Relativized Grothendieck Toposes and Potential Map

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Grothendieck Toposes

Presheaves on a Topological Space

Definition

Suppose T is a topological space with open sets $\mathcal{O}(T)$. A *presheaf* on T is a functor from $\mathcal{O}(T)^{op}$ into the category of sets.

Specifically if F is a presheaf then

- For each open set $U \in \mathcal{O}(T)$ we have a set $F(U)$.
- For each pair of open sets $U \subseteq V$ with $U, V \in \mathcal{O}(T)$ we have a map $F(i_{U,V}) : F(V) \to F(U)$
- If $U \subseteq V \subseteq W$ then $F(i_{U,V}) \circ F(i_{V,W}) = F(i_{U,W})$

Example

An example of a presheaf on $\mathbb{R}$ is the collection C of functions to $\mathbb{R}$. That is

- $C(U) = \{f : U \to \mathbb{R}\}$ if U is an open subset of $\mathbb{R}$.
- If $f \in C(V)$ and $U \subseteq V$ then $f|_U$ is the function whose domain is U and which agrees with f on its domain.

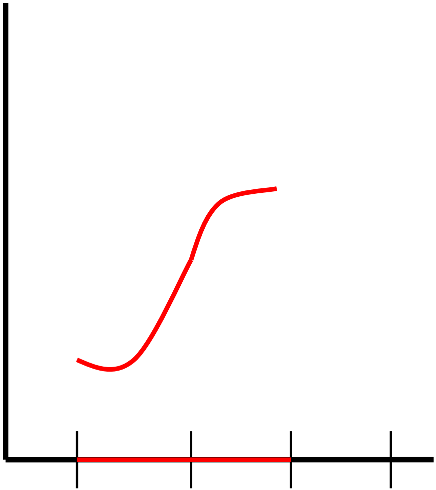
Sheaves on a Topological Space

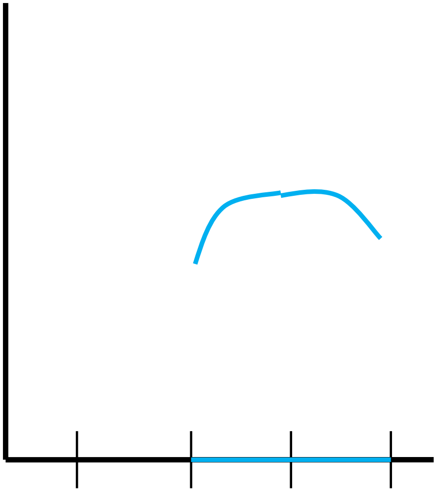
Definition

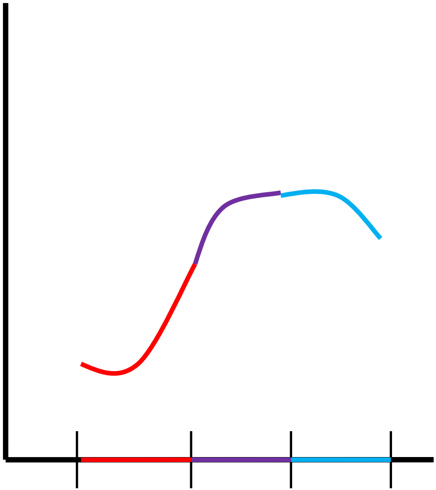
Suppose F is a presheaf on $\mathcal{O}(T)$. We say that F is a *sheaf* if whenever

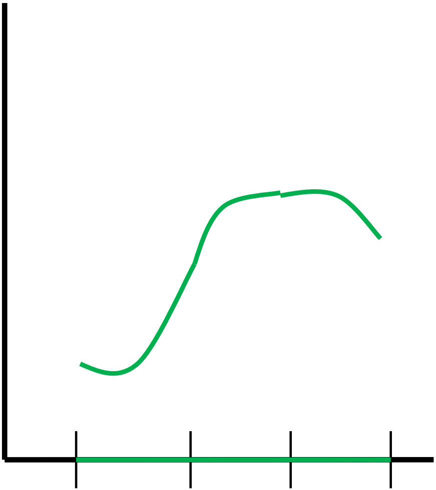
- $U = \bigcup_{i \in I} U_i$
- $a_i \in F(U_i)$ for all $i \in I$
- $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ for all $i, j \in I$

then there is a unique element $a \in F(U)$ such that $a|_{U_i} = a_i$ for all $i \in I$.









