Γ-Ultrametric Spaces and Separated Presheaves

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Γ-Ultrametric Spaces

Lattice of Open Sets

Definition

A partial order $(P, \leq)$ is a *complete lattice* if it has all infima. We denote the infimum of two elements a and b by $a \wedge b$ and there supremum by $a \vee b$ (and similarly for infinite infima and suprema)

Theorem

Any complete lattice is cocomplete (i.e. has all suprema).

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Theorem

Any complete lattice is cocomplete (i.e. has all suprema).

Example

If T is a topological space with $\mathcal{O}(T)$ as the collection of open sets, then $\mathcal{O}(T)$ is a complete lattice.

Γ-Ultrametric spaces

Definition

Let $(\Gamma, \leq, 0)$ be a complete lattice with minimal element 0. A Γ-*ultrametric space* is a pair (M, d_M) such that

- M is a set and $d_M : M \times M \to \Gamma$.
- (Reflexivity) $(\forall x, y \in M) d_M(x, y) = 0 \leftrightarrow x = y$
- (Symmetry) $(\forall x, y \in M) d_M(x, y) = d_M(y, x)$
- (Strong Triangle Inequality) $(\forall x, y, z \in M) d_M(x, y) \vee d_M(y, z) \geq d_M(x, z)$

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A *non-expanding* map between Γ-ultrametric spaces (M, d_M) and (N, d_N) is a function $f : M \to N$ such that

$$(\forall a, b \in M) d(f(a), f(b)) \leq d(a, b)$$

This gives us a category Γ-**UltMet** whose objects are Γ-ultrametric spaces and whose morphisms are the non-expanding maps.

Example of $\mathbb{R}$-Ultrametric spaces

Example

For any $x, y \in \omega^\omega$ we can define

- $d(x,y) = \dfrac{1}{\min\{n : x(n) \neq y(n)\}}$ if $x \neq y$
- $d(x,y) = 0$ if $x = y$.

Then (ω^ω, d) is a $\mathbb{R}$-Ultrametric space.

Notice that the only values of $\mathbb{R}$ which are realized as distances are those of the form $\frac{1}{n}$ where $n \in \mathbb{N} \cup \{\infty\}$. Hence (ω^ω, d) is also a $(\mathbb{N} \cup \{\infty\})^{op}$-Ultrametric space (where $(\mathbb{N} \cup \{\infty\})^{op}$ is the lattice with the same elements as $\mathbb{N} \cup \{\infty\}$ but the opposite order).

Example of $\mathbb{Q}$-Ultrametric spaces

Example (p-adic norm)

If p is a prime number we define $|\cdot|_p$, the *p-adic norm*, as follows

- $|0|_p = 0$
- If $q = p^n \cdot \frac{a}{b}$ where p doesn't divide a or b then $|q| = \frac{1}{p^n}$

For all $q_0, q_1 \in \mathbb{Q}$ let $d_p(q_0, q_1) = |q_0 - q_1|_p$. Then $(\mathbb{Q}, d_p)$ is an $\mathbb{R}$-ultrametric space.

Closed Balls

Definition

For each $x \in M, \gamma \in \Gamma$ we define the *(closed) ball of radius* γ *around* x to be

$$B^M(x,\gamma) = \{y : d_M(x,y) \leq \gamma\}$$

We will omit mention of M when it is clear from context.

Definition

If $A \subseteq M$ the *diameter of* A is $diam(A) = \bigvee\{d_M(x,y) : x,y \in M\}$.

Closed Balls

The following lemmas are immediate.

Lemma

$(\forall \gamma \in \Gamma, x, y \in M) x \in B(y,\gamma) \leftrightarrow B(x,\gamma) = B(y,\gamma)$

Lemma

$(\forall \gamma_x, \gamma_y \in \Gamma)(\forall x, y \in M) \gamma_x \leq \gamma_y \to B(x,\gamma_x) \subseteq B(y,\gamma_y)$ *or* $B(x,\gamma_x) \cap B(y,\gamma_y) = \emptyset$.

Intersection of Balls

Theorem

Suppose

- $\alpha = \bigwedge\{\gamma_i : i \in I\}$ *and* $\{x_i : i \in I\} \subseteq M$
- $B = \bigcap_{i \in I} B(x_i, \gamma_i) \neq \emptyset$

Then $(\forall x \in B) B = B(x, \alpha)$.

Proof.

Suppose $x \in B$. Then $(\forall i \in I) x \in B(x_i, \gamma_i)$ and so $B(x, \gamma_i) = B(x_i, \gamma_i)$. Hence $(\forall i \in I) B \subseteq B(x, \gamma_i)$ and $diam(B) \leq \gamma_i$. Therefore $diam(B) \leq \alpha$ and in particular $B \subseteq B(x, \alpha)$.

But we also have $(\forall i \in I) B(x, \alpha) \subseteq B(x, \gamma_i)$ and so $B(x, \alpha) \subseteq B$. □

Relative Distance Preserving Maps

While much of our focus will be on Γ-ultrametric spaces for a specific Γ, it will at times be useful to be able to compare generalized ultrametric spaces with different sets of distances.

Definition

If (M, d_M) is a Γ-ultrametric space and (N, d_N) is an Λ-ultrametric space we say a map $f : M \to N$ is *relative distance preserving* if for all $a, b, c, d \in M$,

$$d_M(a, b) \leq d_M(c, d) \to d_N(f(a), f(b)) \leq d_N(f(c), f(d))$$

We let **GenUltMet** be the category whose objects are Γ-ultrametric spaces for some Γ and whose morphisms are the relative distance preserving maps.

Spherically Complete

An important property that a Γ-ultrametric spaces may have is what is called spherical completeness. This is the Γ-ultrametric analog of completeness for metric spaces.

Definition

A Γ-ultrametric space (M, d_M) is *spherically complete* if whenever

- $\{\gamma_i : i < \kappa\} \subseteq \Gamma$ and $\{x_i : i < \kappa\} \subseteq M$
- $B(\gamma_i, x_i) \subseteq B(\gamma_j, x_j)$ if $i \geq j$.

Then $\bigcap_{i<\kappa} B(\gamma_i, x_i) \neq \emptyset$.

We define **SComp(Γ)** to be the full subcategory of Γ-**UltMet** whose objects are spherically complete.

