Tranferring model-theoretic results about $\mathcal{L}_{\infty,\omega}$ to a Grothendieck topos

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Definition of $\mathcal{L}_{\infty,\omega}$

Definition

For a language $\mathfrak{L}$, $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ is the smallest collection of formulas which contains the language $\mathfrak{L}$ and is closed under:

- $\wedge$, $\vee$, $\neg$, and $\Rightarrow$.
- $(\forall x)$ and $(\exists x)$.
- Infinite disjunctions and conjunctions, so long as the result only has a finite number of free variables.

Why We Care About $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$

Why We Care:

- $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ is natural a generalization of first order logic to infinite formulas.
- Many theorems of first order logic have analogs for $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$. These include the downward Löwenheim-Skolem theorem, compactness and completeness.
- The satisfaction relation for $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ is absolute.

Definition of Grothendieck Topos

Definition

A Grothendieck topos is a category equivalent to the category of sheaves on a site.

In this talk we will restrict our attention to those Grothendieck toposes which are equivalent to sheaves on a countable lattice.

Why We Care About Grothendieck Toposes

Why We Care:

- Grothendieck toposes play an important role in a wide variety of areas ranging from topology to algebraic geometry to higher order logic.
- Grothendieck toposes are closely related to Kripke models as well as the notion of forcing.
- Grothendieck toposes are natural models of intuitionistic set theory.
- Grothendieck toposes have enough structure to interpret formulas of $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$.

Overall Goal

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- Fix an (appropriate) Grothendieck topos G.
- Represent $\mathfrak{L}$-structure in G by models of a sentence in $\mathcal{L}_{\infty,\omega}(\mathfrak{L}')$ in the category of sets.
- Represent sentences of $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ (interpreted in G) by sentences of $\mathcal{L}_{\infty,\omega}(\mathfrak{L}')$ (interpreted in the category of sets).
- Use these representations to prove theorems about $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ interpreted in G that are analogous to those which are known about $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ interpreted in the category of sets.

Main Problem

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- Being a sheaf is a second order property and may not be absolute.
- Therefore being an $\mathfrak{L}$-structure in a category of sheaves may not be absolute.
- As the satisfaction relation of $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ is absolute, $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ may not be able to capture the class of $\mathfrak{L}$-structures in a (fixed) category of sheaves.

Absoluteness And Grothendieck Toposes

Lattice with Covers

Definition

Define a **lattice with covers** $\mathfrak{Lat} = (L, \preceq, \wedge, \vee, \mathrm{Cov}_{\mathfrak{Lat}})$ to consist of the following:

- A distributive lattice $(L, \preceq, \wedge, \vee)$
- For each $p \in L$ a collection of **covering ideals**, $\mathrm{Cov}_{\mathfrak{Lat}}(p)$.

such that for each $p \in L$:

- $\{q : q \preceq p\} \in \mathrm{Cov}_{\mathfrak{Lat}}(p)$.
- For all $C \in \mathrm{Cov}_{\mathfrak{Lat}}(p)$:
  - For all $x \in C$, $x \preceq p$.
  - $p = \bigvee C$.
- If $C \in \mathrm{Cov}_{\mathfrak{Lat}}(p)$ then for any $q \in L$, $C \wedge q := \{x \wedge q : x \in C\} \in \mathrm{Cov}_{\mathfrak{Lat}}(q)$.

In order to simplify things we will assume $\mathfrak{Lat}$ is countable.

Definition of Separated Presheaves

Definition

A **separated presheaf** $\mathcal{S}$ on $\mathfrak{Lat}$ consists of:

- A collection of sets $\{\mathcal{S}(p) : p \in L\}$.
- A collection of commutative functions

$$\{i^{\mathcal{S}}_{p,q} : \mathcal{S}(p) \to \mathcal{S}(q) \text{ where } q \preceq p\}.$$

such that

- For each $C \in \mathrm{Cov}_{\mathfrak{Lat}}(p)$ and $x, y \in \mathcal{S}(p)$:

$$\text{If } (\forall q \in C)\ i_{p,q}(x) = i_{p,q}(y) \text{ then } x = y.$$

Presheaf of Functions

Example

Notice that any $(T, \mathcal{O}(T))$ gives a lattice with covers where the lattice is $(\mathcal{O}(\mathcal{T}), \subseteq)$ and $C \in \mathrm{Cov}_T(U)$ if and only if $\bigcup_{V \in C} V = U$.

For each $U \in \mathcal{O}(T)$ let $\mathcal{F}(U) \subseteq X^U$ where $i_{U,V}(f) = f|_V$. Then $\mathcal{F}$ is a separated presheaf.

We call such a collection of functions a **presheaf of functions** on $(T, \mathcal{O}(\mathcal{T}))$.