Sieves

Definition

Suppose C is a category and $A \in \text{obj}(C)$. A *sieve on* A is a collection of maps S satisfying

- $(\forall f \in S)\text{codom}(f) = A$
- If $f \in S$ and $g : B \longrightarrow \text{dom}(f)$ then $f \circ g \in S$.

We want to think of a sieve on A as a collection of maps which (might) *cover* A.

Sieves on Topological Spaces

Example

Suppose T is a topological space with open sets $\mathcal{O}(T)$. Consider $\mathcal{O}(T)$ as a category. I.e. $U \to V$ if $U \subseteq V$. Then $S \subseteq \mathcal{O}(T)$ is a sieve on U if

- $(\forall V \in S) V \subseteq U$
- If $V \in S$ and $W \subseteq V$ then $W \in S$.

Sites

Definition

A *site* is a pair (C, J_C) where C is a small category and J_C a function from the objects of C to collections of sieves such that for any $A \in \text{obj}(C)$:

- (Identity) $\{f : \text{codom}(f) = A\} \in J_C(A)$
- (Base Change) If $S \in J_C(A)$ and $f : B \to A$ then $f^*S = \{g : f \circ g \in S\} \in J_C(B)$
- (Local Character) For all sieves T on A, if $S \in J_C(A)$ and $(\forall f \in S) f^*T \in J_C(\text{dom}(f))$ then $T \in J_C(A)$

We say elements of $J_C(A)$ *cover* A.

Example

Suppose T is a topological space with open sets $\mathcal{O}(T)$. Consider $\mathcal{O}(T)$ as a category and let $J_{\mathcal{O}(T)}$ be such that

- For any sieve S on U, $S \in J_{\mathcal{O}(T)}(U)$ if and only if $\bigcup S = U$.

Then $(\mathcal{O}(T), J_{\mathcal{O}(T)})$ is a site

So a sieve S covers an open set U if it's union equals U.

This is a good example to keep in mind if you haven't seen sites before.

Definition of a Grothendieck Topos

Definition

If (C, J_C) is a site, let $Sheaf(C, J_C)$ be the category whose objects are sheaves and whose maps are natural transformations.

A *Grothendieck Topos* is any category equivalent to $Sheaf(C, J_C)$ for some site (C, J_C).

Notice that we don't require a Grothendieck topos to equal the category of sheaves on a site, just that it was equivalent to one.

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Question: Why do we care about Grothendieck toposes?

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- For any fixed group G the collection of sets with a G-action is a Grothendieck topos.
- The category of simplicial sets is a Grothendieck topos.
- The Boolean valued model of sets obtained while forcing is (almost) a Grothendieck topos.

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Answer: They have very nice properties.

- Grothendieck toposes are complete, co-complete, have exponentials and a subobject classifier.
- Given any small category C, it's free co-completion is a Grothendieck topos. (i.e. the category obtained by freely adjoining arbitrary co-products and co-equalizers).
- They are generalized topological spaces (and many of the notions from topology extend to toposes).

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In particular they provide examples of *unusual* mathematical universes.

- There is a Grothendeick topos where infitesimals exist.
- There is a Grothendieck topos where Brouwer's "Theorem":

  "All functions from the reals to the reals are continuous"

  holds.