Theorem

Suppose (M, d_M) is a spherically complete Γ-ultrametric space and

- $\{\gamma_i : i < \kappa\} \subseteq \Gamma$ *and* $\{x_i : i < \kappa\} \subseteq M$
- $B(\gamma_i, x_i) \subseteq B(\gamma_j, x_j)$ *if* $i \geq j$.

is a decreasing chain of closed balls. Then for all $x \in \bigcap_{i<\kappa} B(\gamma_i, x_i)$

$$B(\bigwedge_{i<\kappa} \gamma_i, x) = \bigcap_{i<\kappa} B(\gamma_i, x_i)$$

Intersection in Spherically Complete Spaces

Theorem

Suppose (M, d_M) *is a spherically complete* Γ*-ultrametric space and*

- $\{\gamma_i : i < \kappa\} \subseteq \Gamma$ *and* $\{x_i : i < \kappa\} \subseteq M$
- $B(\gamma_i, x_i) \subseteq B(\gamma_j, x_j)$ *if* $i \geq j$.

is a decreasing chain of closed balls. Then for all $x \in \bigcap_{i<\kappa} B(\gamma_i, x_i)$

$$B(\bigwedge_{i<\kappa} \gamma_i, x) = \bigcap_{i<\kappa} B(\gamma_i, x_i)$$

Review of Category Theory

Functors and Natural Transformations

Definition

Suppose C, D are categories. A *functor* $F : C \rightarrow D$ is a map such that

- $(\forall c \in \text{obj}(C))F(c) \in \text{obj}(D)$
- $(\forall f \in \text{morph}(C))$ if $f : A \rightarrow B$ then $F(f) : F(A) \rightarrow F(G)$.
- $(\forall f, g \in \text{morph}(C))F(f \circ g) = F(f) \circ F(g)$.

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Definition

Suppose $F, G : C \to D$ are functors. A *natural transformation* $\alpha : F \Rightarrow G$ is a sequence of maps

- $(\forall A \in \text{obj}(C))\alpha_A : F(A) \to G(A)$
- $(\forall g : A \to B,\ g \in \text{morph}(C))G(g) \circ \alpha_A = \alpha_B \circ F(g)$

This then gives us a category $\mathbf{Cat}[C, D]$ whose objects are functors and whose maps are natural transformations.

Natural Isomorphism

Definition

Suppose $F, G : C \to D$ and $\alpha : F \Rightarrow G$ is a natural transformation. We say that α is a *natural isomorphism* if either of the following equivalent conditions hold

- For each $c \in \text{obj}(C)$, α_c is an isomorphism.
- There is a natural transformation $\alpha^{-1} : G \Rightarrow F$ such that $\alpha \circ \alpha^{-1} = id_G$ and $\alpha^{-1} \circ \alpha = id_F$

Equivalence of Categories

Definition

If $F : C \to D$, $G : D \to C$ are functors, we say that F and G are each *equivalences of categories* if there are natural isomorphisms

- $\alpha : F \circ G \Rightarrow id_C$
- $\beta : G \circ F \Rightarrow id_D$

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- $\beta : G \circ F \Rightarrow id_D$

Theorem

A functor $F : C \to D$ is an equivalence of categories if and only if

- *F is full and faithful. I.e. For all objects $A, B \in obj(C)$, $F : C[A, B] \to D[F(A), F(B)]$ is an isomorphism.*
- *For all $A \in obj(D)$ there is a $B \in obj(C)$ such that $F(B) \cong A$*

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- *For all $A \in obj(D)$ there is a $B \in obj(C)$ such that $F(B) \cong A$*

Example

Let $Card : SET \to ORD$ be the functor which takes all sets to their cardinality. Then *Card* is an equivalence of categories.

Adjoints

Definition

We say that a functor $F : C \longrightarrow D$ is *Left Adjoint* to $G : D \longrightarrow C$ (and G is *Right Adjoint* to F), written $F \dashv G$, if for any $X \in \mathrm{obj}(C)$ and $Y \in \mathrm{obj}(D)$ maps from $F(Y)$ to X are (naturally isomorphic) to maps from Y to $G(X)$ (i.e. $C[F(Y), X] \cong D[Y, G(X)]$)

Adjoint Example

Example

Let *Forget* : **Group** $\rightarrow$ **Set** be the (forgetful) functor which takes a group and returns the underlying set (i.e. forgets the group structure). Let *Free* : **Set** $\rightarrow$ **Group** be the functor which assigns to any set its free group.

Then *Free* is left adjoint to *Forget* ($Forget \dashv Free$)

We can think of an adjoint functor as a functor which takes an object in one category and returns the "best approximation" to that object in another.

Monomorphisms and Epimorphisms

Definition

We say a map $f : A \rightarrowtail B$ is a *monomorphism* if for any $g, h : D \to A$

$$f \circ g = f \circ h \Rightarrow g = h$$

Monomorphisms are the categorical analog of injective maps.

Definition

We say a map $f : A \twoheadrightarrow B$ is an *epimorphism* if for any $g, h : A \to D$

$$g \circ f = h \circ f \Rightarrow g = h$$

Epimorphisms are the categorical analog of surjective maps.

Epi-Mono Factorization

Definition

We say a category has *epi-mono factorization* if for every map $f : A \longrightarrow B$ there are maps $f_e : A \longrightarrow Im$ and $f_m : Im \longrightarrow B$ such that

- f_e is an epimorphism.
- f_m is a monomorphism.
- $f = f_m \circ f_e$
- If $f = g_m \circ g_e$ where g_m is a monomorphism and g_e is an epimorphism, then there is an isomorphism i such that $g_m = f_m \circ i$ and $g_e = i^{-1} \circ f_e$.

We call Im the *image of* f.

Example

The category **Set** has epi-mono factorization.

Subobjects

Definition

Suppose $f : A \rightarrowtail B, g : A' \rightarrowtail B$ are monomorphisms. We say that $f \leq g$ if there is a map $h : A \longrightarrow A'$ such that $h \circ f = g$.

Definition

A subobject of B is an equivalence class of maps $[f]$ into B under

$$f \sim g \Leftrightarrow f \leq g \text{ and } g \leq f$$

We denote by $Sub(B)$ the collection of subobjects of B with the induced order.

Subobjects are the categorical analog of subsets.

Partial Orders as Categories

Definition

A *partial order* is a category C such that for any two objects $A, B \in \text{obj}(C)$

- There is at most one map in $C[A, B]$ and
- If $C[A, B]$ and $C[B, A]$ are both non-empty then $A = B$

Sheaves and Presheaves

Presheaves on a Complete Lattice

Definition

Suppose Γ is a complete lattice (considered as a category). A *presheaf* on Γ is a functor from Γ^{op} into the category of sets.

If $F : \Gamma^{op} \to \mathbf{Set}$ is a presheaf, $a \in F(U)$ and $i : V \to U$ (i.e $V \leq U$) we will write $a|_V$ for $F(i)(a)$ (which we call the restriction of a to V).

We let $\mathbf{PreSh}(\Gamma)$ be the category of preheaves on Γ along with natural transformations.

Examples

Example

An example of a sheaf on $\mathbb{R}$ is the collection C of functions from $\mathbb{R}$ to a set X. That is

- $C(U) = \{f : U \to X\}$ if U is an open subset of $\mathbb{R}$.
- If $f \in C(V)$ and $U \subseteq V$ then $f|_U$ is the function whose domain is U and which agrees with f on its domain.

