Computability of 0-1 Laws

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0-1 Law For First Order Logic

Lets begin by reviewing what the 0-1 law for first order logic says

Theorem (0-1 Law For First Order Logic)

Suppose L is a finite language and for any first order formula φ of L let

$$F_{n,\varphi} = \frac{|\{M \models \varphi : |M| = n\}|}{|\{M : |M| = n\}|}$$

then $\lim_{n\to\infty} F_{n,\varphi}$ is 0 or 1

In other words any formula of first order logic holds asymptotically almost always or asymptotically almost never.

Since the discovery of the 0-1 law for first order logic there have been many attempts to generalize it. One of the most successful of which has come from limiting the collection of finite models we consider. For example the following all have 0-1 laws

Theorem (0-1 Law For Finite Graphs)

For any first order formula φ in the language of graphs let

$$G_{n,\varphi} = \frac{|\{M \models \varphi : |M| = n \textit{ and } M \textit{ is a graph}\}|}{|\{M : |M| = n \textit{ and } M \textit{ is a graph}\}|}$$

then $\lim_{n\to\infty} G_{n,\varphi}$ is 0 or 1.

Triangle Free Graphs and Bipartite Graphs

Theorem (0-1 Law For Triangle Free Finite Graphs)

For any first order formula φ in the language of graphs let

$$TF_{n,\varphi} = \frac{|\{M \models \varphi : |M| = n \text{ and } M \text{ is a triangle free graph}\}|}{|\{M : |M| = n \text{ and } M \text{ is a triangle free graph}\}|}$$

then $\lim_{n\to\infty} TF_{n,\varphi}$ is 0 or 1.

Triangle Free Graphs and Bipartite Graphs

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then $\lim_{n\to\infty} TF_{n,\varphi}$ *is* 0 *or* 1.

Theorem (0-1 Law For Bipartite Finite Graphs)

For any first order formula φ in the language of graphs let

$$BG_{n,\varphi} = \frac{|\{M \models \varphi : |M| = n \text{ and } M \text{ is a bipartite graph}|\}}{|\{M : |M| = n \text{ and } M \text{ is a bipartite}|\}}$$

then $\lim_{n\to\infty} BG_{n,\varphi}$ *is* 0 *or* 1.

Collections of Finite Structures

Now lets formulate a general version of a 0-1 law.

Definition

We call a collection $\mathbb{A}$ of finite L structures *valid* if for each n the collection $\mathbb{A}_n$, of structures in $\mathbb{A}$ of size n, is finite.

Notice we have not assumed that our language is finite. We have just required that we only consider a finite number of models of any given size.

Collections of Finite Structures

Definition

Suppose $\mathbb{A}$ is a valid collection of finite structures. Let

$$\mathbb{A}_{n,\varphi} = \frac{|\{M \models \varphi : |M| = n \text{ and } M \in \mathbb{A}\}|}{|\{M : |M| = n \text{ and } M \in \mathbb{A}\}|}$$

and let $\varphi_{\mathbb{A}} = \lim_{n\to\infty} \mathbb{A}_{n,\varphi}$ if it exists and be undefined otherwise.

Definition

We define $01Law(\mathbb{A})$ to be the collection of first order formulas φ such that $\varphi_{\mathbb{A}} = 1$. We say $\mathbb{A}$ satisfies the *0-1 law* if $01Law(\mathbb{A})$ is a complete theory.

Possible 0-1 Laws

Now can ask, "what types of theories can we get as the result of 0-1 laws for a finite collection of structures?"

Possible 0-1 Laws

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It turns out that we can get almost any theory.

Theorem (A, Freer, Patel)

Suppose T is a theory. Then the following are equivalent

(a) *There is a valid $\mathbb{A}$ such that $T = 01Law(\mathbb{A})$*

(b) *For all finite $X \subseteq T$ there is an $n_X \in \omega$ such that for all $m > n_X$ there is a model M of size m which satisfies all elements of X.*

Possible 0-1 Laws

Proof.

$\neg(b) \Rightarrow \neg(a)$:

If (b) doesn't hold then cofinally often (in the size of models) there is no model satisfying X of the appropriate size. Hence in the limit $\bigwedge X$ can't be satisfied asymptotically almost surely.

$(b) \Rightarrow (a)$:

List all formulas of T as $\varphi_0, \varphi_1, \ldots$. For each n then find an $m(n)$ such that for all $m > m(n)$ there is a model satisfying $\bigwedge_{i \leq n} \varphi_i$ of size m. Notice that $m(n)$ can be chosen to be a non-decreasing function. In particular for each m we can let M_m be a model which satisfies all $\bigwedge_{i \leq r} \varphi_i$ where r is the largest such that $m_r \leq m$.

We then let $\mathbb{A} = \{M_i : i \in \omega\}$ and it is immediate that $01Law(\mathbb{A}) = T$. □

Theories With Finite Models

A natural next question we might ask is "what collections of finite models do we want to consider?" One option is to consider all finite models of some first order theory. However this turns out to be no more restrictive than considering arbitrary valid collections of finite models.

