Computability of 0-1 Laws

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0-1 Law For First Order Logic

Lets begin by reviewing what the 0-1 law for first order logic says

Theorem (0-1 Law For First Order Logic)

Suppose L is a finite language and for any formula φ of L let

$$F_{n,\varphi} = \frac{|\{M \models \varphi : |M| = n\}|}{|\{M : |M| = n\}|}$$

then $\lim_{n\to\infty} F_{n,\varphi}$ is 0 or 1

In other words any formula of first order logic holds asymptotically almost always or asymptotically almost never.

Graphs

Since the discovery of the 0-1 law for first order logic there have been many attempts to generalize it. One of the most successful of which has come from limiting the collection of finite models we consider. For example the following all have 0-1 laws

Theorem (0-1 Law For Finite Graphs)

For any first order formula φ in the language of graphs let

$$G_{n,\varphi} = \frac{|\{G \models \varphi : |G| = n \text{ and } G \text{ is a graph}\}|}{|\{G : |G| = n \text{ and } G \text{ is a graph}\}|}$$

then $\lim_{n\to\infty} G_{n,\varphi}$ is 0 or 1.

Triangle Free Graphs and Bipartite Graphs

Theorem (0-1 Law For Triangle Free Finite Graphs)

For any first order formula φ in the language of graphs let

$$TF_{n,\varphi} = \frac{|\{TF \models \varphi : |TF| = n \text{ and } TF \text{ is a triangle free graph}\}|}{|\{TF : |TF| = n \text{ and } TF \text{ is a triangle free graph}\}|}$$

then $\lim_{n\to\infty} TF_{n,\varphi}$ *is* 0 *or* 1.

Triangle Free Graphs and Bipartite Graphs

Theorem (0-1 Law For Triangle Free Finite Graphs)

For any first order formula φ in the language of graphs let

$$TF_{n,\varphi} = \frac{|\{TF \models \varphi : |TF| = n \text{ and } TF \text{ is a triangle free graph}\}|}{|\{TF : |TF| = n \text{ and } TF \text{ is a triangle free graph}\}|}$$

then $\lim_{n\to\infty} TF_{n,\varphi}$ is 0 or 1.

Theorem (0-1 Law For Bipartite Finite Graphs)

For any first order formula φ in the language of partial orders let

$$BG_{n,\varphi} = \frac{|\{BG \models \varphi : |BG| = n \text{ and } BG \text{ is a partial order}|\}}{|\{BP : |BP| = n \text{ and } BG \text{ is a partial order}|\}}$$

then $\lim_{n\to\infty} BG_{n,\varphi}$ is 0 or 1.

Collections of Finite Structures

Now lets formulate a general version of a 0-1 law.

Definition

A collection of finite L structures $\mathbb{A}$ is called *valid* if for each n there are a non-zero finite number of structures in $\mathbb{A}$ of size n.

Notice we have not assumed that our language is finite. We have just required that we only consider a finite number of models of any given size.

Collections of Finite Structures

Definition

Suppose $\mathbb{A}$ is a valid collection of finite structures. Let

$$\varphi_{\mathbb{A},n} = \frac{|\{M \models \varphi : |M| = n \text{ and } M \text{ is an } L \text{ structure}\}|}{|\{M : |M| = n \text{ and } M \text{ is an } L \text{ structure}\}|}$$

and let $\varphi_{\mathbb{A}} = \lim_{n\to\infty} \varphi_{\mathbb{A},n}$ if it exists and be undefined otherwise.

Definition

We define $01Law(\mathbb{A})$ to be the collection of formulas φ such that $\varphi_{\mathbb{A}} = 1$. We say $\mathbb{A}$ satisfies the *0-1 law* if $01Law(\mathbb{A})$ is a complete theory.

Possible 0-1 Laws

Now can ask, "what types of theories can we get as the result of 0-1 laws for a finite collection of structures?"

Possible 0-1 Laws

Now can ask, "what types of theories can we get as the result of 0-1 laws for a finite collection of structures?"

It turns out that we can get almost any theory.