Definition

A presheaf F on Γ is a *sheaf* if whenever

- $U = \bigvee_{i \in I} U_i$
- $a_i \in F(U_i)$ for all $i \in I$
- $a_i|_{U_i \wedge U_j} = a_j|_{U_i \wedge U_j}$ for all $i, j \in I$

then there is a unique element $a \in F(U)$ such that $a|_{U_i} = a_i$ for all $i \in I$.

We let $\mathbf{Sheaf}(\Gamma)$ be the full subcategory of $\mathbf{PreSh}(\Gamma)$ consisting of sheaves.

Separated Presheaves

Suppose F is a presheaf on Γ. Given

- $U = \bigvee_{i \in I} U_i$
- $a_i \in F(U_i)$ for all $i \in I$
- $a_i|_{U_i \wedge U_j} = a_j|_{U_i \wedge U_j}$ for all $i, j \in I$

There are two ways that it can fail the sheaf condition

(1) There are no elements $a \in F(U)$ such that $(\forall i) a|_{U_i} = a_i$.

(2) There are two distinct $a, a' \in F(U)$ such that $(\forall i) a|_{U_i} = a_i = a'|_{U_i}$.

Separated Presheaves

Suppose F is a presheaf on Γ. Given

- $U = \bigvee_{i \in I} U_i$
- $a_i \in F(U_i)$ for all $i \in I$
- $a_i|_{U_i \wedge U_j} = a_j|_{U_i \wedge U_j}$ for all $i, j \in I$

There are two ways that it can fail the sheaf condition

(1) There are no elements $a \in F(U)$ such that $(\forall i) a|_{U_i} = a_i$.

(2) There are two distinct $a, a' \in F(U)$ such that $(\forall i) a|_{U_i} = a_i = a'|_{U_i}$.

Definition

A presheaf F is *separated* if condition (2) never happens. We let **Sep**(Γ) be the full subcategory of **PreSh**(Γ) consisting of separated presheaves.

Subsheaves

Definition

If A and B are sheaves on Γ then we say *A is a subsheaf of B* ($A \subseteq B$) if $A(V) \subseteq B(V)$ for all open set $V \in \Gamma$ and

$$(\forall x \in A(V))A(i)(x) = B(i)(x)$$

whenever $i : U \to V$.

Theorem

If B is a sheaf, then there is a bijection between subsheaves of B and subobjects of B in **Sheaf**(Γ).

Terminal Sheaf

Definition

We let 1_Γ be the unique sheaf on Γ such that $|1_\Gamma(U)| = 1$ for all $U \in \Gamma$. We call 1_Γ the *terminal sheaf*

Lemma

For any sheaf A on Γ there is a unique map from A to 1_Γ.

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Lemma

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Lemma

$(\Gamma, \leq)$ *is isomorphic to* $(Sub_{\mathbf{Sheaf}(\Gamma)}(1_\Gamma), \leq)$

Equalizers and Pullbacks

Definition

Suppose we have two maps $f, g : A \longrightarrow B$ of sheaves. The *equalizer* of f and g ($eq(f, g)$) is the subsheaf

$$eq(f, g)(U) = \{x \in A(U) : f(x) = g(x)\}$$

Definition

Suppose we have a maps of sheaves $f : A \longrightarrow B$ and a subsheaf $B' \subseteq B$. We define the *pullback of B' along f* ($f^{-1}[B']$) to be the subsheaf of A where

$$f^{-1}[B'](U) = \{x \in A(U) : f(x) \in B'(U)\}$$

Definition

We say a separated presheaf A on Γ is *flabby* if

$$(\forall \gamma \in \Gamma^{op})(\forall a \in A(\gamma))(\exists a' \in A(1))a'|_{\gamma} = a$$

We define **Flabby(Γ)** to be the full subcategory of **Sep**(Γ) whose objects are the flabby presheaves.

Likewise we define **FlabbySheaf(Γ)** to be the full subcategory of **Sheaf(Γ)** whose objects are flabby presheaves.

Presheaves and Γ-Ultrametric Spaces

Equivalence of Flabby Presheaves

Theorem

There is an equivalence of categories between **Flabby(Γ)** *and* Γ^{op}-**UltMet**.

Equivalence of Flabby Presheaves

Theorem

There is an equivalence of categories between **Flabby(Γ)** *and* Γ^{op}-**UltMet**.

In order to be consistent the names of all elements will be as in Γ. We will use a superscript ^{op} to signify the corresponding notion in Γ^{op}. For example $a \leq b$ if and only if $a \geq^{op} b$ and $1 = 0^{op}$.

Theorem

There is an equivalence of categories between **Flabby(Γ)** *and* Γ^{op}-**UltMet**.

In order to be consistent the names of all elements will be as in Γ. We will use a superscript ^{op} to signify the corresponding notion in Γ^{op}. For example $a \leq b$ if and only if $a \geq^{op} b$ and $1 = 0^{op}$.

Also, for simplicity, we will only show the object portion of the equivalence. I.e. we will show that there is a bijection between the objects of **Flabby(Γ)** and the objects of Γ^{op}-**UltMet**.

We will then leave it to the enthusiastic listener to confirm that this bijection extends to the maps as well (and hence is an equivalence of categories).

Equivalence of Flabby Presheaves:Claim 1

Claim

If (A, d_A) is an Γ^{op}-ultrametric space let

$$F(A)(\gamma) = \{B(a, \gamma) : a \in A\}$$

where $B(a, \gamma)|_{\gamma^*} = B(a, \gamma^*)$ for all $\gamma^* \leq \gamma$. Then $F(A)$ is a separated flabby presheaf on Γ

Proof.

First we need to confirm that restriction is well defined.

Suppose $B(a, \gamma) = B(b, \gamma) \in F(A)(\gamma)$, $\gamma^* \leq \gamma$ and $x \in B(a, \gamma^*)$ (i.e. $d_A(x, a) \leq^{op} \gamma^*$).

$d_A(a, b) \leq^{op} \gamma \leq^{op} \gamma^*$, so $d_A(x, b) \leq^{op} \gamma^*$ and $x \in B(b, \gamma^*)$.

Hence $B(a, \gamma^*) = B(b, \gamma^*)$ and restriction is well defined. □

Equivalence of Flabby Presheaves:Claim 1

Proof.

To see that $F(A)$ is flabby notice that if $B(a, \gamma) \in F(A)(\gamma)$ then $\{a\} = B(a, 0^{op}) \in F(A)(1)$ and $B(a, 0^{op})|_\gamma = B(a, \gamma)$.

To see $F(A)$ is separated, suppose $\gamma = \bigvee_{i \in I} \lambda_i$ and $\boldsymbol{B} = \{B(x_i, \lambda_i) \in A^*(\lambda_i) : i \in I\}$ is such that $B(x_i, \lambda_i)|_{\lambda_i \wedge \lambda_j} = B(x_i, \lambda_i \wedge \lambda_j) = B(x_j, \lambda_i \wedge \lambda_j) = B(x_j, \lambda_j)|_{\lambda_i \wedge \lambda_j}$.

i.e. $\boldsymbol{B}$ is a compatible collection of elements. □

Proof.

