Invariant Measures Concentrated on Countable Models

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Ultrahomogeneous Structures

A countable structure R is **ultrahomogeneous** if each isomorphism between finite subsets of R can be extended to an automorphism of R.

Example

The complete graph on $\mathbb{N}$ is ultrahomogeneous.

Ultrahomogeneous graphs can be thought of as those graphs which contain a maximal amount of symmetry.

Universal Graphs

A countable graph R is **universal** if every other countable graph is isomorphic to a subgraph of R.

There are $2^{\aleph_0}$ many non-isomorphic countable graphs so it is not obvious that a universal countable graph must exist.

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A countable graph R is **universal** if every other countable graph is isomorphic to a subgraph of R.

There are $2_0^{\aleph}$ many non-isomorphic countable graphs so it is not obvious that a universal countable graph must exist.

However not only is there a universal graph, there is one which is ultrahomogeneous.

Definition

The **Rado Graph** is the (unique) countable universal graph which is ultrahomogeneous.

Construction of the Rado Graph

First fix a function $f : \mathbb{N} \rightarrow \text{PowerSet}_{\text{Finite}}(\mathbb{N})$ such that

(1) $f(n) \subseteq n$ for all $n \in \mathbb{N}$.

(2) For all $X \in \text{PowerSet}_{\text{Finite}}(\mathbb{N})$ there are infinitely many $n \in \mathbb{N}$ such that $f(n) = X$.

We then construct the Rado graph with underlying set $\mathbb{N}$.

Construction:

Stage n: For each $m \in \mathbb{N}$ with $m < n$ our graph $\mathfrak{R}$ has an edge between m and n if and only if $m \in f(n)$.

We can think of this construction proceeding in stages where at each stage we add one more element to the graph.

At Stage n we add the element n to our graph and determine which numbers less than n share an edge with n.

Specifically, n is connected to exactly those edges less than n which are contained in $f(n)$.

Alice Restaurant Property

The way f was defined then guarantees that “everything that can happen does happen”, which in turn is enough to force the constructed graph to be both universal and ultrahomogeneous.

The property that “everything that can happen does”, or that *you can find whatever you want* is sometimes called the **Alice’s Restaurant Property**.

Second Construction of the Rado Graph

We construct a graph with underlying set $\mathbb{N}$.

$G(\mathbb{N}, p)$ Construction:
For each pair (m, n) with $m, n \in \mathbb{N}$ independently flip a coin with heads probability $p \in [0, 1]$.

If the coin comes up heads our graph has an edge between m and n. If the coin comes up tails there is no edge between m and n.

Theorem (Erdös, Rényi 1963)

For any $0 < p < 1$ if R_0 and R_1 are two graphs created using the construction $G(\mathbb{N}, p)$, then with probability one R_0 and R_1 are isomorphic to each other and are isomorphic to the Rado graph, $\mathfrak{R}$.

For this reason the Rado graph is sometimes called the **Random Graph**.

Measures Concentrated on the Rado Graph

We now make this precise.
Let S_{Gr} be the measurable space where:

- The set is $\{\mathcal{G} : \mathcal{G}$ is a graph with underlying set $\mathbb{N}\}$.
- The σ-algebra is generated by the sets
  - $U_{n,m} = \{\mathcal{G}$: In $\mathcal{G}$ there is an edge between m and $n\}$

  (for $m, n \in \mathbb{N}$).

The probabilistic construction, $G(\mathbb{N}, p)$, gives us a probability measure μ_p on S_{Gr} generated by $\mu_p(U_{m,n}) = p$ for every $m, n \in \mathbb{N}$.

The previous theorem then says that for any $0 < p < 1$, μ_p is **concentrated** on (the space of models isomorphic to) the Rado graph. I.e. $\mu_p(\{\mathcal{G} : \mathcal{G} \cong \mathfrak{R}\}) = 1$.

Non-Invariant Measures Concentrated on the Rado Graph

Of course finding a measure concentrated on the Rado graph is easy. Choose a specific representation, $\mathfrak{R}$, of the Rado graph on $\mathbb{N}$. Let $\mu_{\mathfrak{R}}$ be the probability measure where $\mu_{\mathfrak{R}}(\mathfrak{R}) = 1$.

However, while this measure is concentrated on models of the Rado graph, it has the unfortunate property that if $\mathfrak{R}^*$ is a different representation of the Rado graph (i.e. $\mathfrak{R} \cong \mathfrak{R}^*$ but $\mathfrak{R} \neq \mathfrak{R}^*$) then $\mu_{\mathfrak{R}}(\mathfrak{R}^*) = 0$.

Invariant Measures Concentrated on the Rado Graph

What makes the measures μ_p special is that they are **invariant**.