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- Every Grothendieck topos is a model of higher order type theory.
- Grothencieck toposes have enough structure to contain models of $\mathcal{L}_{\infty,\omega}$.
- Grothendieck toposes “classify” the geometric theories of $\mathcal{L}_{\infty,\omega}$ (note classically every theory of $\mathcal{L}_{\infty,\omega}$ has a conservative expansion which is geometric).
- Kripke models of intuitionisitc logic are the just models in an appropriate Grothendieck topos.

Relativization

Inner Models

Throughout this talk we will assume we are working inside a model $(Set, \in)$ of ZermeloFraenkel set theory (ZF).

Definition

A *Standard Model* is a subclass $M \subseteq Set$ such that $(M, \in) \models ZF$ and M is transitive.

Intuitively standard models are those models of ZF which agree with Set on the notion "what a set is".

Absoluteness of First Order Logic

One of the most important properties of first order logic, from a model theoretic point of view, is the absoluteness of of the satisfaction relation. Specifically suppose

- V_0 and V_1 are standard models of ZF with $V_0 \subseteq V_1$
- L is a first order language in V_0 and φ is a first order formula in L
- $M \in V_0$ is a model of the language L.

Then $V_0 \models$ "$M \models \varphi$" if and only if $V_1 \models$ "$M \models \varphi$".

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This result allows us to disregard set theoretic concerns when discussing the first order theory of a model.

However, when we move to higher order languages the satisfaction relation may cease to be absolute.

Example of Topological Spaces

Consider the theory, Th_{Top} of Topological Spaces given in the language with one second order unary predicate $\mathcal{O}(\cdot)$. If

- $\mathcal{T} = \langle T, \mathcal{O}^T \rangle \in V_0$
- $V_0 \models$ "$\mathcal{T} \models Th_{Top}$"

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However, there is a "canonical" topological space $\mathcal{T}^* = \langle T, \mathcal{O}^{T^*} \rangle$ associated with $\mathcal{T}$ in V_1.

This is the topological space where $\mathcal{O}^{T^*}$ is the closure of $\mathcal{O}^T$ under finite intersections and arbitrary unions.

We can think of the model $\mathcal{T}^*$ as the *relativization* of $\mathcal{T}$ to V_1.

Definition of Relativizations

Lets make this precise.

Definition

Let $L = \{\equiv, \not\equiv\} \cup \{R_i : i \in I\}$ where

- R_i is a relation with arity $n_i + m_i$.
- $\equiv$ and $\not\equiv$ are binary relations.

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- $\equiv$ and $\not\equiv$ are binary relations.

Let Models(L) be the category where

- The objects of Models(L) are sequences $\mathcal{M} = \langle M, \equiv^M, \not\equiv^M, R_i^M \rangle$ where
  - $\equiv, \not\equiv \subseteq M \times M$ represent being "equal" or "not equal"
  - $R_i \subseteq M^{n_i} \times Powerset(M)^{m_i}$ is a relation on elements and subsets.

Definition of Relativizations

Definition

- If $\mathcal{M} = \langle M, \equiv^M, \not\equiv^M, R_i^M \rangle$ and $\mathcal{N} = \langle N, \equiv^N, \not\equiv^N, R_i^N \rangle$ then $f \in \text{Models}(L)[\mathcal{M}, \mathcal{N}]$ if $f : M \to N$ is a function such that
  - For all $\mathbf{a} \in M$, $\mathbf{b} \in Powerset(M)$ if $\mathcal{M} \models R_i(\mathbf{a}, \mathbf{b})$ then $\mathcal{N} \models R_i(f[\mathbf{a}], f[\mathbf{b}])$ (i.e. f preserves relations)
  - f preserves equality and non-equality of elements as well as the inclusion relation $a \in X$.

