Trees, Sheaves and Definition by Recursion

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Outline

(1) Trees are instances of sheaves on $\omega + 1$.

(2) "Definition by recursion" can be done from the perspective of sheaves on $\omega + 1$.

(3) We can index recursive definitions by sheaves on certain more general topological spaces.

(4) The Suslin-Kleene Separation Theorem can be extended to other sheaves.

Presheaves on a Topological Space

Definition

Suppose T is a topological space with open sets $\mathcal{O}(T)$. A *presheaf* on T is a functor from $\mathcal{O}(T)^{op}$ into the category of sets.

Specifically if F is a presheaf then

- For each open set $U \in \mathcal{O}(T)$ we have a set $F(U)$.
- For each pair of open sets $U \subseteq V$ with $U, V \in \mathcal{O}(T)$ we have a map $F(i_{U,V}) : F(V) \to F(U)$.
- If $U \subseteq V \subseteq W$ then $F(i_{U,V}) \circ F(i_{V,W}) = F(i_{U,W})$.

Sheaves on a Topological Space

Definition

Suppose F is a presheaf on $\mathcal{O}(T)$. We say that F is a *sheaf* if whenever

- $U = \bigcup_{i \in I} U_i$
- $a_i \in F(U_i)$ for all $i \in I$
- $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ for all $i, j \in I$

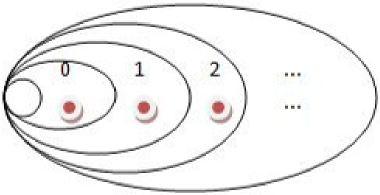
then there is a unique element $a \in F(U)$ such that $a|_{U_i} = a_i$ for all $i \in I$.

$\omega + 1$ As A Topological Space

Definition

Let $\omega + 1$ be the topological space where

- The underlying set is $\omega = \{0, 1, 2, \dots\}$
- Open sets are ordinals $\alpha \in \omega + 1$ (where $\alpha = \{\beta \in \omega : \beta < \alpha\}$)

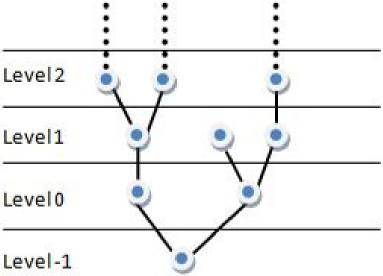


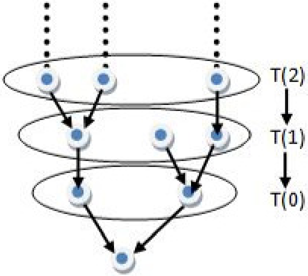
Motivating Observations

We have two important observations about $\omega + 1$:

Theorem

There is a language $L_{TREE} = \{Level_n, \leq_{n,m} : n, m \in \omega\}$ and a first order theory TREE, of trees, such that models of TREE are the "same thing" as presheaves on $\omega + 1$.





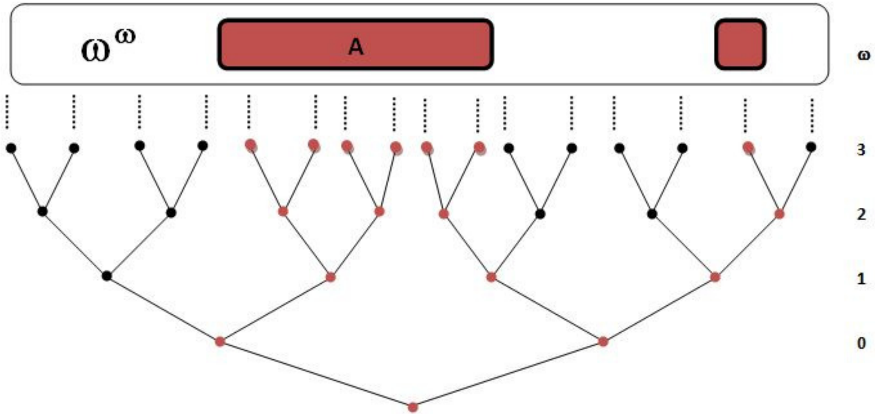
Motivating Observations

Theorem

If $\mathcal{N}$ is the sheaf on $\omega + 1$ where $\mathcal{N}(m) = \{f : m \to \omega\}$ for $m \in \omega + 1$ then