Specifically, if S_∞ is the group of permutations of $\mathbb{N}$, then there is a natural action of S_∞ on S_{Gr} where, for $g \in S_\infty$ and $\mathcal{G} \in S_{\mathrm{Gr}}$, $g \cdot \mathcal{G}$ is the graph on $\mathbb{N}$ obtained by *relabeling* the elements according to g.

We say a measure μ on S_{Gr} is **invariant** if for any Borel subset $A \subseteq S_{\mathrm{Gr}}$ and any $g \in S_\infty$ we have $\mu(A) = \mu(g \cdot A)$.

Invariant Measures Concentrated on the Rado Graph

Theorem

For each $p \in [0,1]$, $G(\mathbb{N}, p)$ is an invariant measure on S_{Gr}.

This follows from fact that each coin in the construction was flipped independently with the same probability. Hence, reorganizing the names for the underlying set has no effect on any of the probabilities.

Invariant Measure

We can think of invariant measures as **abstract probabilistic constructions** which don't depend on the implicit ordering given to the underlying (countable) set.

This suggests that there is a connection between the symmetry of a model and whether or not there exists an invariant measure concentrated on it. In particular we can ask:

Question Are there countable ultrahomogeneous graphs $\mathcal{G}$, other than the Rado graph, for which there is an invariant measure concentrated on those models isomorphic to $\mathcal{G}$?

Complete Graph

There is an easy answer to this question.

Let $\mathfrak{C}$ be the graph on $\mathbb{N}$ where for each m, n there is an edge between m and n. We call this the **complete graph**.

We can think of the construction as having the **complete restaurant property**. I.e. at every stage *you must have everything*.

Theorem

$\mathfrak{C}$ is an ultrahomogeneous graph and μ_1 is an invariant measure concentrated on $\mathfrak{C}$.

Empty Graph

Let $\mathfrak{E}$ be the graph on $\mathbb{N}$ where for each m, n there is not an edge between m and n. We call this the **empty graph**.

We can think of the construction as having the **empty restaurant property**. I.e. at every stage *you don't get to have anything*.

Theorem

$\mathfrak{E}$ is an ultrahomogeneous graph and μ_0 is an invariant measure concentrated on $\mathfrak{E}$.

For over 40 years these were the only known measures concentrated on ultrahomogeneous graphs.

Henson Graph

Definition

The **Henson Graph** is the (unique) countable ultrahomogeneous triangle free graph which is universal for all countable triangle free graphs.

We can construct a representation, $\mathfrak{H}$, of the Henson graph on $\mathbb{N}$ in a manner similar to the Rado graph (and we use the map $f : \mathbb{N} \to \text{Powerset}_{\text{Finite}}(\mathbb{N})$ from that construction).

Henson Graph Construction

Construction:

Stage n, Case 1: There are $m_0, m_1 < n$ with $m_0, m_1 \in f(n)$ and an edge between m_0 and m_1.

There is no edge between any $m < n$ and n.

Stage n, Case 2: Otherwise.

For $m < n$ there is an edge between m and n if and only if $m \in f(n)$.

We can think of the construction as having the **Henson restaurant property**: at every stage *you can find whatever you want, just so long as you don't want a triangle*.

Invariant Measures on the Henson Graph

While the construction of the Henson graph is very similar to that of the Rado graph finding an invariant measure concentrated on it was much harder.

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Theorem (Petrov, Vershik 2010)

For each $n \geq 3$ there when G_n is the unique graph such that:

- *G_n is a countable ultrahomogeneous graph.*
- *G_n is K_n-free.*
- *G_n is universal for countable K_n-free graphs.*

then there is an invariant measure concentrated on G_n

In this paper they also classify all invariant measures on $\mathfrak{R}$ as well as on G_n.

Ultrahomogeneous Graphs

Question Can we classify those countable ultrahomogeneous graphs on which an invariant measure is concentrated?

Ultrahomogeneous Graphs

Question Can we classify those countable ultrahomogeneous graphs on which an invariant measure is concentrated? The first step in answering these questions is the following theorem

Theorem (Lachlan, Woodrow 1980)

If G is a countable ultrahomogeneous graph then either G or the complement of G is isomorphic to one of the following:

- $\mathfrak{R}$, *the Rado graph.*
- $\mathfrak{H}$, *the Henson graph.*
- *The countable ultrahomogeneous universal K_n-free graph for $n > 3$.*
- *The disjoint union of m copies of K_ω, the complete graph ω (for $1 \leq m \leq \omega$).*
- *The countable union of K_n (for $2 \leq n < \omega$).*

Main Question

We will instead answer a more general question.