Theorem

For every collection of finite models $\mathbb{A}$ there is a first order theory $T_{\mathbb{A}}$ whose finite models are exactly those in $\mathbb{A}$.

Theories With Finite Models

Proof.

For each n there are only finitely many models of size n. Call them $M_1, \ldots, M_{r_n}$. In particular for each such M_i there is a quantifier free formulas ψ_{M_i} such that $M_j \models \psi_{M_i}$ if and only if $i = j$ (i.e. ψ_{M_i} separates M_i from the other elements of $\mathbb{A}_n$.)

Let

$$S(n) = [(\exists x_1, \ldots, x_n) \bigwedge_{i \neq j} x_i \neq x_j] \wedge [(\forall x_1, \ldots, x_n, x_{n+1}) \bigvee_{i \neq j} x_i = x_j]$$

Our theory $T_{\mathbb{A}}$ then consists of the axioms

- If $M \models S(n)$ and $M \models \varphi_{M_i}$ then $M \cong M_i$.
- If $S(n)$ then $\bigvee_{i \leq r_n} \varphi_{M_i}$.

It is then immediate that the collection of finite models of $T_{\mathbb{A}}$ is exactly $\mathbb{A}$.

□

Random Models

In order to find a notion which is restrictive but still captures the cases we have seen we need an observation

Theorem

For each of

- *The collection of finite graphs*
- *The collection of finite triangle free graphs*
- *The collection of finite bipartite graphs*

There is a complete theory such that the above collection of finite models are just the collection of finite submodels of some infinite model of our theory.

We call the (unique) model the "random graph", "random triangle free graph", and the "random bipartite graph" respectively.

0-1 Law for a Theory

This suggests a definition

Definition

Suppose T is a (possibly incomplete) first order theory. Then let $\mathbb{A}(T)$ be the collection of finite substructures of infinite models of T.

Definition

We let $01Law(T)$ be the collection of first order formulas

$$\{\varphi : \varphi_{\mathbb{A}(T)} = 1\}$$

We say T satisfies the *0-1 Law* if $01Law(T)$ is a complete theory.

So $01Law(-)$ is a map from first order theories to first order theories.

Basic Properties of 0-1 Law for a Theory

There are some immediate properties of the $01\mathit{Law}$ function.

Lemma

For any T, $01\mathit{Law}(T)$ has only infinite models.

Lemma

If $\varphi \in T$ and φ is Π_1 then $\varphi \in 01\mathit{Law}(T)$.

Lemma

If T_0 and T_1 satisfy the same Σ_1 and Π_1 sentences then $01\mathit{Law}(T_0) = 01\mathit{Law}(T_1)$.

0-1 Law Is Non-Trivial

The first thing we want to show about the $01Law$ operator is that it is non-trivial. I.e. there are complete theories T such that $01Law(T)$ is complete and $01Law(T) \neq T$.

0-1 Law Is Non-Trivial

The first thing we want to show about the $01Law$ operator is that it is non-trivial. I.e. there are complete theories T such that $01Law(T)$ is complete and $01Law(T) \neq T$.

Theorem

Let

- T_{TFG} *be the theory of the random triangle free graph*
- T_{BiP} *be the theory of the random bipartite random graph*

Then $01Law(T_{TFG}) = 01Law(T_{BiP}) = T_{BiP}$

Computability of the Operator

Now we can ask the main question of the talk:

"If T is a complete computable theory with a 0-1 law, how computable can $01Law(T)$ be?"

Upper Bounds

Lemma

If T is a computable, Σ_1 complete theory then $\mathbb{A}(T)$ is computable.

Proof.

Because the Σ_1 theory determines which atomic types are realized. □

Upper Bounds

Theorem

Suppose T is a Σ_1-complete computable theory. Then $01Law(T)$ is Turing reducible to $0'''$.

Outline.

- We know that $\varphi \in 01Law(T)$ if and only if $(\forall\epsilon)(\exists n)(\forall m > n)|\varphi_{\mathbb{A}(T),m} - 1| < \epsilon$
- So the relation "$\varphi \in 01Law(T)$" is Π^0_3 and hence computable from $0'''$

$\square$

Upper Bounds

Theorem

Suppose T is a Σ_1-complete computable theory satisfying the 0-1 law. Then $01Law(T)$ Turing reducible to $0'$.

Proof.

- We know that for all first order formulas φ, either $\varphi_{\mathbb{A}} = 0$ or $\varphi_{\mathbb{A}} = 1$.
- $\varphi \in 01Law(T) \Leftrightarrow (\exists n)(\forall m > n)|\varphi_{\mathbb{A}(T),m} - 1| < 1/3$. Hence "$\varphi \in 01Law(T)$" is a Σ^0_2 formula.
- $\varphi \notin 01Law(T) \Leftrightarrow (\exists n)(\forall m > n)|\varphi_{\mathbb{A}(T),m} - 0| < 1/3$. Hence "$\varphi \in 01Law(T)$" is a Π^0_2 formula.
- So "$\varphi \in 01Law(T)$" is Δ^0_2 and hence computable from $0'$

□

Lower Bound

Theorem (Main Theorem)