Definition of Relativizations

Definition

Suppose

- V is a standard model of ZF
- $\varphi(y)$ is a formula (possibly with parameters in V)
- $M \in \mathrm{obj}(\mathrm{Models}^V(L))$ (i.e. M is a model of L)

We then define $Ext^V(\varphi, M)$ to be the category whose objects are of model of φ extending M. I.e. those models N such that

- $M \subseteq N$ and $R_i^M \subseteq R_i^N$ for all relations.
- $V \models \varphi(N)$

with the maps of $Ext^V(\varphi, M)$ begin those $f \in \mathrm{Models}(L)[N, P]$ such that $f(m) = m$ for all $m \in M$.

Definition of Relativizations

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Suppose

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- φ is a formula of L.
- M is a model of L such that $V_0 \models \varphi(M)$

If N is an initial object of $(Ext(\varphi, M))^{V_1}$ then we say N is a *relativization* of M to V_1 for φ. We will also use the shorthand M^{V_1}.

Relativization of Grothendieck Toposes

We now want to ask "do Grothendieck toposes relativize?"

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Theorem

There is a language L_{Groth} and higher order theory Th_{Groth} such that

- *Every model of Th_{Groth} is a Grothendieck Topos.*
- *Every Grothendieck Topos has a unique expansion to L_{Groth} satisfying Th_{Groth}.*
- *Every model of Th_{Groth} has a relativization for Th_{Groth} for every standard model of ZFC.*

Relativization of Grothendieck Toposes

Proof Outline.

(1) We define a *weak site* as a pair (C, J_C) satisfying all the axioms of a site except local character

(2) We show that every weak site relativizes to an actual site and the property of being a weak site is absolute.

(3) We show that if (C, J_C) and (D, J_D) are weak sites $V_0 \models Sh(C, J_C) \simeq Sh(D, J_D)$ and $V_0 \subseteq V_1$ are standard models then $V_1 \models Sh(C, J_C) \simeq Sh(D, J_D)$

□

Relativization of Grothendieck Toposes

Proof Outline Cont.

(4) We let L_{Groth} be the language of category theory with a distinguished predicate and let Th_{Groth} bet the theory which says our category is a Grothendieck topos and the distinguished predicate picks out weak sites whose category of sheaves are equivalent to our category.

(5) If $V_0 \models$ "$G \models Th_{Groth}$" and $V_1 \models G \subseteq H_1, H_2$ where $H_1, H_2 \models Th_{Groth}$ then $V_1 \models H_1 \simeq H_2$. Then we use the axiom of choice to choose a minimal such H.

□

Relativization of Sheaves

An easy corollary is

Corollary

If $V_0 \models$ "$A \in obj(Sh(C, J_C)$" *then for every standard model* V_1 *containing* V_0*,* A *relativizes (as a sheaf on* (C, J_C)*) to an* A^{V_1} *where* $V_1 \models$ "$A^{V_1} \in obj(Sh(C, J_C))$"*.*

Further the relativization, A^{V_1} *of* A*, is the sheafification of* A *considered as a separated presheaf in* V_1*.*

Potential Isomorphisms

Notion of Potential Isomorphism

One of the most important notions in category theory is that of an isomorphism. Intuitively we can think of two isomorphic objects as "the same for all practical purposes"

However, often there are objects which, in some sense, should be the same except for peculiarities of the underlying set theory.

Example

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Let *DLO* be the first order theory of dense linear orderings without endpoints. It is well know that

Lemma

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Lemma

DLO is $\aleph_0$-categorical. I.e. it has only one countable model up to isomorphism.

However when we look at models of larger cardinality we find there are many non-isomorphic models

Lemma

There are 2^κ many non-isomorphic models of DLO of size κ for any cardinal $\kappa > \omega$.

Potentially Isomorphism

The only reason why two models, M and N, of DLO may not be isomorphic is because the category of models of DLO can't distinguish between sizes of infinity.

Corollary

If

- $M, N \in V_0 \subseteq V_1$
- $V_0 \models$ "$M, N \models DLO$"
- $V_1 \models |M| = |N| = \omega$

then $V_1 \models M \cong N$.