Question Given an arbitrary countable language L and an arbitrary countable L-structure M can we determine if there is an invariant measure concentrated on models of M?

First though we need to make this question precise.

Relational Languages

Definition

A **first order language** L consists of a set of relations symbols $\mathcal{R}_L$ and function symbols $\mathcal{F}_L$.

An L-**structure** $\mathcal{M}$ consists of

- A set M.
- For each relation $R \in \mathcal{R}_L$ of arity n a subset $R^M \subseteq M^n$.
- For each function $f \in \mathcal{F}_L$ of arity n a function $f^M : M^n \to M$.

We write

$$\mathcal{M} \models R(n_1, \ldots, n_m) \text{ if } \langle n_1, \ldots, n_m \rangle \in R^M$$

and we write

$$\mathcal{M} \models f(n_1, \ldots, n_m) = n^* \text{ if } f^M(n_1, \ldots, n_m) = n^*.$$

Most structures we deal with in mathematics can be thought of as L-structures for an appropriately chosen language L.

In particular if L_{Gr} is the language with a single binary relation E then a graph is the same thing as an L-structures where the relation is symmetric and anti-reflexive.

Polish Space of Models

If L is a countable L-structure then we let S_L be the σ-algebra where:

- The underlying set is $\mathbb{N}$.
- The σ-Algebra is generated by sets of the form

  $$\hat{R}(n_1, \ldots, n_m) = \{\mathcal{M} : \mathcal{M} \models R(n_1, \ldots, n_m)\}$$

  and

  $$\hat{f}(n_1, \ldots, n_m, n^*) = \{\mathcal{M} : \mathcal{M} \models f(n_1, \ldots, n_m) = n^*\}$$

  where $n_1, \ldots, n_m$ and n^* are natural numbers.

The group of permutations of $\mathbb{N}$, S_∞, has a natural action on S_L which is given by *relabeling* the underlying set according to the permutation.

Invariant Measures on Space of Models

If μ is a probability measure on S_L then we say μ is **invariant** if for all $g \in S_\infty$ and all Borel subsets $X \subseteq S_L$ we have

$$\mu(X) = \mu(g \cdot X)$$

(where $g \cdot X = \{g \cdot x : x \in X\}$).

We say such a probability measure is **concentrated** on a countable model $\mathcal{M}$ if

$$\mu(\{\mathcal{N} \in S_L : \mathcal{N} \cong \mathcal{M}\}) = 1.$$

Definable Closure

Definition

If $\mathcal{M}$ is an L-structure with underlying set M then a map $\alpha : M \to M$ is an **automorphism** if

- For all $n_1, \ldots, n_m$, $\mathcal{M} \models R(n_1, \ldots, n_m)$ if and only if $\mathcal{M} \models R(\alpha(n_1), \ldots, \alpha(n_m))$.
- For all $n_1, \ldots, n_m, n*$, $\mathcal{M} \models f(n_1, \ldots, n_m) = n^*$ if and only if $\mathcal{M} \models f(\alpha(n_1), \ldots, \alpha(n_m)) = \alpha(n^*)$.

Definition

Suppose $\mathcal{M}$ is an L-structure and $\mathbf{a}$ is a finite subset of $\mathcal{M}$. The **definable closure** of $\mathbf{a}$, denoted $\mathrm{dcl}(\mathbf{a})$, is defined as

$$\mathrm{dcl}(\mathbf{a}) = \{b \in \mathcal{M} : \text{such that for any automorphism } g \text{ of } \mathcal{M} \text{ if } g(\mathbf{a}) = \mathbf{a} \text{ then } g(b) = b\}.$$

Examples of Group Theoretic Definable Closure

Example

Consider the language with a single successor function and the theory which says every element has a unique successor and predecessor.

Models of this theory are $\mathbb{Z}$ chains. If a is an element of one of these models the definable closure of a is the entire $\mathbb{Z}$ chain it is contained in.

Example

Let Gr be the free group with three generators, a, b, c. The $\mathrm{dcl}(\emptyset) = e$ (the unit) and $\mathrm{dcl}(a) = \{a^n : n \in \mathbb{Z}\}$.

Trivial Definable Closure

Definition

We say a model $\mathcal{M}$ has **trivial definable closure** if for all finite tuples $\mathbf{a} \subseteq \mathcal{M}$ we have $\mathrm{dcl}(\mathbf{a}) = \mathbf{a}$.

Example

$(\mathbb{Q}, \leq)$ has trivial definable closure.

Definable Closure Examples

Example

The Rado graph has trivial definable closure.