Note that being a presheaf of functions is absolute between transitive models of set theory.

Definition of Compatible Collection

Definition

A **compatible collection** in a separated presheaf $\mathcal{S}$ over $p \in L$ is a sequence $\langle x_q : q \in C \rangle$ where:

- $x_q \in \mathcal{S}(q)$.
- If $r \preceq q$ then $i_{q,r}(x_q) = x_r$.

We can think of a compatible collection as a **hole** which may be missing from $\mathcal{S}$.

Definition of Sheaf

Definition

A **sheaf** $\mathcal{S}$ is a separated presheaf such that for each compatible collection $\langle x_q : q \in C \rangle$ there is an $x^* \in \mathcal{S}(p)$ where for all $q \in C$, $i_{p,q}(x^*) = x_q$.

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Example

If $\mathcal{F}$ is a presheaf of functions on $(T, \mathcal{O}(\mathcal{T}))$ then $\mathcal{F}$ is a sheaf if whenever:

$$\langle f_i : U_i \to X, i \in I\rangle \text{ is such that } f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

there is a function $f : \bigcup_{i\in I} U_i \to X$ in $\mathcal{F}(\bigcup_{i\in I} U_i)$ which, for each $i \in I$, agrees with f_i on U_i.

Absolute Grothendieck Topos

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Theorem

There is a category $Sh(\mathfrak{Lat})$ such that:

- *The objects of $Sh(\mathfrak{Lat})$ are the separated presheaves on $\mathfrak{Lat}$.*
- *$Sh(\mathfrak{Lat})$ is equivalent to the category of sheaves on $\mathfrak{Lat}$.*
- *Being an object or morphism of $Sh(\mathfrak{Lat})$ is absolute.*

Encoding With Ordinals

Subobjects

There is a **sheafification** functor, **a**, which takes a separated presheaf and returns the *smallest* sheaf containing it.

Definition

For separated presheaves $A \subseteq B$, we say A is **closed** in B if for all $p \in L$ and for all $x \in B(p)$

$$x \in A(p) \text{ if and only if } \{q : i_{p,q}(x) \in A(q)\} \in \mathrm{Cov}_{\mathfrak{Lat}}(p)$$

Subobjects are the analog of subsets in a Grothendieck topos

Lemma (A.)

There is a one-to-one relationship between

- *Closed subpresheaves of B.*
- *Subobjects of* $\mathbf{a}(B)$ *in the category of sheaves on* $\mathfrak{Lat}$.

Constructing The Closure

The logical operations on presheaves don't always preserve being closed. As such we need the notion of a closure of a subpresheaf.

Definition

Suppose $A \subseteq B$. We define A_α by induction as follows:

- $A_0 := A$.
- $A_{\omega\cdot\beta} := \bigcup_{\gamma<\omega\cdot\beta} A_\gamma$.
- $A_{\alpha+1}(p) := \{x \in B(p) : \{q : i_{p,q}(x) \in A_\alpha\} \in \mathrm{Cov}_{\mathfrak{Cat}}(p)\}$.
- $A_\infty := \bigcup_{\gamma\in\mathrm{ORD}} A_\gamma$.

Lemma

For any $A \subseteq B$, A_∞ is the smallest subpresheaf of B which contains A and is closed in B. We call A_∞ the **closure** *of A in B.*

In an $\mathfrak{L}$-structure in a Grothendieck topos

Formulas are interpreted by subobjects.

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The logical operations are operations on the subobjects.

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The logical operations are operations on the subobjects.

We can use the construction of the closure to define the other logical operations.

Example

Suppose each formula $\varphi_i(\mathbf{x})$ interpreted by closed $X_i \subseteq A$.

Let $X^*(p) := \bigcup_{i \in I} X_i(p)$ for each $p \in L$.

Then $\bigvee_{i \in I} \varphi_i(\mathbf{x})$ is interpreted by X^*_∞.

Note even if each X_i is closed in A, X^* may not be.

Encoding Theorem

Putting these together we have the following theorem.

Theorem (A.)

Fix a countable language $\mathfrak{L}$.

Then there is a language $\mathfrak{L}'$ and a theory $T_{Enc} \in \mathcal{L}_{\omega_1,\omega}(\mathfrak{L}')$ such that models $\mathcal{M} \models T_{Enc}$ encode $\mathfrak{L}$-structures $\mathcal{S}_{\mathcal{M}}$ in $Sh(\mathfrak{Lat})$.

Further, for any sentence $\varphi \in \mathcal{L}_{\omega_1,\omega}(\mathfrak{L})$ there is a sentence $Th_\varphi \in \mathfrak{L}_{\omega_2,\omega}(\mathfrak{L}')$ such that $\mathcal{M} \models Th_\varphi$ if and only if $\mathcal{S}_{\mathcal{M}} \models \varphi$ (in $Sh(\mathfrak{Lat})$).