Case 1: $\bigcap_{i \in I} B(x_i, \lambda_i) = \emptyset$
In this case there is no element in $F(A)(\gamma)$ compatible with $\boldsymbol{B}$.

This is because any such element would have be of the form $B(x, \gamma)$ with $B(x_i, \lambda_i) = B(x, \lambda_i)$ for all $i \in I$. But that would imply $x \in \bigcap_{i \in I} B(x_i, \lambda_i) = \emptyset$.

Case 2: $(\exists x \in \bigcap_{i \in I} B(x_i, \lambda_i))$
We then have $B(x, \gamma) = \bigcap_{i \in I} B(x_i, \lambda_i)$ and hence $B(x, \gamma)$ is the unique element of $F(A)(\gamma)$ compatible with $\boldsymbol{B}$. □

Equivalence of Flabby Presheaves: Claim 2

Claim

If A is a flabby separated presheaf on Γ let $E(A) = (A^o, d_A)$ be such that $A^o = A(1)$ and $d_A(a, b) = \bigvee\{\gamma : a|_\gamma = b|_\gamma\}$. Then (A^o, d_A) is an Γ^{op}-ultrametric space.

Equivalence of Flabby Presheaves: Claim 2

Claim

If A is a flabby separated presheaf on Γ let $E(A) = (A^o, d_A)$ be such that $A^o = A(1)$ and $d_A(a, b) = \bigvee\{\gamma : a|_\gamma = b|_\gamma\}$. Then (A^o, d_A) is an Γ^{op}-ultrametric space.

Proof.

First notice $a|_{d_A(a,b)} = b|_{d_A(a,b)}$ because $\{a|_\gamma : \gamma \leq d_A(a, b)\}$ is a compatible collection of elements covering both $a|_{d_A(a,b)}$ and $b|_{d_A(a,b)}$.

In particular this means that if $d_A(a, b) = 0^{op}$ then $a = a|_1 = b|_1 = b$. So (A^o, d_A) satisfies (reflexivity). □

Proof.

(symmetry) is immediate from the definition.

To see the (strong triangle inequality) holds let $a, b, c \in A(1)$ with $d_A(a, b) \leq^{op} \gamma$ and $d_A(b, c) \leq^{op} \gamma$.

Then $a|_\gamma = b|_\gamma = c|_\gamma$ and hence $d_A(a, c) \leq^{op} \gamma$. □

Equivalence of Flabby Presheaves: Claim 3

Claim

For all $A \in obj(\mathbf{Flabby}(\Gamma))$ *there is a natural isomorphism* $\eta_A : A \Rightarrow F(E(A))$ *which is the identity on* $A(1)$.

For all $(B, d_B) \in obj(\Gamma^{op}\text{-}\mathbf{UltMet})$ *there is a natural isomorphism* $\varepsilon_B : (B, d_B) \Rightarrow E(F(B, d_B))$.

Proof.

For all $A \in \text{obj}(\mathbf{Flabby}(\Gamma))$ we let $\eta_A = id$.

If $(A', d_{A'}) = E(F(A, d_A))$ then $A' = \{\{a\} : a \in A\}$ and we can let $\varepsilon_A(x) = \{x\}$ □

It easy to see that η, ε are natural isomorphisms and hence E and F are equivalences of categories.

Spherically Completeness and Flabby Sheaves

Theorem (*)

The equivalence of the theorem above restricts to an equivalence of categories between **FlabbySheaf(Γ)** *and* **SComp(Γ^{op})**.

Spherically Completeness and Flabby Sheaves

Claim

If (A, d_A) is a spherically complete Γ^{op}-ultrametric space then $F(A)$ is a sheaf.

Proof.

Let $I = \langle \gamma_j : j < \kappa \rangle \subseteq \Gamma$ with $\langle a_i : i \in I \rangle$ be a compatible collection of elements of $F(A)$.

Notice that without loss of generality we can assume that if $\gamma \in I$ and $\gamma' < \gamma$, then $\gamma' \in I$. □

Proof.

By definition there are x_i such that $a_i = B(x_i, \gamma_i)$. We define B_i by induction.

- $B_0 = B(x_0, \gamma_0)$.
- $B_{\alpha+1} = B(x_{\gamma_{\alpha+1}}, \gamma_{\alpha+1}) \cap B_\alpha$
- $B_{\omega\cdot\beta} = \bigcap_{j<\omega\cdot\beta} B_j$.

We now want to show $(\forall l < \kappa, \lambda \leq \kappa) B(x_{\gamma_l}, \gamma_l) \cap B_\lambda \neq \emptyset$. □

Spherically Completeness and Flabby Sheaves

Proof.

To get a contradiction assume j is least such that $B(x_{\gamma_l}, \gamma_l) \cap B_j = \emptyset$ for some $l < \kappa$. We break into three cases.

Case 1: $j = 0$
This case can't happen because $\langle a_i : i \in I \rangle$ is a compatible collection of elemetns and hence the a_i's are closed under intersection.

Case 2: $j = \omega \cdot \beta$.
Notice that $B_r \subseteq B_s$ if $s \leq r < \omega \cdot \beta$. So

$$\begin{aligned} B(x_{\gamma_l}, \gamma_l) \cap B_{\omega \cdot \beta} &= B(x_{\gamma_l}, \gamma_l) \cap \bigcap_{h < \omega \cdot \beta} B_h \\ &= \bigcap_{h < \omega \cdot \beta} (B(x_{\gamma_l}, \gamma_l) \cap B_h) \end{aligned}$$

Proof.

But $B(x_{\gamma_l}, \gamma_l) \cap B_h \neq \emptyset$ by the inductive hypothesis and $\langle B(x_{\gamma_l}, \gamma_l) \cap B_h : h < \omega \cdot \beta \rangle$ is a decreasing sequence of balls.

Hence $\bigcap_{h<\omega\cdot\beta} B(x_{\gamma_l}, \gamma_l) \cap B_h \neq \emptyset$ because (A, d_A) is spherically complete. So this case can't happen. □

Proof.

Case 3: $j = \alpha + 1$.
$B(x_{\gamma_I}, \gamma_I) \cap B_{\alpha+1} = B(x_{\gamma_I}, \gamma_I) \cap (B(x_{\gamma_\alpha}, \gamma_\alpha) \cap B_{\alpha+1}) =$
$B(x_{\gamma_I \wedge \gamma_\alpha}, \gamma_I \wedge \gamma_\alpha) \cap B_\alpha \neq \emptyset$.

So this case can't happen and we have our contradiction.

