_2, \cdots x_n) \leftrightarrow P^n(x_{i_1}, x_{i_2}, \cdots x_{i_n})$
and $P^n(x_1, \cdots x_n) \rightarrow P^n(x_1, \cdots, x_{n-1})$.

-So, in particular, we can think about P as a predicate on $M^{<\omega}$, which is upwards closed among tuples.

Draw picture with arities at each level

-We now want to define the "color" of a tuple $x = \|x\|$ as the height of this tree. Or to be more precise we want

$\neg P(\mathbf{x}) \leftrightarrow -\infty$
$P(\mathbf{x}) \rightarrow \|x\| \geq 0$
$P(\mathbf{x}y) \rightarrow \|\mathbf{x}\| \geq \|\mathbf{x}y\| + 1.$
$\|\mathbf{x}\|$ =sup$\{\alpha : \|\mathbf{x}\| \geq \alpha$ if it exists and ∞ otherwise.

1.1.1 Everything That Can Happen Does

-So, now that we have a way of defining the height of a tuple, we can start to talk about what these models will look like.

-First off, it is worth mentioning that in this talk we only have to consider models where there are no elements of non-wellfounded height, i.e. of color ∞. This is a very nice simplification and is very important to allow us to deal with these models.
-Now, we are going to want these models to be very homogeneous. We are going to want in some sense that everything that can happen will.
-To be more precise we are going to want that for all $\mathbf{x}$ and for all $\langle \gamma_S : S \subseteq |\mathbf{x}|, \gamma_S < |S| \rangle$ there exists a such that $\|\mathbf{x}a\| = \gamma_S$.

Draw a picture of a tree where everything that can happen does

-So, in particular if we define $\text{Spec}(M) = \{\alpha : \exists \mathbf{x} \|\mathbf{x}\| = \alpha\}$ then this homogeneity will give us

$$\text{Spec}(M)|\gamma * \omega = \text{Spec}(N)|\gamma * \omega \Leftrightarrow M \equiv^{gamma*\omega} N$$

-The reason is that essentially if two tuples $\mathbf{x} \in M, \mathbf{y} \in N$ have $\|x\||\gamma\|\mathbf{y}\||\gamma$ then for all extensions of $\mathbf{x}$ we can find an extension of $\mathbf{y}$ with the same color, and vice versa. And we can extend this to a sequence of partial isomorphisms of length at least γ.

Maybe hand wave a little more

-Now, before we go on we will need to define some notation which will be very useful. We say that $C(\mathbf{x})|\gamma = C(\mathbf{y})|\gamma$ if for all subsets $S \subseteq n = |x| = |y| \|\{x_i : i \in S\}\||\gamma = \|\{y_i : i \in S\}\||\gamma$. Essentially what this says is up to γ all the subtuples of $\mathbf{x}$ and $\mathbf{y}$ have the same colors.

1.2 R

-Now, one of the nicest things about the homogeneity of these models is that the above back and forth argument can actually be coded by a predicate inside of the model.

-Or to be more precise we can add a predicate R to our language with the following definition:

Definition 1.1. $R(\mathbf{x}, \mathbf{y}) \leftrightarrow P(\mathbf{x}) \leftrightarrow P(\mathbf{y}) \wedge (\forall a)(\exists b)R(\mathbf{x}a, \mathbf{y}b) \wedge (\forall b)(\exists a)R(\mathbf{x}a, \mathbf{y}b)$

-We now have a our first theorem.

Theorem 1.2. $R(\boldsymbol{x}, \boldsymbol{y}) \rightarrow \|\boldsymbol{x}\| = \|y\|$

Proof. Assume $\|x\| = -\infty$ and $R(\mathbf{x}, \mathbf{y})$. Then $\neg P(x)$ and hence $\neg P(y)$, so $\|y\| = -\infty$.
Assume $\|x\| < \alpha \wedge R(x, y) \rightarrow \|y\| = \|x\|$.

-$R(x, y) \rightarrow (\forall a)(\exists b)R(xa, yb)$.
-But, if $\|x\| = \alpha$, $\|xa\| < \alpha$ and so by assumption $\|yb\| = \|xa\|$.
-Hence, $\|y\| \geq \|x\|$ by the definition of $\| \cdot \|$.
-And, by the symmetry of R we know that $\|y\| \leq \|x\|$

□

-What is more we actually know by the homogeneity that if $\|\mathbf{x}\| < \infty$ and $\|\mathbf{x}\| = \|\mathbf{y}\| \rightarrow R(\mathbf{x}, \mathbf{y})$. This is because if we can find an tuple extending one of them of a given color we can find a tuple extending the other. However, to show this will require a more detailed understanding of the nature of Knight's model then I have time to give.
-But one important consequence of this which is worth mentioning is that if we extend our language to include R we don't actually get any new models.

