Cute and Cudly Topoi And The Models That Live In Them

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Sketches of an Elephant

Four men, who had been blind from birth wanted to know what an elephant was like, so they asked an elephant-driver for information. he led them to an elephant and invited them to examine it, so one man felt the elephant's leg, another its trunk, another its tail and the fourth its ear. Then they attempted to describe the elephant to one another. The first man said 'The elephant is like a tree'. 'No', said the second, 'the elephant is like a snake'. Nonsense!' said the third, 'the elephant is like a broom'. 'You are all wrong', said the fourth, 'the elephant is like a fan'. And so they went on arguing amongst themselves, while the elephant stood watching them quietly.

Sketches of an Elephant

The notion of a topos is like the elephant in that there are many different ways of describing what a topos is.

In fact the definitive source on topos theory is called “Sketches of an Elephant” by Peter Johnstone.

What is a Topos?

A topos is:

(i) a model of (bounded) intuitionistic set theory.

(ii) (the embodiment of) an intuitionistic higher order theory.

(iii) a category with finite limits and all power-objects.

(iv) a semantics for intuitionistic formal systems.

(v) (the extensional essence of) a first order (infinitary) geometric theory.

(vi) a generalized space.

(vii) a category of sheaves on a site.

(viii) a setting for synthetic differential geometry.

(ix) a setting for synthetic domain theory.

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Example: **SET**

The most basic example of a topos is the category **SET** (or V).

<u>Category **SET**:</u>

The objects are:

All sets.

The morphisms between objects A and B:

All functions with domain A whose range is contained in B.

When trying to understand abstract topos theoretic ideas a good first step is to figure out what they mean in **SET**.

Example: $\mathbf{SET} \times \mathbf{SET}$

Category $\mathbf{SET} \times \mathbf{SET}$:

The objects are:

All pairs of sets $\langle A_0, A_1 \rangle$

The morphisms between objects $< A_0, A_1 \rangle$ and $\langle B_0, B_1 \rangle$ are:

All pairs of functions $\langle f_0, f_1 \rangle$ where $f_0 : A_0 \to B_0$ and $f_1 : A_1 \to B_1$.

This generalizes the category whose objects and maps are arbitrary sequences (of a fixed length) of sets and functions.

Example: Category of Functions

Category of Functions:

The objects are:

Functions between sets.

The morphisms between objects $f : A_0 \to A_1$ and $g : B_0 \to B_1$ are:

All pairs of functions $\langle \alpha_0, \alpha_1 \rangle$ where $\alpha_0 : A_0 \to B_0$, $\alpha_1 : A_1 \to B_1$ and $\alpha_1 \circ f = g \circ \alpha_0$.

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So a map between f and g is a pair of arrows such that

$$\begin{array}{ccc} f : A_0 & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ g : B_0 & \longrightarrow & B_1 \end{array}$$

commutes.

Example: Trees

Category of Trees:

The objects are:

Trees.

The morphisms f between a tree $\langle T_0, \leq_{T_0} \rangle$ and a tree $\langle T_1, \leq_{T_1} \rangle$ are:

Functions $f : T_0 \to T_1$ such that whenever $a \leq_{T_0} b$ then $f(a) \leq_{T_1} f(b)$ and $level(a) = level(f(a))$.

In particular the maps are those functions which preserve the levels of a tree as well as the tree relation.

Example: Sets through $\mathbb{R}^+$-time

Category of Sets through $\mathbb{R}^+$-time:

The objects are:

A set through $\mathbb{R}^+$-time X contains, for each real $r \in \mathbb{R}^+$, a set X_r and for each $s \geq r$ a map $i_{r,s} : X_r \to X_s$.

We can think of the set X_r as the state of each element at time r and the map $i_{r,s}$ as taking an element at time r and returning it's state at time s.

The morphisms f between sets through $\mathbb{R}^+$-time X and Y are:

Collections of functions $f_r : X_r \to Y_r$ indexed by $\mathbb{R}^+$, that commute with i.

So a morphism is a collection of maps such that whenever $r \leq s$

$$\begin{array}{ccc} f_r : X_r & \longrightarrow & Y_r \\ \downarrow & & \downarrow \\ f_s : X_s & \longrightarrow & Y_s \end{array}$$

commutes.