We therefore have $B_\kappa = \bigcap_{i<\kappa} B_i \neq \emptyset$ and so $B_\kappa = B(x, \bigvee I)$ for any $x \in B_\kappa$.

Hence B_κ is the unique element of $F(A)(\bigvee I)$ which is covered by $\langle (a_i, i) : i \in I \rangle$.

So, because $\langle a_i : i \in I \rangle$ was arbitrary, $F(A)$ is a sheaf. □

Spherically Completeness and Flabby Sheaves

Claim

If A is a flabby sheaf on Γ then (A^o, d_A) is a spherically complete Γ^{op}-ultrametric space.

Proof.

Suppose

- $\{\gamma_i : i < \kappa\} \subseteq \Gamma^{op}$ and $\{x_i : i < \kappa\} \subseteq A$
- $B^{A^o}(\gamma_i, x_i) \subseteq B^{A^o}(\gamma_j, x_j)$ if $i \geq j$.

□

Proof.

Whenever $j \leq i$ we then have $x_i \in B^{A^o}(\gamma_j, x_j)$ and $\gamma_i \leq \gamma_j$ and so $x_j|_{\gamma_i} = x_i|_{\gamma_i}$.

Hence $(\forall i, j < \kappa) x_j|_{\gamma_i \wedge \gamma_j} = x_i|_{\gamma_i \wedge \gamma_j}$ and $\langle x_i|_\zeta : i < \kappa, \zeta \leq \gamma_i \rangle$ is a compatible collection of elements.

But because A is a sheaf there is an $y \in A(\bigvee_{i<\kappa} \gamma_i)$ covered by $\langle x_i|_\zeta : i < \kappa, \zeta \leq \gamma_i \rangle$. □

Proof.

Further, because A is a flabby sheaf we know there is an $x^* \in A(1) = A^o$ such that $x^*|_{\bigvee_{i<\kappa} \gamma_i} = y$.

So for all $i < \kappa$, $x^*|_{\gamma_i} = x_i|_{\gamma_i}$ hence $x^* \in B^{A^o}(\gamma_i, x_i)$ and $x^* \in \bigcap_{i<\kappa} B^{A^o}(\gamma_i, x_i) \neq \emptyset$.

So (A^o, d_A) is spherically complete (as our sequence of balls was arbitrary). □

Flabby Presheaves has a Right Adjoint

Now that we have a way to describe the category of Γ-ultrametric spaces it is natural to ask what properties this category has.

Flabby Presheaves has a Right Adjoint

Now that we have a way to describe the category of Γ-ultrametric spaces it is natural to ask what properties this category has.

Theorem

The inclusion map $i : \mathbf{Flabby(\Gamma)} \to \mathbf{Sep}(\Gamma)$ *has a right adjoint* r*.*

Proof.

If $A \in \text{obj}(\mathbf{Sep(\Gamma)})$, let $r(A)(\gamma) = \{x \in A : (\exists y \in A(1))y|_\gamma = x\}$.

It is not hard to see that this extends to a functor and that this functor is the right adjoint to the identity. □

Flabby Presheaves has a Right Adjoint

Now that we have a way to describe the category of Γ-ultrametric spaces it is natural to ask what properties this category has.

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Proof.

If $A \in \text{obj}(\mathbf{Sep(\Gamma)})$, let $r(A)(\gamma) = \{x \in A : (\exists y \in A(1)) y|_\gamma = x\}$.

It is not hard to see that this extends to a functor and that this functor is the right adjoint to the identity. □

Notice though that r does not restrict to an adjunction for the inclusion map $i : \mathbf{FlabbySheaf(\Gamma)} \to \mathbf{Sheaf(\Gamma)}$ because even if A is a sheaf, there is no guarantee that $r(A)$ will be as well.

Contracting Maps

Contracting Maps

Definition

We say a map $a : A \to B$ of Γ-Ultrametric spaces is *contracting* if

$$(\forall x, y \in A) x \neq y \to d(a(x), a(y)) < d(x, y)$$

As we will see, these will play an important role in the study of two player games.

Contracting Maps

Theorem

If $a : A \to B$ is a contracting map and $b : B \to D$ is any map, then $b \circ a : A \to D$ is a contracting map.

Proof.

Suppose $x, y \in A$ are such that $x \neq y$.

Then $d(b \circ a(x), b \circ a(y)) \leq d(a(x), a(y)) < d(x, y)$

Hence $b \circ a$ is contracting. □

Contracting Maps

Theorem

If $a : A \to B$ is a contracting map and $e : E \to A$ is any map, then $a \circ e$ is a contracting map.

Proof.

Suppose $x, y \in A$ are such that $x \neq y$.

Then $d(a \circ e(x), a \circ e(y)) < d(e(x), e(y)) \leq d(x, y)$

Hence $a \circ e$ is contracting. □

Fixed Point Theorem

Theorem (Priess-Crampe, Ribeuboim (*))

Suppose A is a spherically complete ultrametric space and $a : A \to A$ is a contracting map. Then there is a unique element $x \in A$ such that $a(x) = x$. We call x the fixed point of a.

Proof.

We will prove the existence of a fixed point x by repeated applying the contracting map a. Define $x_\alpha \in A$ and $Dist_\alpha \in \Gamma$ as follows

- x_0 is any point in A, $Dist_1 = 1$
- $x_{\alpha+1} = a(x_\alpha)$ and $Dist_{\alpha+1} = d(x_\alpha, x_{\alpha+1})$
- $Dist_{\omega\cdot\gamma} = \bigwedge_{i<\omega\cdot} Dist_i$ and $x_{\omega\cdot\gamma} \in \bigcap_{i<\omega\cdot\gamma} B(x_i, \gamma_i)$ is any element in the intersection of the $B(x_i, \gamma_i)$

Notice that for any $\omega \cdot \gamma, \bigcap_{i<\omega\cdot\gamma} B(x_i, \gamma_i)$ is non-empty as A is spherically complete, and hence $x_{\omega\cdot\gamma}$ is well defined. □

Fixed Point Theorem

Claim

For all $\alpha < \beta \in Ord$, $Dist_\alpha \geq Dist_\beta$

Proof.

First notice that if $\beta = \omega \cdot \gamma$ then (trivially) $(\forall \alpha < \beta) Dist_\alpha \geq Dist_\beta$

So it suffices to show that for all α, $Dist_\alpha \geq Dist_{\alpha+1}$. This breaks into two cases.

<u>Case 1:</u> $\alpha = \alpha^* + 1$

$$\begin{aligned} Dist_{\alpha+1} &= d(x_\alpha, x_{\alpha+1}) = d(x_\alpha, a(x_\alpha)) \\ &= d(x_{\alpha^*+1}, a(x_{\alpha^*+1})) = d(a(x_{\alpha^*}), a(a(x_{\alpha^*}))) \\ &< d(x_{\alpha^*}, a(x_{\alpha^*})) = d(x_{\alpha^*}, x_{\alpha^*+1}) \\ &= Dist_{\alpha^*+1} = Dist_\alpha \end{aligned}$$

Fixed Point Theorem

Proof.