Potentially Isomorphism

This suggests a definition

Definition

Suppose $C \in V_0$ is a category with $A, B \in \text{obj}(C)$. If it is consistent that there is a standard model V_1 and relativizations $A^{V_1}, B^{V_1} \in \text{obj}(C^{V_1})$ such that $A^{V_1} \cong B^{V_1}$, then we say A and B are *Potentially Isomorphic*.

Potentially Isomorphism

A remarkable result about first order models is, if two models are potentially isomorphic then there is a witness to that fact. Specifically

Lemma

There is a sentence φ_{PI}, which is absolute, such that $V \models \varphi_{PI}(M, N)$ if and only if it is consistent that M and N are isomorphic in some larger universe.

In particular this means if it is consistent that in SOME standard model that M and N are isomorphic then that fact can be determined without looking outside of the standard model where M and N first appear.

Potentially Isomorphism

We can now ask a similar question for sheaves. Specifically

Definition

Suppose $V_0 \models M, N \in Sh(C, J_C)$. We say M and N are *potentially isomorphic* if it is consistent that there is a standard model V_1 where A^{V_1} is isomorphic to B^{V_1}

Potentially Isomorphism

Even though a sheaf is a higher order notion we still have

Theorem

For any sheaves M and N on a weak site (C, J_C) (and in V) there is an absolute formula φ_{PI} such that $V \models \varphi_{PI}(M, N)$ if and only if

$$V \models M \text{ is potentially isomorphic to } N$$

i.e. it is consistent that there is some standard model V_1 such that $V_1 \models M^{V_1} \cong N^{V_1}$.

Potential Isomorphisms of Sheaves

Proof Outline.

We want to define a *partial isomorphisms* between sheave M, N as follows

(1) Given a weak site, an element of its relativization can be thought of as a tree. We then consider finite approximations to such a tree.

(2) A *partial isomorphism* is a pair, (p, q), of labeled approximations. We also require p to be labeled by elements in $M \times N$ and q to be labeled with elements in $N \times M$ such that p and q are "compatible".

□

Potential Isomorphisms of Sheaves

Proof Outline.

We let $\varphi_{PI}(M, N)$ be the formula which says

(3) There is a collection of partial isomorphisms, I, such that whenever $(p, q) \in I$ and X is a finite collection of valid conditions which extensions of p might satisfy, then we can find a $(p', q') \in I$ where p' satisfies X. Likewise for q.

□

Potential Isomorphisms of Sheaves

Proof Outline.

We then show

(4) If there exists a model V_1 where M and N are isomorphic then $\varphi_{PI}(M, N)$ holds (i.e. such a collection of partial isomorphisms exists)

(5) If $\varphi_{PI}(M, N)$ holds then we can force with the collection of partial isomorphisms to find a model $V[G]$ such that $V[G] \models M^{V_1} \cong N^{V_1}$.

□

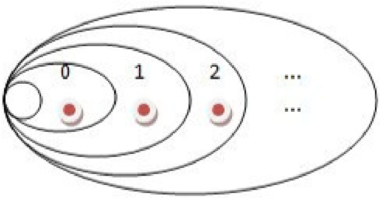
Generalized Ordinals and Grothendieck Toposes

$\omega + 1$ As A Topological Space

Definition

Let $\omega + 1$ be the topological space where

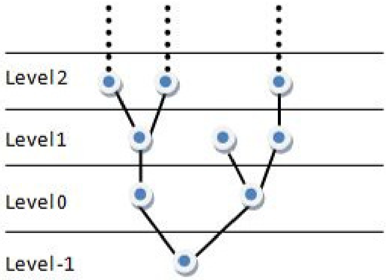
- The underlying set is $\omega = \{0, 1, 2, \ldots\}$
- Open sets are ordinals $\alpha \in \omega + 1$ (where $\alpha = \{\beta \in \omega : \beta < \alpha\}$)

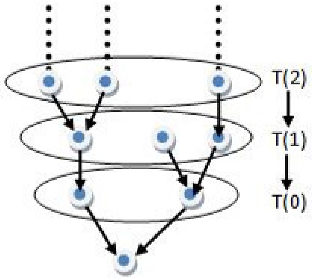


Motivating Observations

Theorem

There is a language $L_{TREE} = \{Level_n, \leq_{n,m} \colon n, m \in \omega\}$ and a first order theory TREE, of trees, such that models of TREE are the "same thing" as presheaves on $\omega + 1$.