Example: Sets with a group action

Fix a group $(G, \circ)$ with identity e.

<u>Category of Sets with a G-action:</u>

The objects are:

Pair $(X, *)$ where

- X is a set, $* : X \times G \to X$ is a function.
- $(\forall x \in X) x * e = e * x = x$
- $(\forall x \in X)(\forall g, h \in G) x * (g \circ h) = (x * g) * h$

The morphisms f between a $(X, *)$ and $(Y, \Box)$ are:

Functions $f : X \to Y$ such that
$(\forall x \in X)(\forall g \in G) f(x * g) = f(x) \Box g$.

Example: Topology valued sets

We now want to describe one of the most important types of topoi: Topology valued sets.

Specifically we are interested in pairs (X, E_X) where X is a set and E_X a function which takes values which are open sets in a (fixed) topological space T.

The fundamental example to keep in mind is a set of continuous functions from a topological space. I.e. let $B \subseteq \{f | f : \mathbb{R} \to \mathbb{R}, f$ is a partial function$\}$ and let $E_B(f, g) = \{x | f(x) = g(x)\}$

Lets make this precise. Fix a topological space T with collection of open sets $O(T)$.

Example: Topology valued sets

Category of $O(T)$-Sets :

The objects are:

Pairs (X, E_X) satisfying the following (where $d(a) = E_X(a, a)$):

- (symmetry) $(\forall a, b \in X)E_X(a, b) = E_X(b, a)$
- (transitivity) $(\forall a, b, c \in X)E_X(a, b) \cap E_X(b, c) \subseteq E_X(a, c)$
- $(\forall \text{ open } U \subseteq E_X(a, a))(\exists b)E_X(b, a) = d(b) = U$. We call b the "restriction of a to U" and write b as $a|_U$.
- Whenever $\langle a_i : i \in I\rangle$ is a collection such that
  - $E_X(a_i, a_j) = d(a_i) \cap d(a_j)$

  then there exists a unique a such that
  - $d(a) = \bigcup_{i\in I} d(a_i)$
  - $a|_{d(a_i)} = a_i$

Example: Topology valued sets

The morphisms f between a tree (X, E_X) and (Y, E_Y) are:

Functions $f : X \to Y$ such that

- $(\forall a \in X) d(a) = d(f(a))$
- $E_X(a, b) \subseteq E_Y(f(a), f(b))$

Properties of Topoi

Now that we have our examples we can ask "what is a topos?"

Definition

A category is a *topos* if

- It has a terminal object.
- It has all (binary) pullbacks.
- All power-objects.

We won't go into what each of these means, but we will discuss some common properties of all topoi have.

Terminal object

Definition

An object 1 in a category is a *terminal object* if

For all objects X there is a unique map $! : X \to 1$.

Example:

- In **SET** every singleton $\{*\}$ is a terminal object.
- In **SET** $\times$ **SET** every pair of singletons $\{(*, *)\}$ is a terminal object.
- In "sets through $\mathbb{R}^+$-time" if X is a terminal object, $X_r = \{*\}$ is a singleton for each $r \in \mathbb{R}^+$.

Pullbacks

Definition

A *pullback* of $f : A \to C$ and $g : B \to C$ is a triple (D, α, β) where

- $\alpha : D \to A, \beta : D \to B$.
- $g \circ \beta = f \circ \alpha$
- Whenever (X, x, y) is such that
  - $x : X \to A$ and $y : X \to B$.
  - $g \circ y = f \circ x$.

  there is a unique map $\zeta : X \to D$ with $\beta \circ \zeta = g$ and $\alpha \circ \zeta = f$.

Pullbacks

Example:

- In **SET** if $f : A \to C$, $g : B \to C$ then $(A \times_C B, \pi_A, \pi_B)$ is a pullback where

$$A \times_C B = \{(a, b) : f(a) = g(b)\}.$$

It turns out that any category with (binary) pullbacks and a terminal object has all finite limits.