To see this suppose $\mathbf{a} \subseteq \mathfrak{R}$ and $b \not\in \mathbf{a}$. We can then find a second element c and an isomorphism $f : \mathbf{a}b \cong \mathbf{a}c$. But then, because $\mathfrak{R}$ is ultrahomogeneous, there is an automorphism f^* of $\mathfrak{R}$ with $f^*(\mathbf{a}b) = \mathbf{a}c$. Hence $b \not\in \mathrm{dcl}(\mathbf{a})$ as f^* fixes $\mathbf{a}$ but doesn't fix b.

Example

If T_3 is the graph consisting of infinitely many triangles then T_3 does not have trivial definable closure.

Main Theorem

Theorem (Ackerman, Freer and Patel)

If L is a countable language and $\mathcal{M}$ is an L-structure then the following are equivalent

- *There is an invariant measure concentrated on $\mathcal{M}$.*
- *$\mathcal{M}$ has trivial definable closure.*

Note that while the statement of the theorem does not appear to involve much logic, and while the special case of ultrahomogeneous graphs dealt with in Petrov-Vershik (2010) can be done in a purely combinatorial manner, the proof of the general case requires the use of several ideas from model theory and infinitary logic.

Ultrahomogeneous Graphs

Which countable ultrahomogeneous graphs have an invariant measure concentrated on them:

- ✓ $\mathfrak{R}$, the Rado graph.
- ✓ $\mathfrak{H}$, the Henson graph.
- ✓ The countable ultrahomogeneous universal K_n-free graph for $n > 3$.
- ✓ The disjoint union of m copies of K_ω, the complete graph ω (for $1 \leq m \leq \omega$).
- × The countable union of K_n (for $2 \leq n < \omega$).

Examples

Example

The structure $(\mathbb{Q}, \leq)$ has trivial definable closure and hence there is an invariant measure concentred on it.

Example

Let $(T, \leq)$ be (any) infinite tree with root r. Then for any element $a \in T$ we have $r \in \mathrm{dcl}(a)$. Hence $(T, \leq)$ does not have trivial definable closure and hence there does not exist an invariant measure concentred on $(T, \leq)$.

Example

If B is the complete bipartite graph then B has trivial definable closure. Hence there is an invariant measure concentrated on B.

Blow Up

Theorem

Every relational structures $\mathcal{M}$ is the quotient of a relational structure $\mathcal{N}$ where there is an invariant measure concentrated on $\mathcal{N}$.

Proof.

Let $\mathcal{N}$ be the structure where:

- The underlying set of $\mathcal{N}$ is $M \times \omega$ (where M is the underlying set of $\mathcal{M}$).
- $\mathcal{N} \models R(\langle a_i, n_1 \rangle, \ldots, \langle a_m, n_m \rangle)$ if and only if $\mathcal{M} \models R(a_1, \ldots, a_m)$.

□

Ages

Definition

An **age** for a language L is a collection of finite subsetsets closed under isomorphism.

Definition

The **age** of a structure $\mathcal{M}$ consists of all finite structures isomorphic to finite subsets of $\mathcal{M}$.

Ages of Models

Theorem (Fraissé)

The following are equivalent for an age K of a language L.

- *K is the age of an L-structure.*
- *K satisfies:*

  (Hereditary Property) *If $A \in K$ and B is a finitely generated substructure of A then $B \in K$.*

  (Joint Embedding Property) *If A and B are in K then there is a C in K such that both A and B are embeddable in C.*

Fraissé's theorem

Theorem (Fraissé)

If K is an age of a countable language L that has the Hereditary Property, the Joint Embedding Property and

> **(Amalgamation Property)** *For any structures A, B and C in K and any injective embeddings $f : A \to B$, $g : A \to C$ there is a D in K and embeddigns $f' : B \to D$ and $g' : C \to D$ such that $f' \circ f = g' \circ g$.*

then there is a (unique) ultrahomogeneous countable structure whose age is K.

Further the age of any ultrahomogeneous structure has the Amalgamation Property.

If an age has the amalgamation property then the unique model is called the **Fraissé Limit** of the age.

Strong Amalgamation

Theorem

If L is a relational language and $\mathcal{M}$ is a countable ultrahomogeneous structure then the following are equivalent.

- *$\mathcal{M}$ has trivial definable closure.*
- *The age of $\mathcal{M}$ satisfies:*

  (Strong Amalgamation Property) *For any structures A, B and C in K and any injective embeddings $f : A \to B$, $g : A \to C$ there is a D in K and injective embeddigns $f' : B \to D$ and $g' : C \to D$ such that $f' \circ f = g' \circ g$.*

Example

Let P be the partial order where:

- The underlying set is $\mathbb{Q} \times n$ for a finite n.
- $(q_0, a_0) < (q_1, a_1)$ if and only if $q_0 < q_1$.

Thank You!