Case 2: $\alpha = \omega \cdot \gamma$
For all $\zeta < \alpha$ we have $B(x_\alpha, Dist_\zeta) = B(x_\zeta, Dist_\zeta)$. So in particular we have $d(x_\alpha, x_\zeta) \leq Dist_\zeta$ and hence

$$\begin{aligned} d(x_{\alpha+1}, x_{\zeta+1}) &= d(a(x_\alpha), a(x_\zeta)) \\ &\leq d(x_\alpha, x_\zeta) \\ &\leq Dist_\zeta \end{aligned}$$

But this means that $x_{\alpha+1} \in B(x_\alpha, Dist_\zeta)$ for all $\zeta < \alpha$.

Therefore $Dist_{\alpha+1} = d(x_\alpha, x_\alpha + 1) \leq \bigwedge_{\zeta \leq \alpha} Dist_\zeta = Dist_\alpha$ □

Fixed Point Theorem

Proof.

Therefore there is some α such that $Dist_{\alpha+1} = Dist_{\alpha+2}$ and hence such that $d(x_\alpha, x_{\alpha+1}) = d(x_{\alpha+1}, x_{\alpha+2}) = d(a(x_\alpha), a(x_{\alpha+1}))$.

But, as a is contracting, this can only happen if $x_\alpha = x_{\alpha+1} = a(x_\alpha)$ and hence if x_α is a fixed point of a.

Fixed Point Theorem

Proof.

Therefore there is some α such that $Dist_{\alpha+1} = Dist_{\alpha+2}$ and hence such that $d(x_\alpha, x_{\alpha+1}) = d(x_{\alpha+1}, x_{\alpha+2}) = d(a(x_\alpha), a(x_{\alpha+1}))$.

But, as a is contracting, this can only happen if $x_\alpha = x_{\alpha+1} = a(x_\alpha)$ and hence if x_α is a fixed point of a.

Now suppose x, y are both fixed points of a. Then we have $d(x, y) = d(a(x), a(y))$ and so $x = y$ as a is contracting. Hence our fixed point is unique. □

Corollary (*)

Suppose $f : A \longrightarrow A$ is a map of flabby sheaves on Γ such that $E(f)$ is a contracting map. Then there is a unique $x \in A(1)$ such that $f(x) = x$.

Definition

If $f : A \longrightarrow A$ is a map of flabby sheaves and $f(x) = x$ for $x \in A(1)$, then we say x is a *fixed point of f*

Infinite Two Player Games

Two Player Games

Definition

An *infinite two player game*, $G_{A,B}(W)$, consists of the following

- A set A of moves for Player I
- A set B of moves for Player II
- A set $W \subseteq A^\omega \times B^\omega$ of winning conditions for Player I

Definition

A *run* of a game $G_{A,B}(W)$ consists of an infinite sequence $r = \langle (a_i, b_i) : i \in \omega \rangle \in A^\omega \times B^\omega$.

We say that Player I *wins a run* r if $r \in W$. We say Player II wins r otherwise.

Definition

A *Strategy for Player II in* $G_{A,B}(W)$ is a function $\tau : A^{<\omega} \to B^{<\omega}$. I.e. given an initial segment of moves Player I has made, τ tells Player II how to play. We also call this a *Type II* strategy

A *Strategy for Player I in* $G_{A,B}(W)$ is a function $\sigma : B^{<\omega} \to A^{<\omega}$ such that $\sigma(\langle b_i : i \in n \rangle)$ only depends on $\langle b_i : i \in n - 1 \rangle$. I.e. given an initial segment of moves Player II has made, σ tells Player I how to play. However, as Player I played first, on the nth round he only has access to the first n-1 moves of Player II. We also call this a *Type I* strategy.

Strategies For Two Player Games

Lemma

If σ is a strategy for Player I in $G_{A,B}(W)$, then σ is also a strategy for Player II in $G_{B,A}(A^\omega \times B^\omega - W)$

Definition

A *Type II* Strategy for Player I in $G_{A,B}(W)$ is a map σ which is a strategy for Player II in $G_{B,A}(A^\omega \times B^\omega - W)$

Strategies For Two Player Games

Definition

Suppose

- σ is a Type II strategy for Player I in $G_{A,B}(W)$
- τ is a Type II strategy for Player II in $G_{A,B}(W)$.
- $r = \langle (a_i, b_i) : i \in \omega \rangle \in (A \times B)^\omega$ is a run of $G_{A,B}(W)$.

then we say

- r is according to σ if $(\forall n) a_n = \sigma(\langle b_i : i \leq n \rangle)$
- r is according to τ if $(\forall n) b_n = \tau(\langle a_i : i \leq n \rangle)$

ω As Lattice

Definition

Let $\hat{\omega} = \omega \cup \{\infty\}$.

Theorem

There is a sheaf $\mathcal{N}$ on $\hat{\omega}$ such that

- $\mathcal{N}(\infty) = \omega^\omega$
- *For every subset $A \subseteq \omega^\omega$ there is a subpresheaf A^* of $\mathcal{N}$ which is a sheaf if and only if A is closed.*
- *For every subpresheaf $\mathcal{A} \subseteq \mathcal{N}$, $\mathcal{A}$ is a sheaf if and only if $\mathcal{A}(\infty)$ is closed.*

Player II Strategies and Maps of Trees

Theorem

Suppose $\mathcal{A}, \mathcal{B}$ are the sheaves on $\hat{\omega}$ with $\mathcal{A}(\infty) = A^\omega$ and $\mathcal{B}(\infty) = B^\omega$. Then

- *A Type II strategy, σ, for Player I in $G_{A,B}(W)$ is the same as a map of sheaves $\sigma : \mathcal{A} \to \mathcal{B}$*
- *A Type II strategy, τ, for Player II in $G_{A,B}(W)$ is the same as a map of sheaves $\tau : \mathcal{A} \to \mathcal{B}$*

Player II Strategies and Maps of Trees

Theorem

Suppose $\mathcal{A}, \mathcal{B}$ are the sheaves on $\hat{\omega}$ with $\mathcal{A}(\infty) = A^\omega$ and $\mathcal{B}(\infty) = B^\omega$. Then

- *A Type II strategy, σ, for Player I in $G_{A,B}(W)$ is the same as a map of sheaves $\sigma : \mathcal{A} \to \mathcal{B}$*
- *A Type II strategy, τ, for Player II in $G_{A,B}(W)$ is the same as a map of sheaves $\tau : \mathcal{A} \to \mathcal{B}$*

This suggests the question:

> "How can we characterize what it means for a sheaf map to be a strategy for Player I?"