Examples of Pullbacks:Products

Definition

A *product* of A and B is a triple $(A \times B, \pi_A, \pi_B)$ where

- $\pi_A : A \times B \to A, \pi_B : A \times B \to B$.
- Whenever $f : C \to A$ and $g : C \to B$ there is a unique map $(f, g) : C \to A \times B$ with $\pi_A \circ (f, g) = f$ and $\pi_B \circ (f, g) = g$.

Example:

- In **SET** if $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.
- In "sets through $\mathbb{R}^+$-time" $(X \times Y)_r = X_r \times Y_r$.

The evilness of subsets

Definition

A property of a category is *evil* if it is not preserved by isomorphism.

The evilness of subsets

Definition

A property of a category is *evil* if it is not preserved by isomorphism.

Lemma

$\subseteq$ *is an evil relation.*

We want to find "non-evil" concept which captures the idea of the subset relation.

Monics

Definition

A map $m : A \to B$ is a *monic* if whenever $f, g : D \to A$ and $m \circ f = m \circ g$ then $f = g$.

Example:

- In **SET** if a map $m : A \to B$ is monic if and only if it is injective.
- In **SET** $\times$ **SET** a map (f, g) is monic if and only if both f and g are injective.
- In "sets through $\mathbb{R}^+$-time" a map $f : X \to Y$ is monic if and only if $f_r : X_r \to Y_r$ are injective for all $r \in \mathbb{R}^+$.

Examples of Pullbacks:Equalisers

There is another important type of pullback:

Definition

An *equaliser* of $f, g : A \to B$ is a (monic) map $eq : E \to A$ where for all $m : O \to A$:

- If $f \circ m = g \circ m$ then there is a unique u with $eq \circ u = m$.

Example:

- In **SET** if a map $eq : E \to A$ is an equalizer of f and g if and only if eq is an injection and $eq[E] = \{a \in A : f(a) = g(a)\}$

Subobjects

As we have seen the notion of a monic captures the idea of an injection. We are now ready to define a “non-evil” notion of a subset.

Definition

Suppose $m : A \to X$ and $n : B \to X$ are monics. We say $m \leq n$ if there is a monic $i : A \to B$ such that $n \circ i = m$.

Lemma

$m \leq n$ and $n \leq m$ if and only if there is an isomorphism $i : A \to B$ such that $n \circ i = m$.

Definition

A *subobject* of an object A is an equivalence class of monics under the quasi-order $\leq$. We let $Sub(A)$ be the collection of all subobjects of A.

Subobjects

Example:

- In **SET** two monics $m : A \to B$ and $m' : A' \to B$ belong to the same subobject if and only if they have the same image.

  So subobjects are classified by subsets.
- In "sets through $\mathbb{R}^+$-time" a map $f : X \to Y$ is monic if and only if $f_r : X_r \to Y_r$ are injective for all $r \in \mathbb{R}^+$
  SET two monics $m : A \to B$ subobjects of X are indexed by subset $A_r \subseteq X_r$ where $i_{r,s}[A_r] \subseteq A_s$.

We can now ask "what structure must the partial order $(Sub(A), \leq)$ have?"

Heyting Algebras

Definition

A partial order $(H, \leq)$ is *Heyting Algebra* if

- It is a lattice with a largest element $\top$ and a least element $\bot$.
- For all a and b in H the set $\{x : a \wedge x \leq b\}$ has a greatest element (which we refer to as $a \Rightarrow b$).

In any Heyting algebra we let $\neg a = a \Rightarrow \bot$.

Boolean algebras are to Heyting algebras what classical logic is to intuitionistic logic. In particular we have in any Heyting algebra

$$(\forall a) a \wedge \neg a = \bot$$

But we don't necessarily have

$$(\forall a) a \vee \neg a = \top$$

Stone Duality

There is a close relationship between Heyting algebras and topological spaces. In particular we have the following theorem

Theorem (Stone Duality)

There is an equivalence of categories between

- *Sober topological spaces and continuous maps.*
- *Spatial complete Heyting algebras.*

where a topological space gets mapped to its partial order of open sets.

(Don't worry about the definition of sober/spatial except that they are mild "niceness" conditions).

Subobjects and Heyting Algebras

Lemma

In any topos $(Sub(A), \leq)$ *is a Heyting algebra for any object* A.

In this way, as we will see, the "internal logic" of a topos is usually intuitionistic.