Suppose B^* is a Δ^0_2 subset of ω. Then there is

- *A computable theory T_{0-1Law} (independent of B^*)*
- *A computable set of formulas Φ (independent of B^*)*
- *A computable complete theory $T_{Basic}(B)$*

such that

- $01Law(T_{Basic}(B))$ *is complete.*
- $01Law(T_{Basic}(B)) = T_{0-1Law} \cup X$ *for an* $X \subseteq \Phi$
- X *is computably isomorphic to* B^*

Lemma

Before we start the construction there is an important lemma we will use later

Lemma

If $L = \{R\}$ is an m-ary relation. Then there are 2^{n^m} many models of L of size n

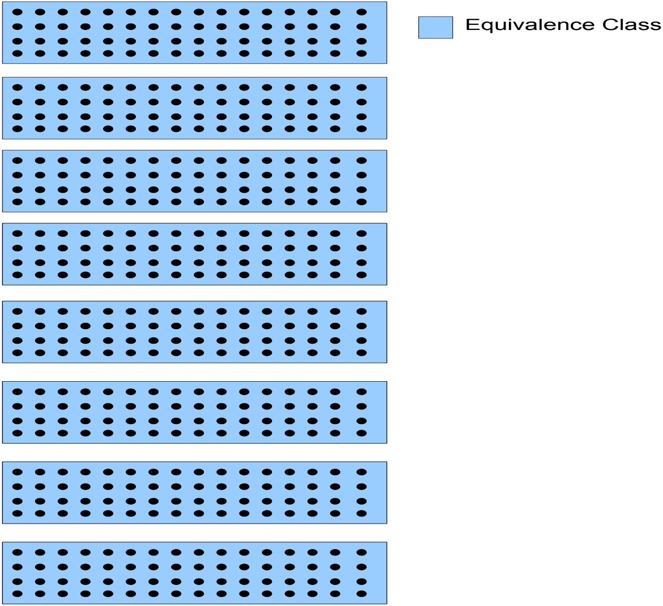
In particular increasing the arity of the relation dramatically increases the number of models.

Outline of Method

An intuitive outline of our construction is as follows

- We have a background model consisting of infinitely many equivalence classes each of which has infinitely many objects. We also have a successor relation which preserves the equivalence classes.
- We represent our Δ^0_2 set B^* as a limit of a recursive set $B(s, n)$
- We divide each standard equivalence class $[n]$ into infinitely many partitions each of which is infinite. We then add a random relation of arity s on an infinite subset of $[n]$ so as to characterize $B(s, n)$.
- We finally ensure that the relations of arity s are only realized if the first s equivalence classes each have s elements each.

Picture of the Infinite Model



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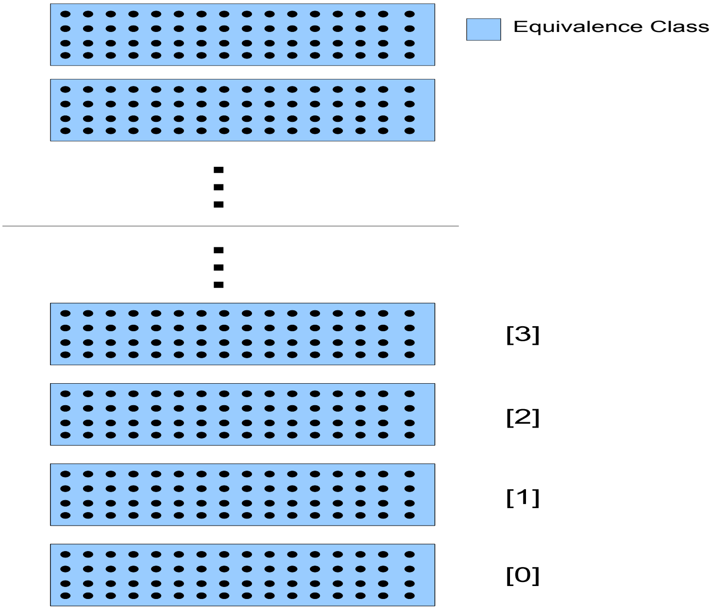

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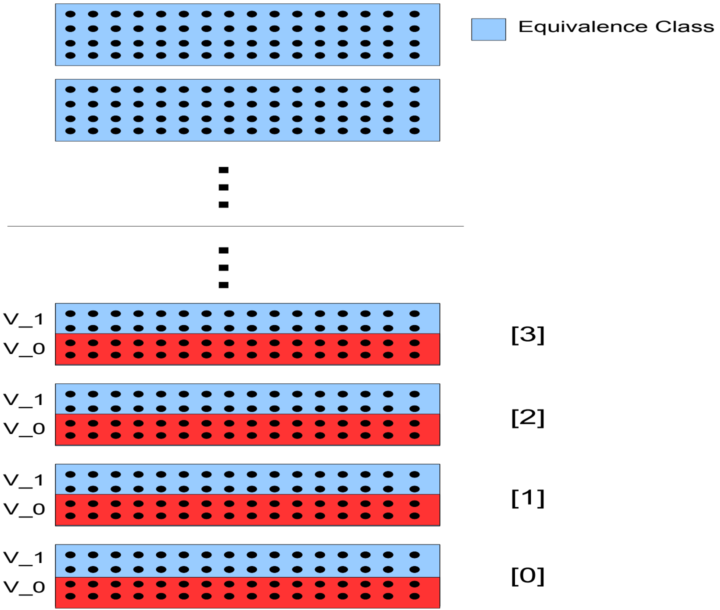