- *For every subpresheaf $\mathcal{B} \subseteq \mathcal{N}$ we have $\mathcal{B}(\omega) \subseteq \omega^\omega$*
- *For every subset $A \subseteq \omega^\omega$ there is a subpresheaf A^* of $\mathcal{N}$ with $A^*(\omega) = A$*
- *A^* is a sheaf if and only if A is a closed subset of ω^ω.*

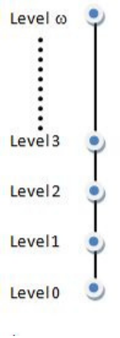
Motivating Observations



Definition of Well-Founded Trees

Definition

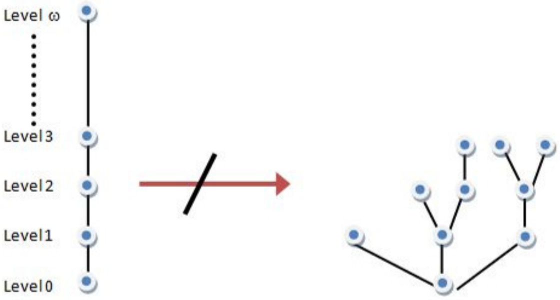
We call the unique sheaf with one element at each element the *terminal sheaf* and denote it by $1_{\omega+1}$.



Definition of Well-Founded Trees

Definition

We say a sheaf T is *well-founded* if there is no map from the terminal tree $1_{\omega+1}$ into T.



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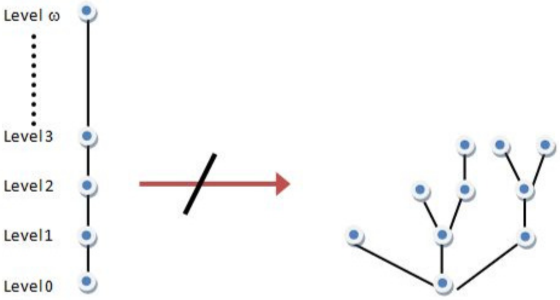


Notice that a sheaf T is well-founded if and only if $T(\omega) = \emptyset$.

Outline

(2) "Definition by recursion" can be done from the perspective of sheaves on $\omega + 1$.

Classical Definition By Recursion

We can now translate the notion of "definition by recursion" into the language of sheaves.

Definition (Recursion)

Suppose $T \subseteq \mathcal{N}$ is a well-founded sheaf, X is a set, and G, F are partial functions such that:

- $G : \bigcup_{n \leq \omega} \mathcal{N}(n) \to X$ and is total on $\bigcup_{n \leq \omega} \mathcal{N}(n) - T(n)$
- F takes two arguments and returns a value in X. The first argument is an element of $\bigcup_{n \leq \omega} \mathcal{N}(n)$ and the second is a partial function $I^* : \bigcup_{n \leq \omega} \mathcal{N}(n) \to X$
- Further if $x \in \mathcal{N}(n)$ and I^* is defined on $\{y \in \mathcal{N}(n+1) : y|_n = x\}$ then $F(x, I^*)$ is defined.

Classical Definition By Recursion

Definition (Recursion Cont.)

We then define the partial functions I_α for $\alpha \in \mathrm{ORD}$ as follows:

- $I_0(x) = G(x)$ if $x \in \bigcup_{n \le \omega} \mathcal{N}(n) - T(n)$ and undefined otherwise.
- $I_{\omega \cdot \alpha} = \bigcup_{\gamma < \omega \cdot \alpha} I_\gamma$.
- $I_{\alpha+1}(x)$ breaks into three cases:
  1. If $I_\alpha(x)$ is defined then $I_{\alpha+1}(x) = I_\alpha(x)$.
  2. Otherwise if $x \in \mathcal{N}(n)$ and $(\forall y \in \mathcal{N}(n+1)) y|_n = x$ implies $I_\alpha(y)$ is defined, then $I_{\alpha+1}(x) = F(x, I_\alpha)$.
  3. Otherwise $I_{\alpha+1}(x)$ is undefined.

We then let $I = \bigcup_{\alpha \in \mathrm{ORD}} I_\alpha$.