1.2.1 Infinity

If I have time maybe give a brief description of why the above isn't en

2 Axioms of my theory

2.1 Cells

-We are now ready to talk about the last big part of Knight's paper that we will black box.
-Now first observe while we know what models we want, there is no 1st order way to describe them in the language we have so far. So what need to do is extend the language.
-Knight does this by adding what he calls "Cells". And, these have four nice properties.

1. If A is a cell and $A(\mathbf{x}), A(\mathbf{y})$ then $\|\mathbf{x}\||\omega = \|\mathbf{y}\||\omega$.
2. If A, and B are cells such that it is consistent for B to extend A ($B \leq A$) and $A(x)$ then $\exists \mathbf{y} B(\mathbf{xy})$.
3. (Generalized Saturation) If for all colors $\gamma + n$ there exists a cell B such that if $\|\mathbf{x}\||\gamma{+}n = \|\mathbf{y}\||\gamma{+}n$ $B(\mathbf{x}\overline{a}\overline{b})$ and $B(\mathbf{y}\overline{c})$ then we know $\|\mathbf{x}\overline{a}\||\gamma + n - |\overline{a}| = \|\mathbf{y}\overline{b}\||\gamma + n - |a|$
4. $\nexists \phi$ such that $\phi(\mathbf{x}, \mathbf{y}) \leftrightarrow \|\mathbf{x}\||\gamma = \|\mathbf{y}\||\gamma$ if $\gamma > \omega$.

-One of the main ideas behind the construction is that if two tuples have the same color then there are witnesses which say that they can be extended to have the same color. But, if they don't, then there is no way to know that until you have extended the witnesses down to the finite case (where you actually have a handle on the colors via the cells).

-As we will see, that we can't know when two tuples are equal with out looking at the whole model is very important. And, the fact that can manage to still get the homogeneity of the model with out actually knowing when two tuples have the same color is one of the hardest things of Knight's paper (and is crucial for his proof as well).

2.2 List Axioms T(M)

-We are now ready to present the theory we will use. The language consists of two copies L, L' of the language of Knight's theory Θ with predicates R, Q and constants $c_i : i \in \omega$.

2.2.1 Axioms

$Q(x) \leftrightarrow \bigvee_{i\in\omega} x = c_i$
$Q \cong M$
$\neq Q \models \Theta$

$(\forall x)(\exists c)R(x,c) \wedge Q(c)$
$\neq Q' \models \Theta$
If $(I,I') \leq (J,J')$ (and both are pairs of cells) and $(I,I')(\mathbf{x})$ then $\exists \mathbf{y}(J,J')(\mathbf{xy})$
$P'(\mathbf{x}) \to P(\mathbf{x})$

2.3 Intuition

-So, before we go on, lets talk about what the intuition behind these axioms are. Let $\mathcal{M} \models T(M)$
-First off lets consider the language L. We want $\mathcal{M}|L$ to isomorphic to the model M.
-But, we want to be able to say this without actually naming all the elements. Because, if we name all the elements then we can determine the entire model with a formula of finite quantifier (just say what relations hold on which constants).
-So, what we do is we fix a portion our model and then require the whole model to have the same spectrum (by the R predicate). Hence, because models of the same spectrum are isomorphic we know that $\mathcal{M}|L$ will be isomorphic to M.
-Now lets look at the language L'. We also want our model restricted to L' to be a model of Knight's theory. But we then have

Theorem 2.1. $[(\forall x)P'(x) \to P(x)] \to \|x\| \leq \|x\|'$ *and hence Spec*$(\mathcal{M}|L) \geq$*Spec*$(\mathcal{M}|L')$.

Now, for the axioms which deal with pairs of cells. An important feature of these axioms which I won't prove here (as it makes heavy use of Knight's machinery) is that they allow Generalized Saturation to be extended to pairs of color. Or, to be more precise,

Theorem 2.2 (Generalized Saturation)**.** *If for all colors* $\gamma + n$ *there exists a pair of cells* (B, B') *such that if* $C(x)|\gamma+n = C(\boldsymbol{y})|\gamma+n$, $C'(\boldsymbol{x})|\gamma'+n' = C'(\boldsymbol{y})|\gamma'+n'$, $(B, B')(\boldsymbol{x}\overline{a}\overline{b})$ *and* $(B, B')(\boldsymbol{y}\overline{c})$ *then we know* $C(\boldsymbol{x}\overline{a})|\gamma+n-|\overline{a}| = C(\boldsymbol{y}\overline{b})|\gamma+n-|a|$ *and* $C'(\boldsymbol{x}\overline{a})|\gamma'+n'-|\overline{a}| = C'(\boldsymbol{y}\overline{b})|\gamma'+n'-|a|$
-For all $\boldsymbol{x}$ *and for all* $\langle(\gamma_S, \gamma'_S) : S \subseteq |\boldsymbol{x}|, (\gamma_S, \gamma'_S) < (\|S\|, \|S\|')\rangle$ *there exists* a *such that* $(\|\boldsymbol{x}a\|, \|\boldsymbol{x}a\|') = (\gamma_S, \gamma'_S)$.