Theorem (A, Freer, Patel)

Suppose T is a theory. Then the following are equivalent

(a) *There is an $\mathbb{A}$ such that $T = 01Law(\mathbb{A})$*

(b) *For all finite $X \subseteq T$ there is an $n_X \in \omega$ such that for all $m > n_X$ there is a model M of size m which satisfies all elements of X.*

Random Models

In order to find a notion which is restrictive but still captures the cases we have seen we need an observation

Theorem

For each of

- *The collection of finite graphs*
- *The collection of finite triangle free graphs*
- *The collection of finite bipartite graphs*

There is an complete theory such that the above collection of finite models are just the collection of finite submodels of some infinite model of our theory.

We call the (unique) model the "random graph", "random triangle free graph", and the "random bipartite graph" respectively.

0-1 Law for a Theory

This suggests a definition

Definition

Suppose T is a first order theory. Then let $\mathbb{A}(T)$ be the collection of finite substructures of infinite models of T.

Definition

We let $01Law(T)$ be the collection of first order formulas

$$\{\varphi : \varphi_{\mathbb{A}(T)} = 1\}$$

We say T satisfies the *0-1 Law* if $01Law(T)$ is a complete theory.

So $01Law(-)$ is a map from first order theories to first order theories.

Basic Properties of 0-1 Law for a Theory

There are some immediate properties of the $01Law$ function.

Lemma

For any T, $01Law(T)$ has only infinite models.

Lemma

If $\varphi \in T$ and φ is Π_1 then $\varphi \in 01Law(T)$.

Lemma

If T_0 and T_1 satisfy the same Σ_1 and Π_1 sentences then $01Law(T_0) = 01Law(T_1)$.

0-1 Law Is Non-Trivial

The first thing we want to show about the $01Law$ operator is that it is non-trivial. I.e. there are complete theories T such that $01Law(T)$ is complete and $01Law(T) \neq T$.

0-1 Law Is Non-Trivial

The first thing we want to show about the $01Law$ operator is that it is non-trivial. I.e. there are complete theories T such that $01Law(T)$ is complete and $01Law(T) \neq T$.

Theorem

Let

- T_{TFG} *be the theory of the random triangle free graph*
- T_{BiP} *be the theory of the random bipartite random graph*

Then $01Law(T_{TFG}) = 01Law(T_{BiP}) = T_{BiP}$

Computability of the Operator

Now we can ask the main question of the talk:

"If T is a complete computable theory with a 0-1 law, how computable can $01Law(T)$ be?"

Upper Bounds

Lemma

If T is a computable, Σ_1 complete theory then $\mathbb{A}(T)$ is computable.

Proof.

Because the Σ_1 theory determines which atomic types are realized. □

Upper Bounds

Theorem

Suppose T is a Σ_1-complete computable theory. Then $01Law(T)$ is Turing reducible to $0'''$.

Outline.

(1) We know that $\varphi \in 01Law(T)$ if and only if $(\forall\epsilon)(\exists n)(\forall m > n)|\varphi_{\mathbb{A}(T),m} - 1| < \epsilon$

(2) So the relation "$\varphi \in 01Law(T)$" is Π^0_3 and hence computable from $0'''$

□

Upper Bounds

Theorem

Suppose T is a Σ_1-complete computable theory satisfying the 0-1 law. Then $01Law(T)$ is Turing reducible to $0'$.

Proof.

(1) We know that for all first order formulas φ, either $\varphi_{\mathbb{A}} = 0$ or $\varphi_{\mathbb{A}} = 1$.

(2) $\varphi \in 01Law(T) \Leftrightarrow (\exists n)(\forall m > n)|\varphi_{\mathbb{A}(T),m} - 1| < 1/3$. Hence "$\varphi \in 01Law(T)$" is a Σ^0_2 formula.

(3) $\varphi \notin 01Law(T) \Leftrightarrow (\exists n)(\forall m > n)|\varphi_{\mathbb{A}(T),m} - 0| < 1/3$. Hence "$\varphi \in 01Law(T)$" is a Π^0_2 formula.

(4) So "$\varphi \in 01Law(T)$" is Δ^0_2 and hence computable from $0'$

□

Lower Bound

Theorem (Main Theorem)

Suppose B^* is a Δ^0_2 subset of ω. Then there is

- *A computable theory T_{0-1Law} (independent of B^*)*
- *A computable set of formulas Φ (independent of B^*)*
- *A computable complete theory $T_{Basic}(B)$*

such that

- $01Law(T_{Basic}(B))$ *is complete.*
- $01Law(T_{Basic}(B)) = T_{0-1Law} \cup X$ *for an* $X \subseteq \Phi$
- X *is computably isomorphic to* B^*