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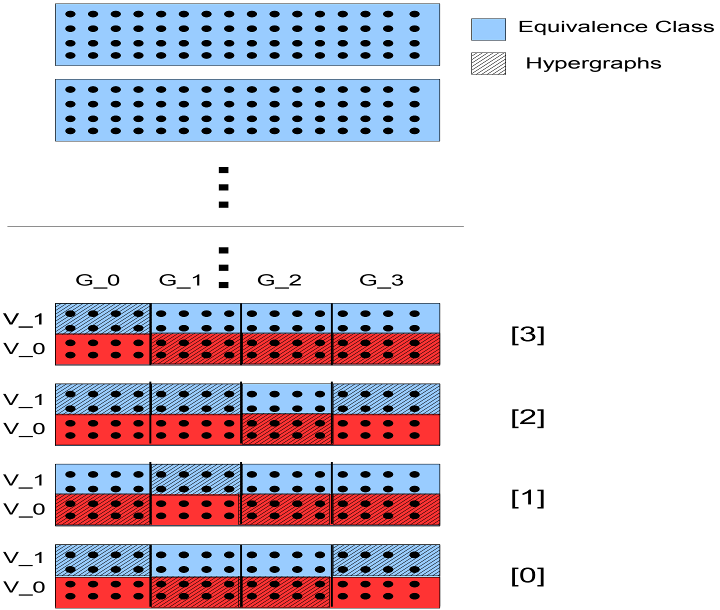


The Language

Definition

Let L_{Basic} be a language containing relations of the following type

- $E(x, y), S(x, y)$ which are meant to capture that E is an equivalence relation and S is a successor relation.
- $C_n(z_0, \ldots, z_n), D_n(z_1, \ldots, z_n)$ witness properties of particular elements or equivalence classes.
- $V_n^0(z_0, \ldots, z_n)$, $V_n^1(z_0, \ldots, z_n)$ represent value of our function B^*.
- $W_{n,m}(\mathbf{x}_0, \ldots, \mathbf{x}_n, \mathbf{y})$ represents a cumulative witness to the fact that $\mathbf{y}$ is in the mth equivalence class and is "allowed" to have a relation of arity n ($m \leq n$)
- $G^i_{n,m}(\mathbf{x}_0, \ldots, \mathbf{x}_n, \mathbf{y})$ represents that $\mathbf{y}$ is in the mth equivalence class, in the hypergraph of arity n, and has "value" $i \in \{0, 1\}$.

Absolute Theory

Definition

The theory, $T_{abs} \subseteq T_{basic}(B)$, which is preserved when we apply $01Law(-)$ to $T_{Basic}(B)$, says

- Equivalence Axioms
  - $E(x, y)$ is an equivalence relation.
  - Each E equivalence class has infinitely many elements.
- Successor Axioms
  - $S(x, y)$ is a successor relation which respects the equivalence relation
  - There is a unique equivalence class without a predecessor which we call [0]

An element x (or equivalence class) is *standard* in a model if $\bigvee_{n \in \omega} x = [x]$ (i.e. it represents some finite number of successors of 0).

Absolute Theory

Definition

- Witness Axioms
  - If we let $[n]$ be the nth successor of $[0]$ then $C_n(z_0, \ldots, z_n)$ is equivalent to $\bigwedge_{i \leq n} z_i \in [i]$. We also let $C_n^*(z)$ be equivalent to $(\exists z_0, \ldots, z_{n-1}) C_n(z_0, \ldots, z)$, i.e. $z \in [n]$.
  - $D_n(z_1, \ldots, z_n)$ holds if and only if all of the z_i are distinct and in the same equivalence class. We also let $D_n^*(z)$ be equivalent to $(\exists z_0, \ldots, z_{n-1}) D_n(z_0, \ldots, z)$, i.e. the equivalence class of z has at least n elements.
  - $W_{n,m}(\mathbf{x}_0, \ldots, \mathbf{x}_n, \mathbf{y})$ holds if and only if
    - $D_n(\mathbf{x}_i)$ for all i.
    - For each i there is an $x_i \in \mathbf{x}_i$ such that $C_n(x_0, \ldots x_n)$
    - For each $y, y' \in \mathbf{y}$, $E(y, y')$
    - $\mathbf{y} \subseteq [m]$

Absolute Theory

Definition

- Value Axioms
  - $C_n(z_0, \ldots, z_n)$ if and only if $V_n^0(z_0, \ldots, z_n)$ or $V_n^1(z_0, \ldots, z_n)$
  - At most one of $V_n^i(z_0, \ldots, z_n)$ is realized and it depends only on z_n. We let $V_{i,n}^*(z)$ be equivalent to $(\exists z_0, \ldots, z_{n-1}) V_n^i(z_0, \ldots, z)$

Basic Theory

Definition

Next let T_{basic} be the theory which says

- T_{abs}
- Successor Axioms
  - Every equivalence class has an S successor.
- Value Axioms
  - For each $n \in \omega, i \in \{0, 1\}$ there are infinitely many z which satisfy $V^*_{i,n}(z)$.