Example of Rank

Example

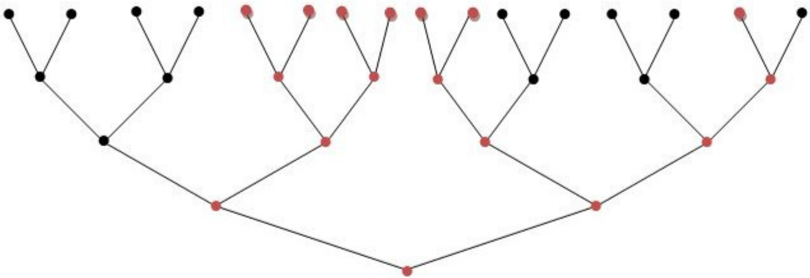
As an example lets suppose $T \subseteq \mathcal{N}$ is a well-founded tree and we want to compute the rank of T, $\rho(r_T)$ (where $r_T \in T(0)$ is the root of T which is represented as the unique function from $\emptyset$ into ω).

To do this we say

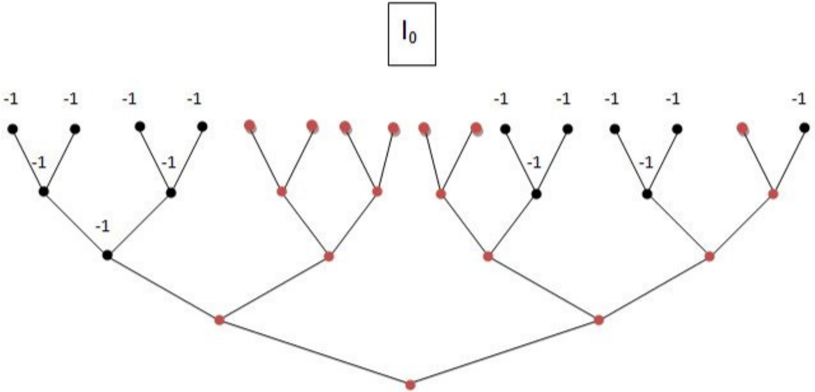
- If $x \in \bigcup_{n\in\omega} \mathcal{N}(n) - T(n)$ then $G(x) = -1$.
- If $x \in T(n)$ and $l^*(y)$ is defined for all $y \in \mathcal{N}(n+1)$ with $y|_n = x$ then we let $F(x, \rho^*) = \sup\{\rho^*(y) + 1 : y \in \mathcal{N}(n+1), y|_n = x\}$.

Then applying the previous method we get a function ρ_T which is total on $T(0)$ and hence defines the rank of the tree T.

Example of Rank



Example of Rank



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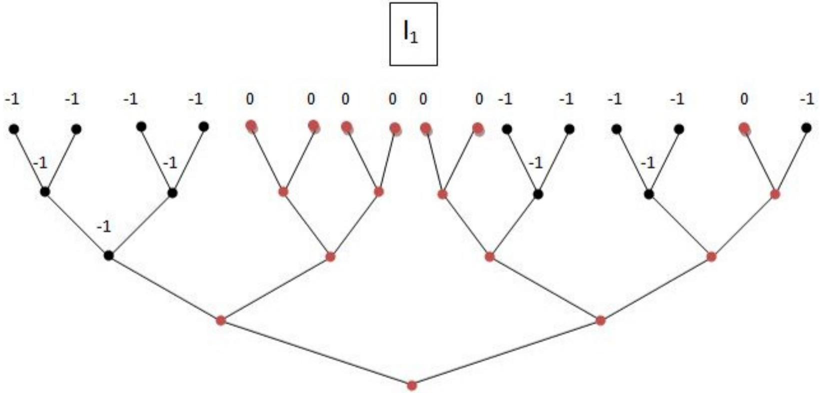

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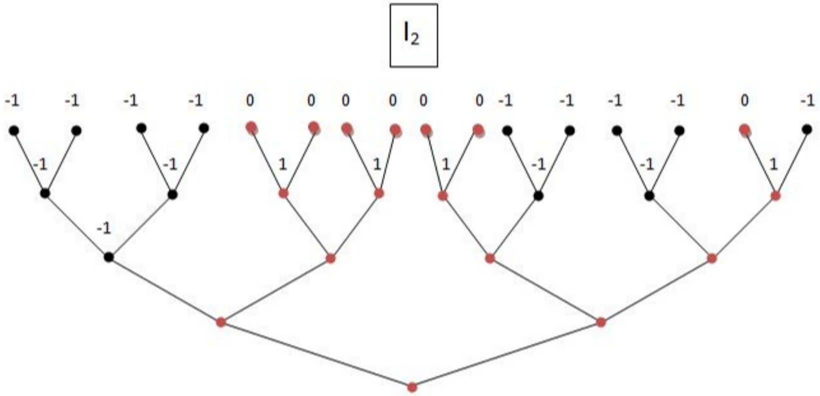