In this way we can think of Th_φ as encoding the sentence φ.

Encoding Theorem

Corollary (A.)

Suppose $\mathcal{M}$ is an $\mathfrak{L}$-structure in $Sh(\mathfrak{Lat})$, and V_0, V_1 are transitive well-founded models of set theory with $\mathfrak{Lat}, \mathcal{M}, \omega_1 \in V_0 \cap V_1$.

For each $\varphi \in \mathcal{L}_{\omega_1,\omega}(\mathfrak{L}) \cap V_0 \cap V_1$ we have:

$$(\mathcal{M} \models \varphi)^{V_0} \text{ if and only if } (\mathcal{M} \models \varphi)^{V_1}.$$

In particular this implies the **satisfaction relation for $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ in $\mathrm{Sh}(\mathfrak{Lat})$ is absolute**.

Transfer Theorems

Downward Löwenheim-Skolem Theorem

Definition

A separated presheaf S is of **generated size** at most κ if

$$\left|\bigcup_{p \in L} S(p)\right| \leq \kappa$$

Theorem (Downward Löwenheim-Skolem Theorem (A.))

Let $\mathcal{M}$ be an $\mathfrak{L}$-structure in $Sh(\mathfrak{Lat})$, $X \subseteq \mathcal{M}$, and $\varphi \in \mathcal{L}_{\omega_1,\omega}(\mathfrak{L})$. Then there is an $\mathfrak{L}$-structure in $Sh(\mathfrak{Lat})$, $\mathcal{M}_X$, where:

- $X \subseteq \mathcal{M}_X \subseteq \mathcal{M}$.
- $\mathcal{M}_X$ *is of generated size* $|X|$.
- $\mathcal{M}_X \models \varphi$

Downward Löwenheim-Skolem Theorem

Proof.

Let V be a Σ_n-elementary submodel of the universe of sets (for some sufficiently large n) where $X \in V$ and $|V| = |X|$.

Then there is an $\mathfrak{L}$-structure $\mathcal{M}_X$ in $\mathrm{Sh}(\mathfrak{Cat})$ such that $\mathcal{M}_X \in V$, $X \subseteq \mathcal{M}_X$ and $\mathcal{M}_X \models \varphi$.

But then in the actual universe $\mathcal{M}_X$ is of generated size $|X|$ and $\mathcal{M}_X \models \varphi$ as the satisfaction relation is absolute. □

Barwise Compactness

Recall the following theorem in the category of sets

Theorem (Barwise Compactness)

Suppose A is a countable admissible set and $T \subseteq \mathcal{L}_{\omega_1,\omega}(\mathfrak{L}) \cap A$ where:

- *T is Σ_1 definable over A.*
- *For every $T_0 \subseteq T$ which is Δ_1 definable over A there is an $\mathcal{M}$ with $\mathcal{M} \models T_0$.*

Then there is a model which satisfies T.

Sheaf Compactness

Theorem (Sheaf Compactness (A.))

Suppose A is a countable but not locally countable Σ_1-admissible set for which there is a well-ordering Σ_1-definable over A.

If $T \subseteq \mathcal{L}_{\omega_1,\omega}(\mathfrak{L}) \cap A$ is such that:

- *T is Σ_1 definable over A.*
- *For every $T_0 \subseteq T$ which is Δ_1 definable over A there is an $\mathfrak{L}$-structure $\mathcal{M}$ in $\mathit{Sh}(\mathfrak{Lat})$ with $\mathcal{M} \in A$ and $\mathcal{M} \models T_0$.*

Then there is an $\mathfrak{L}$-structure in $\mathit{Sh}(\mathfrak{Lat})$ which satisfies T.

Directed Limits

Theorem (Directed Limits (A.))

Suppose $F \subseteq \mathcal{L}_{\infty,\omega}(\mathfrak{L})$ is a fragment and $\langle \mathcal{M}_i : i \in \kappa \rangle$ is a chain of $\mathfrak{L}$ structures in $Sh(\mathfrak{Lat})$ where for each $i < j$:

- *$\mathcal{M}_i \subseteq \mathcal{M}_j$ with inclusion map $\iota_{i,j}$.*
- *$\iota_{i,j}$ preserves all formulas in F.*

Let $\mathcal{M}^*$ be the directed limit of $\langle \mathcal{M}_i : i \in \kappa \rangle$ in $Sh(\mathfrak{Lat})$. Then each inclusion map $\iota_i^* : \mathcal{M}_i \to \mathcal{M}^*$ preserves each formula in F as well.

Proof.

This follows from the corresponding result for fragments of $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ in the category of sets. □

Thank You!