Basic Theory

Definition

- (Hyper)graph Axioms
  - If $G^i_{n,m}(\mathbf{x}_0,\ldots,\mathbf{x}_n,\mathbf{y})$ holds then so does $W_{n,m}(\mathbf{x}_0,\ldots,\mathbf{x}_n,\mathbf{y})$
  - $G^i_{n,m}(\mathbf{x}_0,\ldots,\mathbf{x}_n,\mathbf{y})$ and $W_{n,m}(\mathbf{x}'_0,\ldots,\mathbf{x}'_n,\mathbf{y})$ implies $G^i_{n,m}(\mathbf{x}'_0,\ldots,\mathbf{x}'_n,\mathbf{y})$

  We let $\overline{G}^i_{n,m}(\mathbf{y}) \Leftrightarrow (\exists \mathbf{x}_0,\ldots,\mathbf{x}_n) G^i_{n,m}(\mathbf{x}_0,\ldots,\mathbf{x}_n,\mathbf{y})$.
- If $\overline{G}^i_{n,m}(\mathbf{y})$ holds then $V^*_{i,n}(y)$ holds for all $y \in \mathbf{y}$.
- If $n \neq n'$, $\overline{G}^i_{n,m}(\mathbf{y})$ and $\overline{G}^i_{n',m}(\mathbf{y}')$ hold then $\mathbf{y} \cap \mathbf{y}' = \emptyset$
- $(\exists \mathbf{y})\overline{G}^0_{n,m}(\mathbf{y}) \leftrightarrow (\forall \mathbf{y})\neg\overline{G}^1_{n,m}(\mathbf{y})$
- If the domain of $\overline{G}^i_{n,m}$ is non-empty then $\overline{G}^i_{n,m}$ is an infinite n-hypergraph on its domain (i.e. on the elements that are a member of at least one edge).

Basic Theory with Parameter

Definition

Let $B : \omega \times \omega \to 2$ and let $T_{basic}(B)$ be the theory with

- T_{basic}
- For all m, n $(\exists \mathbf{x}_0, \ldots, \mathbf{x}_n, \mathbf{y}) G_{n,m}^{B(n,m)}(\mathbf{x}_0, \ldots, \mathbf{x}_n, \mathbf{y})$

Theorem

$T_{basic}(B)$ *is consistent.*

Theorem

$T_{basic}(B)$ *is complete.*

Theory of Asymptotically Almost Sure Theory

Definition

Let T_{0-1Law} be the following theory

- T_{abs}
- Successor Axioms
  - There is a unique equivalence class with out a successor
- Value Axioms
  - For each n, $x, y \in [n]$ and $i \in \{0, 1\}$ we have $V^*_{i,n}(x) \leftrightarrow V^*_{i,n}(y)$.
- (Hyper)graph Axioms
  - For each n, m, no $\mathbf{y}$ satisfies $\overline{G}^i_{n,m}(\mathbf{y})$.

Theory of Asymptotically Almost Sure Theory with Parameters

Once again, we almost have a complete theory. But this time we need to determine what which value is taken at each standard level.

Definition

Let $B^* : \omega \to 2$ be a function and let $T_{0-1Law}(B^*)$ be the theory with

- T_{0-1Law}
- If $B^*(n)$ is defined then $(\exists x_n) V^*_{B^*(n),n}(x_n)$

Theorem

$T_{0-1Law}(B^*)$ is complete.

Relations Between Theories

Theorem

Suppose $B : \omega \times \omega \to 2$ *and* $B^* : \omega \to 2$ *is such that*

$$(\forall n) \lim_{s \to \infty} B(s, n) = B^*(n)$$

Then $01Law(T_{basic}(B)) = T_{0-1Law}(B^*)$ *and* $T_{0-1Law}(B^*)$ *is complete.*

Relations Between Theories

Outline.

Suppose we select randomly a structure of size N. Then with asymptotic probability 1 the following will be true

(1) There will be exactly $Floor(\sqrt{N})$ many equivalence classes which represent the first $Floor(\sqrt{N})$ standard equivalence classes.

(2) There will be at least $Floor(\sqrt{N})$ many elements of each of the equivalence classes

□

Proof.

The reason is that the number of possible models of size N which have a single $n+1$-ary relation vastly dwarfs the number of structures which have a single n-ary relation. Hence the our structures will all tend to have the maximal arity random relation which is allowed by their size. I.e. a relation of arity $Floor(\sqrt{N})$.