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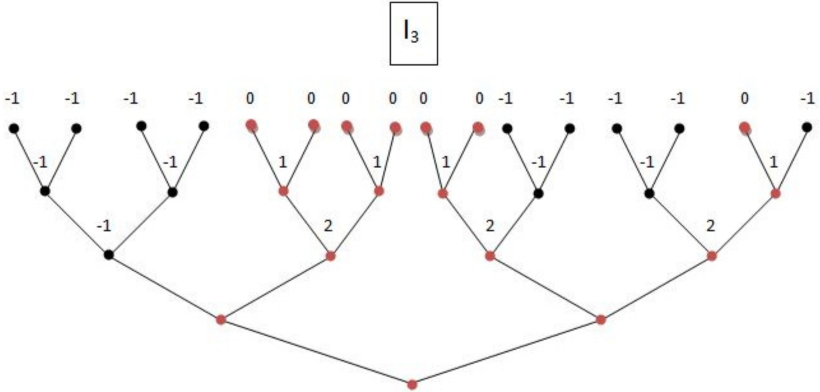

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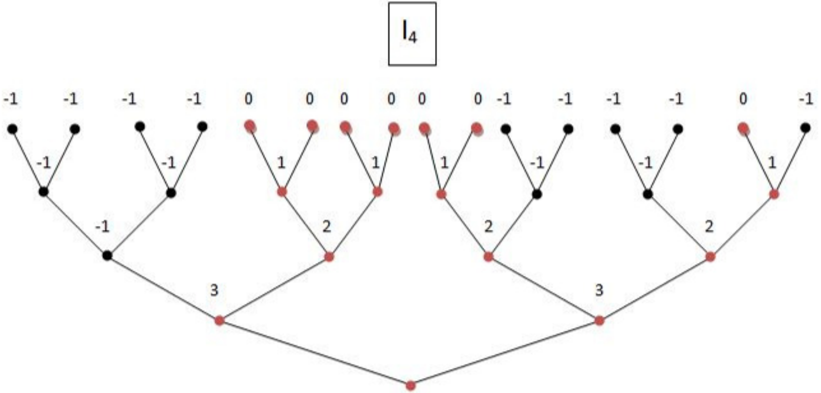

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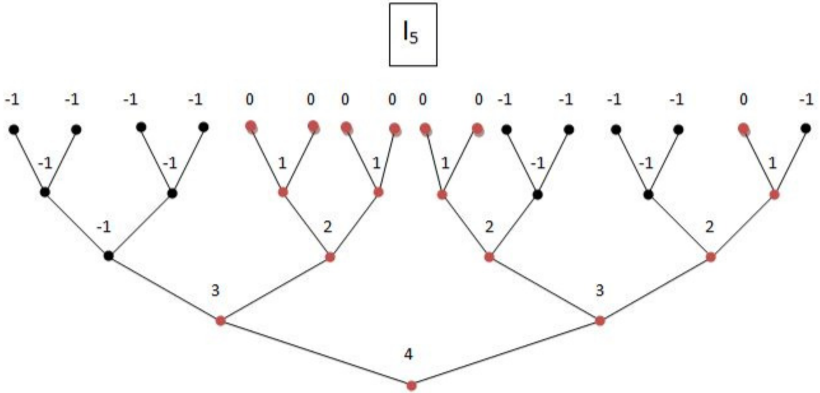


Classical "Definition By Recursion" Is Well-Defined

Theorem

I is a total function on $T(0)$.

Proof

Assume not.

Then there is some $x_0 \in T(0)$ such that $I(x_0)$ is undefined.

Then because of how I was defined at successor stages we must have some $x_1 \in T(1)$ such that $x_1|_0 = x_0$ and $I(x_1)$ is undefined (because otherwise $I(x_0)$ must be defined). Similarly for all $n \in \omega$ we get $x_{n+1} \in T(n+1)$ such that $x_{n+1}|_n = x_n$.

Classical "Definition By Recursion" Is Well-Defined

Proof Cont.

Now because T is a sheaf and $\langle x_n : n < \omega \rangle$ is a compatible collection of elements there must be some element $x^* \in T(\omega)$ such that $x^*|_n = x_n$.

But this contradicts the fact that T is well-founded (i.e. that $T(\omega) = \emptyset$). □

Outline

(3) We can index recursive definitions by sheaves on other topological spaces.