Player I Strategies

Lemma

Suppose

- σ *is a Type I strategy for Player I*
- τ *is a Type II strategies for Player II*

Then there is a unique run $\sigma * \tau$ *according to both* σ *and* τ*.*

Player I Strategies

Lemma

Suppose

- σ *is a Type II strategy for Player I*
- *For all Type II strategies,* τ*, for Player II there is a unique run* $\sigma * \tau$ *according to both* σ *and* τ

Then σ *is a Type I strategy.*

Proof.

Suppose σ is as in the lemma. We will prove by induction that $\sigma(\langle b_i : i \in n \rangle)$ doesn't depend on b_n.

Base Case:

Suppose $\sigma(\langle x \rangle) \neq \sigma(\langle y \rangle)$ for some $x, y \in B$. Then let $\tau(\sigma(x)) = x$ and $\tau(z) = y$ for all $z \in A - \{\sigma(x)\}$. Further let $\tau(\mathbf{a}) = x$ for all $\mathbf{a} \in A^n$. We then have the following two sequences

$$\langle (\sigma(x), x)(\sigma(xx), xx), \ldots \rangle$$

and

$$\langle (\sigma(y), y), (\sigma(yx), yx), \ldots \rangle$$

are two runs according to σ and τ $\Rightarrow\Leftarrow$. □

Player I Strategies

Proof.

Inductive Case:
Assume that for $n < m$, $\sigma(\langle b_i : i \in n \rangle)$ only depends on $\langle b_i : i \in n-1 \rangle$ but there are $x, y \in B$ and $\langle b_i : i \in m-1 \rangle \in B^{m-1}$ such that

$$\sigma(\langle b_i : i \in m-1 \rangle x) \neq \sigma(\langle b_i : i \in m-1 \rangle y)$$

This then reduces to the base case by letting τ return $\langle b_i : i \in m-1 \rangle$ on the first $m-1$ moves no matter what σ does. □

Now suppose we want to extend this notion of an infinite two player game to where the length of a game is no longer ω but is instead an arbitrary complete (linearly ordered) lattice L. We then have the following definitions

Definition

A *two player game* on sheaves in $\textbf{Sheaf}(L)$, $G_{A,B}(W)$, consists of the following

- A flabby sheaf A of moves for Player I
- A flabby sheaf B of moves for Player II
- A set $W \subseteq A \times B(1)$ of winning conditions for Player I

Type II Strategies For Sheaf Games

Definition

- A *(Type II) Strategy for Player I in* $G_{A,B}(W)$ is a map of sheaves $\sigma : B \longrightarrow A$
- A *(Type II) Strategy for Player II in* $G_{A,B}(W)$ is a map of sheaves $\tau : A \longrightarrow B$.

Definition

A *run* of a game $G_{A,B}(W)$ consists of a map $1 \longrightarrow A \times B$. Or equivalently an element of $A \times B(1)$.

- If $\sigma : B \longrightarrow A$ is a Type II strategy for Player I
- If $\tau : A \longrightarrow B$ is a Type II strategy for Player II

We say that

- r is according to σ if $r = (\sigma(b), b)$ for some $b \in B(1)$.
- r is according to τ if $r = (a, \tau(a))$ for some $a \in A(1)$.

In particular $r = (a, b)$ is according to σ and τ if $a = \sigma(\tau(a))$ and $b = \tau(\sigma(b))$

Type I Strategies For Sheaf Games

Definition

We say a strategy σ for Player I is a *Type I Strategy* if for all (Type II) strategies τ for Player II, there is a unique run $\sigma * \tau$ which is according to both σ and τ.

Type I Strategies For Sheaf Games

Definition

We say a strategy σ for Player I is a *Type I Strategy* if for all (Type II) strategies τ for Player II, there is a unique run $\sigma * \tau$ which is according to both σ and τ.

This suggests the questions:

- For what linear orders do Type I strategies exist?
- When they do exist how can we recognize/characterize Type I strategies?

Type I Strategies and Contracting Maps

Theorem

A strategy $\sigma : B \to A$ is of Type I if and only if $E(\sigma)$ is a contracting map.

Proof.

$\Leftarrow$:
If $E(\sigma)$ is contracting then for all maps τ, $E(\sigma \circ \tau) = E(\sigma) \circ E(\tau)$ and $E(\tau \circ \sigma) = E(\tau) \circ E(\sigma)$ are contracting maps. Therefore, as A, B are flabby sheaves, we have $\sigma \circ \tau$ has a unique fixed point $a \in A(1)$ and $\tau \circ \sigma$ has a unique fixed point $b \in B(1)$.

But we also have that $\sigma(b)$ is a fixed point of $\sigma \circ \tau$ and $\tau(a)$ is a fixed point of $\tau \circ \sigma$. Hence $a = \sigma(b)$ and $b = \sigma(a)$. So (a, b) is the unique run according to σ and τ. □

Type I Strategies and Contracting Maps

For this part of the proof we will identify maps σ, τ with their images under E

Proof.

$\Rightarrow$:

Suppose σ is a strategy of type Type I and σ is not contracting. Then there are $b_0, b_1 \in E(B) = B(1)$ such that $d(\sigma(b_0), \sigma(b_1)) = d(b_0, b_1)$.

Let $\tau : E(A) \to E(B)$ be the map such that $\tau(x) = b_0$ if $d(x, b_0) \leq d(\sigma(b_0), \sigma(b_1))$ and $\tau(x) = b_1$ otherwise.

We then have $\tau(\sigma(b_0)) = b_0$ and $\tau(\sigma(b_1)) = b_1$.

Therefore $\tau \circ \sigma$ is not contracting and hence σ is not contracting. $\Rightarrow\Leftarrow$ □

Sets of Maps as Γ-Ultrametric Spaces

Generating Set and Hom Sets

We will now show that in many categories the Hom sets (i.e. sets of the form $C[A, B]$ where $A, B \in \text{obj}(C)$) can be viewed as generalized ultrametric spaces.

Definition

Let C be a category with $G \subseteq \text{obj}(C)$ and let $P(G, A) = Powerset(\bigcup_{X \in G} C[X, A])$.

For every $A, B \in \text{obj}(C)$ define $d_G : C[A, B] \times C[A, B] \to P(G, A)$ as follows:

- $d_G(f, g) = \{h : X \to A \text{ such that } f \circ h = g \circ h\}$.

For every $k : B \to D$ we define a map $k_! : (C[A, B], d_G) \to (C[A, D], d_G)$ by $k_!(g) = k \circ g$

Definition

Suppose C is a category and $G \subseteq \text{obj}(C)$. We say that G *generates* C if

$$(\forall f, g \in C[A, B])(\exists X \in G, h \in C[X, A]) f \neq g \rightarrow f \circ h \neq g \circ h$$

Maps, Generating Sets and Hom Sets

Definition

Suppose C is a category and $G \subseteq \text{obj}(C)$. We say that G *generates* C if

$$(\forall f, g \in C[A, B])(\exists X \in G, h \in C[X, A]) f \neq g \rightarrow f \circ h \neq g \circ h$$

Theorem

If G is a generating set of objects for C then $(C[A, B], d_G)$ is a $P(G, A)$-ultrametric space (where $\alpha \leq \beta$ if and only if $\beta \subseteq \alpha$ and $0 = \bigcup_{X \in G} C[X, A]$). Further if $f : B \rightarrow D$ then each map $f_!$ is non-expanding.