However, conditions (1) and (2) are exactly the conditions which need to be satisfied to allow a finite structure to have a $Floor(\sqrt{N})$-ary random relation. □

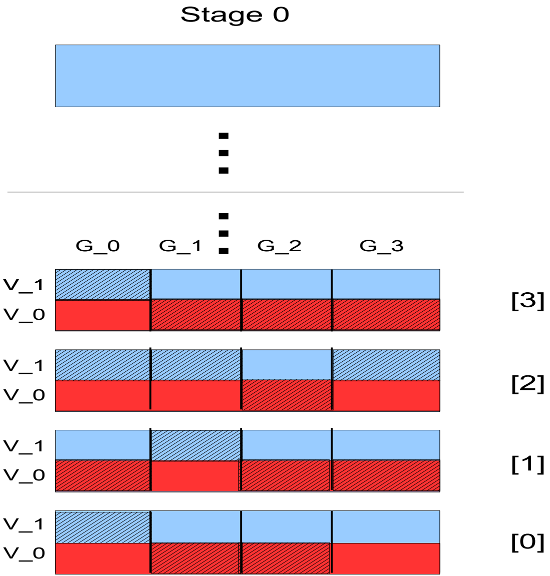
Proof.

Further we have

(3) All but one equivalence class will contain exactly $Floor(\sqrt{N})$ many elements and the remaining equivalence class will contain the rest.

(3) follows because among the models of size N which have $Floor(\sqrt{N})$ many equivalence classes (each with its own random relation) there are vastly more models which only have one equivalence relation without the minimual number of elements. □

Almost Sure Growth of Finite Models



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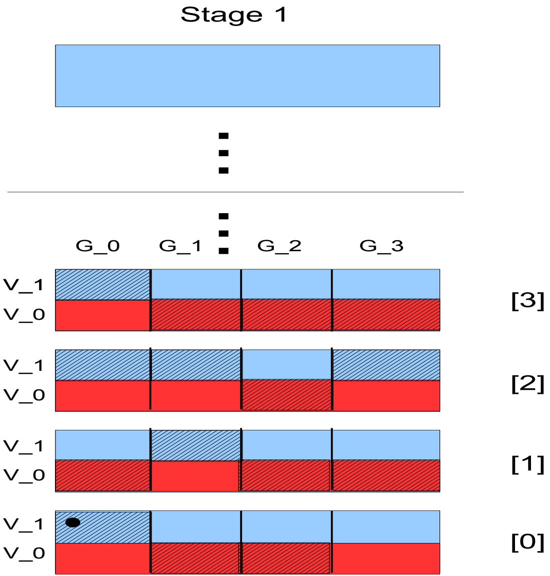

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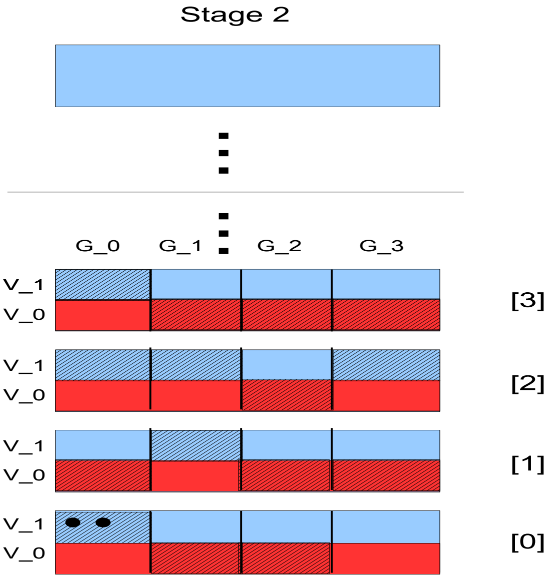

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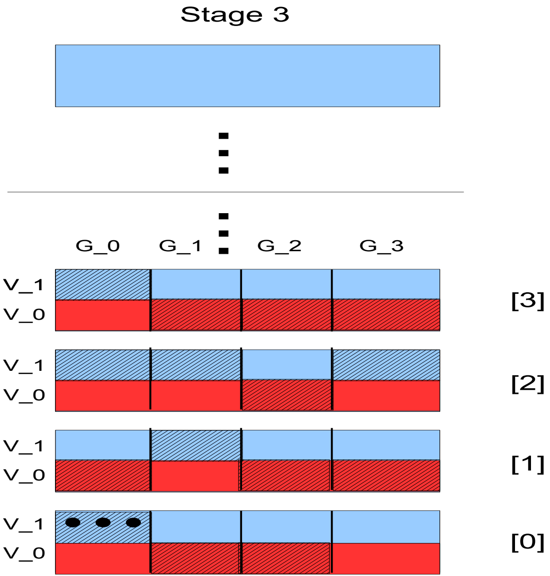

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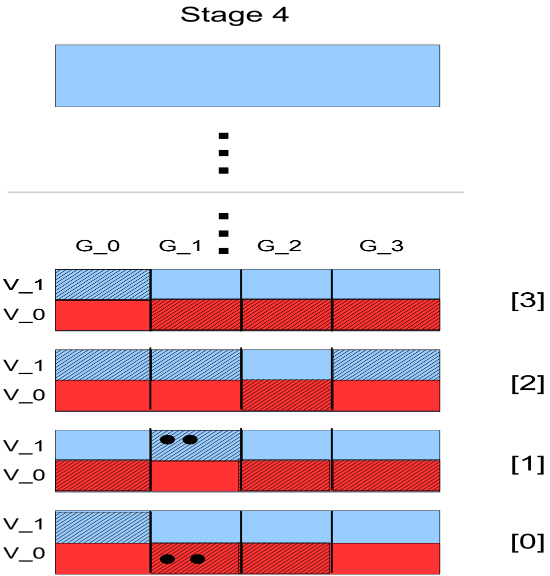