Second Order Tree

We have seen how trees can be thought of as sheaves on the topological space $\omega + 1$. This then suggests the following definitions

Definition

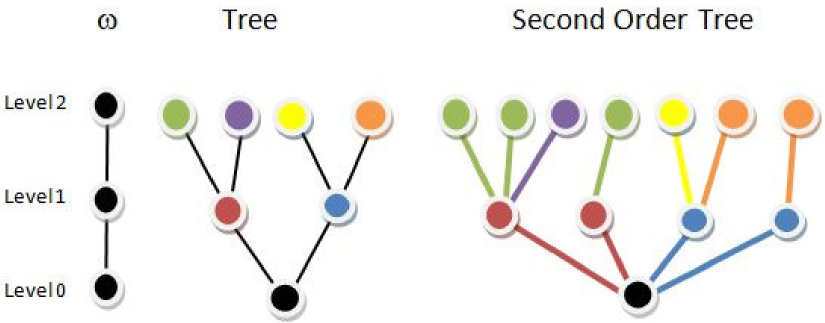
If T is a tree let $\hat{T}$ be the topological space such that

- The underlying set of $\hat{T}$ is T.
- The sets $\hat{x} = \{y \in T : y < x\}$, for $x \in T$, are a sub-basis $\mathcal{O}(\hat{T})$

Definition

If T is a tree and S is a sheaf on $\hat{T}$ then we say that S is a *Second Order Tree*.

Example of a Second Order Trees



An important fact to notice is that if S is a second order tree then S is also a tree.

Well-Founded Sheaves

The classical construction of recursion suggests the following important definitions for a sheaf S on a topological space $(T, \mathcal{O}(T))$.

Definition

We say that S is *well-founded* if $S(T) = \emptyset$.

Definition

We say that S is *flabby* if for every $U \in \mathcal{O}(T)$ and every element $a \in S(U)$ there is an element $a^* \in S(T)$ with $a^*|_U = a$.

Notice that $\mathcal{N}$ is a flabby sheaf.

Example

Example

Let $T = 2^{<\omega}$ be a binary branching tree.

Let S be the sheaf on T generated as follows:

- $S(\emptyset)$ has a single point.
- For any sequence $s \in 2^{<\omega}$, $S(s^\wedge\langle 0\rangle) = \emptyset$.
- For any sequence s, if $x \in S(s)$ then there are ω many $y \in S(s^\wedge\langle 1\rangle)$ such that $y|_s = x$.

We then have

- S is a well-founded second order tree.
- S, treated as a tree, is isomorphic to $2^{<\omega}$.

Recursion on a Second Order Tree

We can give our definition of recursion on sheaves

Definition (Sheaf Recursion)

Suppose T is a tree, N is a flabby sheaf on $\hat{T}$, and $W \subseteq N$ is a well-founded subsheaf. Let X be a set and G, F_V (for each $V \in T$) be partial functions such that:

- $G : \bigcup_{V \in T} N(V) \to X$ and is total on $\bigcup_{V \in T} N(V) - W(V)$.
- F_V takes two arguments and returns a value in X. The first argument is an element of $N(\hat{U})$, where $U <_1 V$, and the second is a partial function $I^* : \bigcup_{V \in T} N(\hat{V}) \to X$.
- Further if $U <_1 V$, $x \in N(\hat{U})$ and I^* is defined on $\{y \in N(\hat{V}) : y|_{\hat{U}} = x\}$ then $F_{\hat{V}}(x, I^*)$ is defined.

Sheaf Recursion

Definition (Sheaf Recursion Cont.)

We next define the partial functions I_α for $\alpha \in \mathrm{ORD}$ as follows:

- $I_0(x) = G(x)$ if $x \in \bigcup_{V \in T} N(\hat{V}) - W(\hat{V})$ and undefined otherwise.
- $I_{\omega \cdot \alpha} = \bigcup_{\gamma < \omega \cdot \alpha} I_\gamma$.
- $I_{\alpha+1}(x)$ breaks into three cases:
  (1) If $I_\alpha(x)$ is defined then $I_{\alpha+1}(x) = I_\alpha(x)$.
  (2) Otherwise if
    - $x \in N(U)$, $U <_1 V$ and $(\forall y \in N(\hat{V}) y|_V = x$ implies $I_\alpha(y)$ is defined,

    then there is a $Q \in T$ with $U <_1 Q$ such that
    - $I_{\alpha+1}(x) = F_Q(x, I_\alpha)$.

  (3) Otherwise $I_{\alpha+1}(x)$ is undefined.