Proof.

First lets show that $(C[A,B], d_G)$ is a $P(G,A)$-ultrametric space.

(Symmetry) is immediate.

For (Reflexivity) notice that because G is a generating set of objects if $f, g \in C[A,B]$ and $f \neq g$ then there is some $a : X \to A$ with $X \in G$ such that $f \circ a \neq g \circ a$.

In particular if $f \neq g$ then $d_G(f,g) \neq 0$. □

Maps, Generating Sets and Hom Sets

Proof.

To show the strong triangle inequality suppose $f, g, h \in C[A, B]$.

Whenever $a : D \to A$ where $a \in d_G(f, g) \cap d_G(g, h)$ we have $f \circ a = g \circ a = h \circ a$ and hence $a \in d_G(f, h)$.

Therefore $d_G(f, g) \cap d_G(g, h) \subseteq d_G(f, h)$ or equivalently $d_G(f, g) \vee d_G(g, h) \geq d_G(f, h)$.

To see that any $k_!$ is a non-expanding map notice that if $a \in d_G(f, g)$ then $f \circ a = g \circ a$ and hence $k \circ f \circ a = k \circ g \circ a$ and $a \in d(k \circ f, k \circ g) = d(k_!(f), k_!(g))$.

So $d(k_!(f), k_!(g)) \supseteq d(f, g)$ or equivalently $d(k_!(f), k_!(g)) \leq d(f, g)$. □

Maps, Generating Sets and Hom Sets

Corollary

*If C is a category and G is a generating set then for each $A \in obj(C)$ there are functor $U_{G,A} : C \to P(G, A)$-**UltMet** given by*

- $(\forall B \in obj(C)) U_{G,A}(B) = (C[A, B], d_G)$
- $(\forall k \in C[B, D]) U_{G,A}(k) = k_!$

Proof.

This follows from the fact that $k_! \circ j_! = (k \circ j)_!$. □

Subobjects and Γ-Ultrametric Spaces

If our categories have epi-mono factorizations then, up to isomorphism in **GenUltMet**, our choice of generating set doesn't matter in determining the generalized ultrametric space structure put on Hom sets of the category.

Definition

Suppose C is a category with epi-mono factorizations.

Let $P(Sub_C) = Powerset(Sub_C(A))$.

For every $A, B \in \text{obj}(C)$ define $d_S : C[A, B] \times C[A, B] \longrightarrow P(S)$ as $d_S(f, g) = \{[h] : f \circ h = g \circ h\}$.

For every $k : B \longrightarrow D$ we define a map $k_! : (C[A, B], d_S) \longrightarrow (C[A, D], d_S)$ by $k_!(g) = k \circ g$

Equivalence of Structure on Hom Sets

Theorem

If $G \subseteq obj(C)$ then for each $A, B \in obj(C)$, and $f, g, x, y \in C[A, B]$ we have

(a) $d_S(x, y) \leq d_S(f, g) \Rightarrow d_G(x, y) \leq d_G(f, g)$.

(b) *If G is a generating set for C then* $d_G(x, y) \leq d_G(f, g) \Rightarrow d_S(x, y) \leq d_S(f, g)$.

Equivalence of Structure on Hom Sets

Proof.

Part (a):

Suppose $a : D \to A$ with $D \in G$.

a then has an epi-mono factorization $a = a_m \circ a_e$.

Now $a \in d_G(x, y)$ if and only if $x \circ (a_m \circ a_e) = y \circ (a_m \circ a_e)$ if and only if $x \circ a_m = y \circ a_m$ if and only if $[a_m] \in d_S(x, y)$ (the second to last equivalence follows because a_e is an epimorphism).

So if $[a_m] \in d_S(x, y) \to [a_m] \in d_S(f, g)$ then $a \in d_G(x, y) \to a \in d_G(f, g)$. Hence $d_S(x, y) \leq d_S(f, g)$ implies $d_G(x, y) \leq d_G(f, g)$. □

Proof.

Part (b):

To get a contradiction suppose

- G is a generating set for C
- $d_G(x, y) \leq d_G(f, g)$
- $a : D \rightarrowtail A$ with $[a] \in d_S(f, g)$
- $[a] \notin d_S(x, y)$

Then $x \circ a \neq y \circ p$ and hence there must be a map $p : D \rightarrow B$ such that $x \circ a \circ p \neq y \circ a \circ p$ and $D \in G$.

So $a \circ p \notin d_G(x, y)$ and hence by assumption $a \circ p \notin d_G(f, g)$ and $f \circ a \circ p \neq g \circ a \circ p$. But $f \circ a = g \circ a$ (by definition of $[a] \in d_S(f, g)$) and so $f \circ a \circ p = g \circ a \circ p \Rightarrow\Leftarrow$. □

Equivalence of Structure on Hom Sets

Corollary

If G is a generating set for C and $A, B \in obj(C)$ then $(C[A, B], d_S)$ is isomorphic to $(C[A, B], d_G)$ in **GenUltMet**.

Corollary

*If C is a category then for each $A \in obj(C)$ there is a functor $U_{Sub,A} : C \to P(Sub, A)$-***UltMet** *given by*

- $(\forall B \in obj(C))U_{Sub,A}(B) = (C[A, B], d_S)$
- $(\forall k \in C[B, D])U_{Sub,A}(k) = k_!$

Equivalence of Structure on Hom Sets

Theorem

If

- $E : \textbf{Flabby}(\Gamma) \to \textbf{UltMet}(\Gamma^{op})$ *is the equivalence of categories.*
- $r : \textbf{Sep}(\Gamma) \to \textbf{Flabby}(\Gamma)$ *is the right adjoint to the inclusion map.*

then there exists a natural isomorphism $\eta : E \circ r \Rightarrow U_{Sub,1}$.

Corollary (*)

Suppose $A \in obj(\textbf{Sep}(\Gamma))$ *is flabby. Then* $(\textbf{Sep}[1, A], d_S) \cong E(A)$.

Equivalence of Structure on Hom Sets

Corollary

Suppose $A, B \in \mathbf{Sep}(\Gamma)$ *and* $a : A \to B$*. Then* a *is contracting if and only if there exists*

- $d : D \rightarrowtail 1_\Gamma$ *a monic*
- $x, y : 1_\Gamma \to A$

such that

- $x \circ d \neq y \circ d$
- $a \circ x \circ d = a \circ y \circ d$

Thank You