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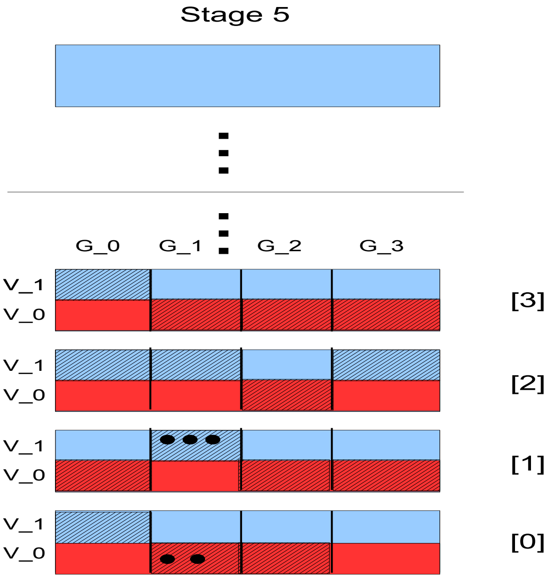

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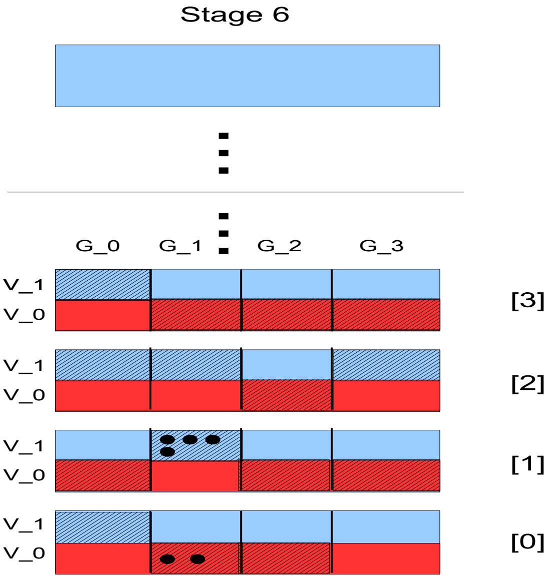

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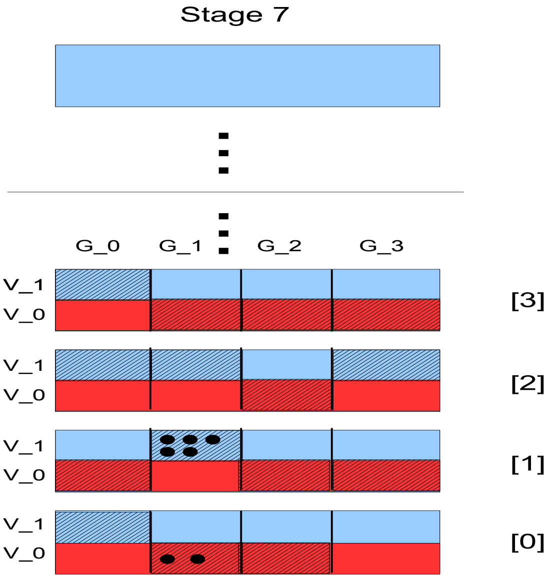

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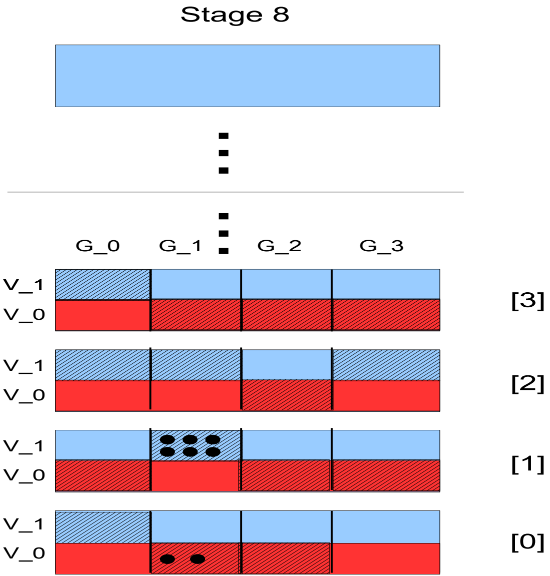