-So, to clarify what this means, it is saying that not only for each tuple can we find an extension with every possible color, but we can do it both for colors in L and in L' simultaneously. And, if two tuples have the same colors in both L, L'

then there is a pair of cells which witness that for any extension of one, there is an extension of the other.

3 Result

-Now, once we have all the results above, we can then proceed to prove the main result of this talk. That, our models of $T(M)$ are determined by their restriction to L'. Before we state and prove this it is important to understand why this isn't obvious. It is conceivable that we could have two models each of whose restrictions are the same as each other, but such that they are put together in different ways. For example, one might be our model where every possible pair of things happens, and another might be a model where either for all tuples $\|\mathbf{x}\|' = \|\mathbf{x}\|$ or $-\infty$.
Do a better job of explaining why they could be different, and maybe

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Theorem 3.1. *If* $\mathcal{M}, \mathcal{N} \models T(M)$ *then* $\mathcal{M}|L' \equiv^{\gamma * \omega} \mathcal{N}|L' \leftrightarrow \mathcal{M} \equiv^{\gamma * \omega} \mathcal{N}$

Proof. -To see this lets first assume $\mathcal{M}|L' \equiv^{\gamma * \omega} \mathcal{N}|L'$.
-Now, consider $I_\eta = \{f : M \rightarrow N, |dom(f)| < \omega, \bigwedge_{S \subseteq |dom(f)|} \|S(dom(f))\|'|\eta + |dom(f)| = \|S(range(f))\|'|\eta +$

$|range(f)| \wedge \|S(dom(f)\| = \|S(range(f))\|\}$
-I claim that I_η is a sequence of partial isomorphisms witnessing the equivalence.
-First observe that $\eta < \eta' \to I_{\eta'} \subseteq I_\eta$.
-So all that is needed now is to show that if I have $a \in M, f \in I_{\eta+1}$ then there is a $b \in N$ such that if I let $g(a) = b$ $g \supseteq f$ and $g \in I_\eta$.
-But, if as $\mathcal{M}|L' \equiv^{\gamma*\omega} \mathcal{N}|L'$ we know by the previous results that either Spec(M) = Spec(N) $\leq \gamma * \omega$ or $\gamma * \omega \leq Spec(M), Spec(N)$.
-In the first case $M \cong N$ so we are done. And, in the second case we know every possible extension happens. So, we can just look at all subsets $S(a^\wedge dom(f))$ we can find a b such that in $b^\wedge range(f)$ if the corresponding subset $S(b^\wedge range(f))$ contains b, then such that $S(a^\wedge dom(f))|\eta - 1 + |dom(f)+1| = S(b^\wedge range(f))|\eta - 1 + |range(f) + 1|$. And if a isn't in the subset, then we know by assumption this is already true.
-And, so we are done. $\square$

-But, this is enough to see that if Spec(M) $= -\infty \cup \alpha * \omega^2$ then the quantifier ranks of the models of $T(M)$ are cofinal in $\alpha * \omega^2$ and bounded by $\alpha * \omega^2$

-To see this let us first prove they are bounded.

Well, if $\mathcal{M}, \mathcal{N} \models T(M)$ then Spec($\mathcal{M}$—L'), Spec($\mathcal{N}$—L') $\leq \alpha$. So, $\mathcal{N}|L' \equiv^{\alpha*\omega^2} \mathcal{M}|L' \Rightarrow Spec(\mathcal{N}|L')|\alpha * \omega^2 = Spec(\mathcal{M}|L')|\alpha*\omega^2 \Rightarrow Spec(\mathcal{N}|L') = Spec(\mathcal{M}|L') \Rightarrow \mathcal{N}|L' \cong \mathcal{M}|L' \Rightarrow \mathcal{M} \cong \mathcal{N}$.

-To see that the quantifier ranks of models are cofinal in $\alpha * \omega^2$ assume they aren't. Then they are bounded by some $\gamma < \alpha * \omega^2$. Let $Spec\mathcal{M}|L' = \gamma + 2 * \omega Spec\mathcal{N}|L' = \gamma + 3 * \omega$. Then $\mathcal{M} \equiv^{\gamma+\omega} \mathcal{N}$ but $\mathcal{M} \ncong \mathcal{N}$ which is a contradiction ($\Rightarrow\Leftarrow$).

4 More Results

If I have time, state other results, and why this P predicate is the "ri