However, if $Sub(A)$ happens to be a Boolean algebra for all objects A then the internal logic will be Boolean. In this case we call the topos a *Boolean topos*.

Model in a topos

Suppose $L = \langle \mathcal{F}, \mathcal{R} \rangle$ is a first order language where

- $\mathcal{F}$ is a collection of function symbols such that for each $f \in \mathcal{F}$, $arity(f) < \omega$.
- $\mathcal{R}$ is a collection of Relation symbols such that for each $R \in \mathcal{R}$, $arity(R) < \omega$.

Definition

A *Model* $\mathbb{M}$ *of* L *in a Topos* T consist of

- An object M.
- For each $f \in \mathcal{F}$ of arity n a map $f^{\mathbb{M}} : M^n \to M$
- For each $R \in \mathcal{R}$ of arity n a monic $r : R^{\mathbb{M}} \to M^n$

Terms

We now define terms in a model

- For every $f \in \mathcal{F}$ of arity n, $f^{\mathbb{M}} : M^n \to M$ is a term.
- $\pi_i^n : M^n \to M$, the projections onto the ith coordinate, are terms.
- If $f : M^n \to M^m$ and $g : M^m \to M^s$ are terms, then $g \circ f : M^n \to M^s$ is a term.
- If $f : M^n \to M^m$ and $g : M^n \to M^r$ are terms then so is $(f, g) : M^n \to M^{m+r}$.

Internal quantifier free formula

We now define $\{\varphi(\mathbf{x}) : \mathbf{x} \in M^n\}$ as a subobject of M^n for any quantifier free formula:

- For $R \in \mathcal{R}$ of arity n let $\{\mathbf{x} : R(\mathbf{x})\}^{\mathbb{M}}$ be the subobject containing $R^{\mathbb{M}}$.
- For terms $f, g : M^n \to M^m$ we let $\{\mathbf{x} : f(\mathbf{x}) = g(\mathbf{x})\}^{\mathbb{M}}$ be the subobject of M^n containing the equalizer of f and g.
- If $\varphi(\mathbf{x})$ and $\psi(\mathbf{x})$ are first order formulas of arity n then

$$\{\mathbf{x} : \varphi \to \psi(\mathbf{x})\}^{\mathbb{M}} = \{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathbb{M}} \Rightarrow \{\mathbf{x} : \psi(\mathbf{x})\}^{\mathbb{M}}.$$
$$\{\mathbf{x} : \varphi \wedge \psi(\mathbf{x})\}^{\mathbb{M}} = \{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathbb{M}} \wedge \{\mathbf{x} : \psi(\mathbf{x})\}^{\mathbb{M}}.$$
$$\{\mathbf{x} : \varphi \vee \psi(\mathbf{x})\}^{\mathbb{M}} = \{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathbb{M}} \vee \{\mathbf{x} : \psi(\mathbf{x})\}^{\mathbb{M}}.$$
$$\{\mathbf{x} : \neg\varphi(\mathbf{x})\}^{\mathbb{M}} = \neg\{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathbb{M}}.$$

Internal Π_1 formula

We are now able to say when a model satisfies a Π_1 formulas.

Definition

Suppose $\varphi(\mathbf{x})$ is a formula of arity n and $\mathbb{M}$ is a model in a topos. We say that $\mathbb{M} \models (\forall \mathbf{x})\varphi(\mathbf{x})$ if and only if $\{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathbb{M}} = \top^{Sub(M^n)}$, the maximal subobject of M^n (i.e. $\{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathbb{M}}$ is isomorphic to M^n).

In order to define the satisfaction relation for all first order formulas we first have to take a slight detour.

Adjoint

Suppose $(A, \leq_A)$ and $(B, \leq_B)$ are partial orders.

Definition

Given (order preserving) maps $F : A \to B$ and $G : B \to A$ we say that F is *left adjoint* to G (or G is *right adjoint* to F), written $F \dashv G$, if

$$(\forall a \in A)(\forall b \in B)F(a) \leq b \text{ if and only if } a \leq G(b)$$

In particular we have, whenever $F \dashv G$, that

$$(\forall b \in B)G \circ F \circ G(b) = G(b)$$

$$(\forall a \in A)F \circ G \circ F(a) = F(a)$$

So if $F \dashv G$ we can think of F and G as "almost" inverses.