We then let $I = \bigcup_{\alpha \in \mathrm{ORD}} I_\alpha$.

Sheaf Recursion

Theorem

I is a total function on $N(0)$

Proof

Assume the theorem fails.

Then there is some $x_0 \in N(0)$ such that $I(x_0)$ is undefined. By construction we must have $x_0 \in W(0)$.

For each $V \in T(1)$ there must then be an $x_V \in W(\hat{V})$ such that $I(x_V)$ is undefined and $x_V|_0 = x_0$. (Otherwise, because of how I was defined, $I(x_0)$ must have a value).

Sheaf Recursion

Proof Cont.

Repeating this process we have for each $n \in \omega + 1$, $U \in T(n)$ and $V \in T(n+1)$ such that $V|_n = U$, there must then be an $x_V \in W(\hat{V})$ such that $I(x_V)$ is undefined and $x_V|_{\hat{U}} = x_U$. (Otherwise, because of how I was defined, $I(x_U)$ must have a value).

We then have constructed $\langle x_V : V \in T \rangle$ where $x_V \in W(\hat{V})$ for each $V \in T$ and $x_V|_{\hat{U} \cap \hat{V}} = x_{U \cap V} = x_U|_{\hat{U} \cap \hat{V}}$.

So $\langle x_V : V \in T \rangle$ is a compatible collection of elements and hence, because W is a sheaf, there must be an element $x \in W(\bigcup_{V \in T} \hat{V}) = W(\hat{T})$.

But this contradicts the well-foundedness of W. Hence I must be total on $N(0)$. □

Outline

(4) The Suslin-Kleene Separation Theorem can be extended to other sheaves.

κ-Suslin and κ-Borelian Sets

Definition

The κ-*Borelian* subsets of ω^ω, $B_\kappa(\omega^\omega)$, is the smallest collection of sets such that

- All closed subsets of ω^ω are in $B_\kappa(\omega^\omega)$.
- $B_\kappa(\omega^\omega)$ is closed under κ-unions and κ-intersections.

κ-Suslin and κ-Borelian Sets

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Definition

A set $X \subseteq \omega^\omega$ is said to be κ-*Suslin* if there is a closed set $D \subseteq \kappa^\omega \times \omega^\omega$ with $X = \{x : (\exists y \in \kappa^\omega)\ (y, x) \in D\}$.

Suslin-Kleene Separation Theorem

Theorem

Given any two disjoint sets $X, Y \subseteq \omega^\omega$, there is a $|\omega^\omega|$-Borelian set B such that $X \subseteq B$ and $Y \cap B = \emptyset$.

Theorem (Suslin-Kleene Separation Theorem)

Given any two disjoint κ-Suslin $X, Y \subseteq \omega^\omega$, there is a κ-Borelian set B such that $X \subseteq B$ and $Y \cap B = \emptyset$.

Any two disjoint subsets of ω^ω can be separated by a κ-Borelian set for some κ. The size of κ needed though is a measure of the complexity of the sets.

Suslin And Borelian Presheaf

These notions suggest the following definitions

Definition

Suppose N is a sheaf. The κ-*Borelian* subpresheaves of N, $B_\kappa(N)$, is the smallest collection of presheaves such that

- All subsheaves of N are in $B_\kappa(N)$.
- $B_\kappa(N)$ is closed under κ-unions and κ-intersections (taken in the category of presheaves).

Suslin And Borelian Presheaf

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Definition

A presheaf $X \subseteq N$ is said to be K-*Suslin* if there is a sheaf $D \subseteq K \times N$ with $X(U) = \{x : (\exists y \in K(U))\ (y, x) \in D(U)\}$.

Splitting Numbers

Definition

Suppose S is a topological space, $T \subseteq \mathcal{O}(S)$ is a tree such that $\bigcup T = S$, and N a sheaf on S. We then define the *splitting number of N relative to T* to be

$$Split_T(N) = \sup\{|\{f \in N(V) : f|_U = g\}| \text{ such that } U, V \in T, U <_1 V, \text{ and } g \in N(U)\}$$

Suslin-Kleene Separation Theorem on Sheaves

We then get the following variant of the Suslin-Kleene separation theorem.

Theorem

Suppose $X, Y \subseteq N$ are K-Suslin subpresheaves such that $X(S) \cap Y(S) = \emptyset$. Then there is a $Split_T(K \times N)$-Borelian presheaf B such that $X(S) \subseteq B(S)$ and $Y(S) \cap B(S) = \emptyset$.