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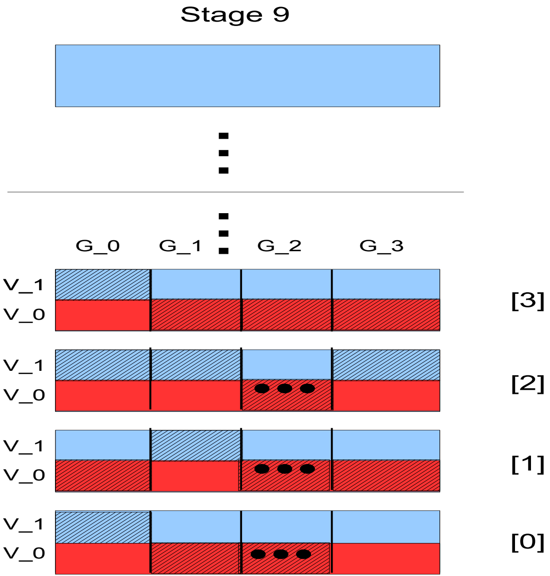

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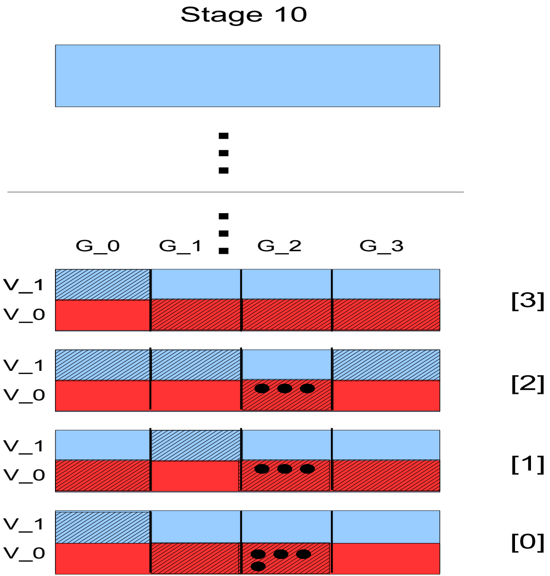

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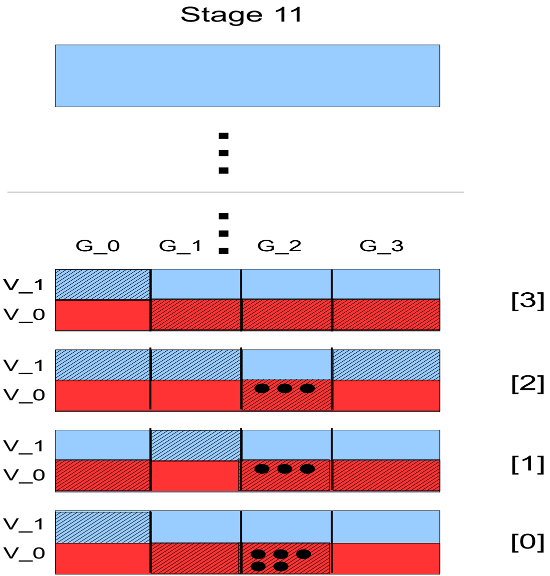

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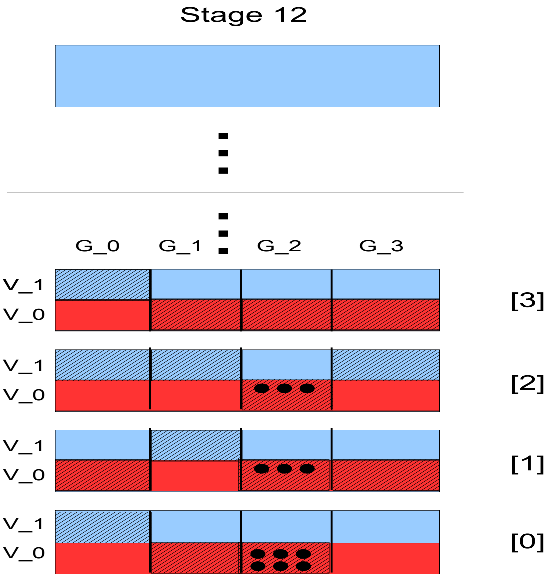

1-Reducibility

First lets review the definition of being 1-reducible.

Definition

We say that a set A is 1-reducible to B ($A \leq_1 B$) if there is an injective computable function $f : \omega \to \omega$ such that $f''[A] \subseteq B$ and $f''[\omega/A] \subseteq \omega/B$.

This is one of the strongest forms of computable reduction and in particular we have

Theorem (Myhill Isomorphism Theorem)

If $A \leq_1 B$ and $B \leq_1 A$ then A and B are computably isomorphic.

Computability of $T_{0-1Law}(B^*)$

Theorem

There is a recursive collection of formulas Φ such that if $\overline{B}^*$ is any subset of Φ then $T_{0-1Law} \cup \overline{B}^*$ is complete and consistent.

Proof.

Let $\Phi = \{(\exists x)V^*_{1,n}(x) : i \in \{0,1\}, n \in \omega\}$. □

Theorem

Fix a computable enumeration of Φ. For every $B^* : \omega \to 2$ let $\overline{B}^* = T_{0-1Law}(B^*) \cap \Phi$. Then there is $\overline{B}^*$ is 1-equivalent to B^* (considered as a subset of ω). In particular $\overline{B}^*$ is computably isomorphic to B^*.

Thank You