Pullbacks and Monics

Theorem

Suppose $f : A \to D$, $m : B \to D$, *m is monic and*

$$\begin{array}{ccc} n : X & \longrightarrow & A \\ \downarrow & & \downarrow f \\ m : B & \longrightarrow & D \end{array}$$

is a pullback diagram. Then n is also monic.

Pullbacks and Monics

In particular

Corollary

If $f : A \to B$ then taking the pullback along f gives us a map $f^{-1} : Sub(B) \to Sub(A)$.

Example:

- In **SET** if $m : A \to B$ is a monic with image A^* then $f^{-1}(m)$ is in the same subobject as the inclusion $f^{-1}[A^*]$ into B.

We are now ready to consider quantifiers. Lets look at the category **SET** first.

Left Adjoints and the Existential Quantifier

Let $\pi : A \times B \to A$ be a projection map. We then have a map $\pi^{-1} : \mathcal{P}(A) \to \mathcal{P}(A \times B)$

Lemma

π^{-1} has a left adjoint $\exists_\pi : \mathcal{P}(A \times B) \to \mathcal{P}(A)$

Proof.

If $\exists_\pi \dashv p^{-1}$ then for all $X \subseteq A \times B$ and $Y \subseteq A$

$$\exists_\pi(X) \subseteq Y \Leftrightarrow X \subseteq \pi^{-1}(Y)$$

and hence $a \in \exists_\pi(X)$ if and only if $(\exists b)(a, b) \in X$.

so $(\exists_\pi X) = \{a : (\exists b)(a, b) \in X\}$. □

Right Adjoints and the Universal Quantifier

Lemma

π^{-1} has a right adjoint $\forall_\pi : \mathcal{P}(A \times B) \to \mathcal{P}(A)$

Proof.

If $\pi^{-1} \dashv \forall_\pi$ then for all $X \subseteq A \times B$ and $Y \subseteq A$

$$Y \subseteq \forall_\pi(X) \Leftrightarrow \pi^{-1}(Y) \subseteq X$$

and hence $a \in \forall(X)$ if and only if $\pi^{-1}(\{a\}) \subseteq X$ if and only if $(\forall b)(a, b) \in X$.

So $(\forall_\pi X) = \{a : (\forall b)(a, b) \in X\}$. □

We can now generalize this to arbitrary topoi.

Internal First Order Formulas

Lemma

In any topos, if $\pi : A \times B \to A$ *is a projection map then* $\pi^{-1} : Sub(A) \to Sub(A \times B)$ *has both a left adjoint* $\exists_\pi$ *and and right adjoint* $\forall_\pi$.

We now can define the subobjects of a model associated to an arbitrary first order formula.

Suppose $\mathbb{M}$ is a model,, $\varphi(\mathbf{x})$ is a first order formula of arity n and $\pi_i^n : M^n \to M$ is projection onto the ith coordinate.

- $\{\mathbf{y} : (\forall x_i)\varphi(\mathbf{x})\}^{\mathbb{M}} = \forall_{\pi_i^n}(\{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathbb{M}})$
- $\{\mathbf{y} : (\exists x_i)\varphi(\mathbf{x})\}^{\mathbb{M}} = \exists_{\pi_i^n}(\{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathbb{M}})$

The Satisfaction Relation for Models in a Topos

Finally suppose $\varphi(\mathbf{x})$ is any formula of arity n (possibly 0). Then we say a model $\mathbb{M} \models (\forall\mathbf{x})\varphi(\mathbf{x})$ if either of the following two equivalent conditions are satisfied.

- $\{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathbb{M}}$ is the maximal subobject of M^n (i.e. is isomorphic to M^n).
- $\{\mathbf{x} : (\forall\mathbf{x})\varphi(\mathbf{x})\}^{\mathbb{M}}$ is the maximal subobject of the terminal object 1.

In particular we can think of the "truth value" of a sentence φ as being subobjects of 1 (i.e. elements of $Sub(1)$).

But we have already seen that the lattice of subobjects of an object is a Heyting algebra. Hence we can think of the internal logic of such a model as being intuitionistic.