Ordinals and Sheaves

As is well known we can define the class of ordinals from the category of trees. First recall

Definition

A *quasi-partial order* is a category such that the set of morphism between any two objects is either $\emptyset$ or $1 = \{*\}$ the one point set.

Definition

Let $P : \mathbf{SET} \to \{\emptyset, 1\}$ be the functor where $P(\emptyset) = \emptyset$ and otherwise $P(x) = 1$.

Specifically if $A, B \in \text{obj}(C)$ let $P(C[A, B]) = 1$ if and only if there is a morphism from A to B.

Ordinals and Sheaves

Lemma

For any category C, by applying P to the hom-sets of C we can turn C into a quasi-partial order $P^*[C]$.

Theorem

$P^*[PreSh(\hat{\omega})]$ *is equivalent to the partial order* $ORD \cup \{\infty\}$

Generalized Ordinals

This suggests a generalization of the notion of ordinal

Definition

Let $\mathrm{ORD}(C, J_C) = P^*[Sheaf(C, J_C)]$

And in fact, if $\hat{\omega}_1$ is defined in the natural way, then $\mathrm{ORD}(\hat{\omega}_1)$ plays an important role in the study of $\omega_1^{\omega_1}$ as well as in certain types of infinitary logics. A role very similar to the one the actual ordinals play in the study of ω^ω and the study of $\mathcal{L}_{\infty,\omega}$

Non-Absoluteness of Generalized Ordinals

Unfortunately though, there is one big difference between ORD($\hat{\omega}$) and ORD($\hat{\omega_1}$) (and more general ORD(C, J_C)). Unlike the actual ordinals, these general ordinals are not absolute. Specifically

Lemma

For all standard models V there is an $R \in ORD(\hat{\omega_1}^V)$ such that $V \models 1 \not\lesssim R$ but there is a forcing extension $V[G]$ such that $V[G] \models 1 \cong R^{V_1}$.

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This suggest the question: "If $A, B \in \mathrm{ORD}(C, J_C)^V$, can we tell if A is *potentially less than* B?", i.e. is it possible to tell when it is consistent that there is an extension V_1 of V where $V_1 \models (\exists f) f : A^{V_1} \to B^{V_1}$?

Potential Ordering Exists

Theorem

For any sheaves M and N on a weak site (C, J_C) (and in V) there is an absolute formula φ_{PM} such that $V \models \varphi_{PM}(M, N)$ if and only if

$$V \models \text{ there potentially is a map from } M \text{ to } N$$

i.e. it is consistent that there is some standard model V_1 such that $V_1 \models Sh(C, J_C)[M^{V_1}, N^{V_1}] \neq \emptyset$.

Proof.

The proof is very similar to the proof that there is a witness to there being a potential isomorphism. □

Potential Ordering Exists

Corollary

For any standard model V there is a partial order $ORD^*(C, J_C)$ such that

- $obj(ORD^*(C, J_C)) = obj(ORD(C, J_C))$
- *For $A, B \in ORD^*(C, J_C)$, $V \models A \leq_{ORD^*(C,J_C)} B$ if and only if it is consistent that there is some extension V_1 such that $V_1 \models A \leq_{ORD(C,J_C)} B$.*

In other words $\mathrm{ORD}^*(C, J_C)$ is an absolute analog of $\mathrm{ORD}(C, J_C)$

Thank You