ON RELATIVIZATION OF COMPLETE METRIC SPACES

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Abstract. We make precise a notion of *relativization* of a complete metric structure. We then consider when a property of such a complete metric structure is absolute, i.e. preserved by moving to a larger or smaller model of set theory. In particular we show all Σ_1 sentences of $\mathcal{L}_{\infty,\infty}$ are absolute provided in prenex disjunctive normal form there are no uncountable conjunctions. We show how this can be used to prove that all first order formulas of continuous logic are absolute as well as produce a generalization of Mostrowski absoluteness to uncountable cardinals. We also show for any abstract property $\mathcal{P}$ (with minor assumptions), if $\mathcal{P}$ is absolute so is the statement that *locally* $\mathcal{P}$ *holds*.

1. Introduction

Define a complete metric structure to be a first order structure all of whose sorts are complete metric spaces and all of whose functions are continuous. This paper is motivated by the question: "What properties of complete metric structures are absolute between models of set theory?". However, before we can even make sense of this question, we run into a difficulty: given two transitive models of set theory $(V_0, \in) \subseteq (V_1, \in)$ and a structure $\mathcal{M} \in V_0$ such that $V_0 \models$"$\mathcal{M}$ is a complete metric structure", it is not necessarily the case that $V_1 \models$"$\mathcal{M}$ is a complete metric structure."

This difficulty arises because being complete is not a first order property of a metric space and hence is not (necessarily) absolute. In order to get around this difficulty we need a notion of what it means for a complete metric structure in V_1 to be "the same as" a complete metric structure in V_0. In Section 2 we make precise this notion of a structure satisfying a higher order property being the same as a structure satisfying the same higher order property in a larger model of set theory. We do this by giving a definition of the relativization of a structure as well as a definition of absoluteness. Our notion of relativization is a special

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case of the notion given in [1] where the relativization of Grothendieck toposes are considered.

With a precise definition of relativization and absoluteness in hand we proceed in Section 3 to give our representation of complete metric structures. Having a precise representation of our complete metric structures is important if we wish to have a precise notion of what a relativization of a complete metric structure is and hence determine which properties are absolute. In particular we will see in Lemma 3.23 that if the only requirements on our complete metric structure is that the sorts are complete metric spaces and the maps are continuous, then even first order Π_1 sentences need not be upwards absolute. This non-absoluteness stems from the fact that we have not placed any requirement on the structure of the set of elements satisfying a relation. In Section 3.3 we consider what happens to relativizations of complete metric structures if we add the assumption that the relations are open, closed, or Borel.

In Section 4 we prove our main absoluteness results. We begin in Section 4.1 by showing one of the most important result of this paper; that, under the assumption that relations are closed, every Σ_1 sentence of $\mathcal{L}_{\infty,\infty}(\mathrm{L})$ which contains no uncountable conjunctions is absolute for all complete metric structures. In Section 4.2 we show that Σ_2 sentences with no uncountable disjunctions are upward absolute as well as giving conditions when certain Σ_2 sentences are downwards absolute.

In Section 5 we give several applications of the absoluteness results from Section 4. In particular in Section 5.1 we show that the inf operator is absolute and also discuss how to deduce from this that all formulas of continuous first order logic (as defined in [2]) are absolute. In Section 5.2 we will show how our results give a generalization of the Mostrowski absoluteness theorem for κ^κ. Finally in Section 5.3 we consider specific properties of complete metric spaces which may or may not be absolute.

We end this paper by applying the absoluteness results of Section 4 to show that (under minor assumptions), if $\mathcal{P}$ is an upward or downward absolute property of a complete metric space then the property "locally $\mathcal{P}$ holds" is also upward or downward absolute (respectively).

1.1. **Notation and Conventions.** Let $(*_{\mathbf{cc}})$ be the statement: "The inclusion functor from the category of Cauchy complete metric spaces and continuous maps into the category of metric spaces and Cauchy-continuous maps has a left adjoint $\mathbf{cc}$". Let $(+_{\mathbf{cc}})$ be the statement: "For every complete metric space $\mathcal{M}$ with dense set D and every element $x \in \mathcal{M}$ there is a distinguished Cauchy sequence $\langle x_i : i \in \omega\rangle \subseteq D$

which converges to x." In this paper we will work in a fixed background model, $(\mathrm{SET}, \in)$, of ZF + $(*_{\mathbf{cc}})$ + $(+_{\mathbf{cc}})$. Note that this $(*_{\mathbf{cc}})$ + $(+_{\mathbf{cc}})$ follows from the global axiom of choice. By a **standard model** of set theory we mean a transitive subclass M of SET such that $(M, \in)$ is a model of $\mathrm{ZF} + (*_{\mathbf{cc}}) + (+_{\mathbf{cc}})$. In what follows it will be useful to fix two such models V_0 and V_1 with $V_0 \subseteq V_1$.

Suppose $\varphi(x)$ is a formula in the language of set theory and V is a standard model of set theory. By $\varphi(x)^V$ we mean the formula obtained by bounding all quantifiers in φ by V. We also sometimes abuse notation and use $\varphi(x)$ for the class $\{x \in \mathrm{SET} : \mathrm{SET} \models \varphi(x)\}$ and likewise abuse notation and use $\varphi(x)^V$ for class $\{x \in V : V \models \varphi(x)\}$. We will denote the non-negative real numbers by $\mathbb{R}^{\geq 0}$ and the non-negative and positive rational numbers by $\mathbb{Q}^{\geq 0}$ and $\mathbb{Q}^{>0}$ respectively. We let $\mathfrak{P}(x)$ denote the powerset of x.

In this paper all functions and relations will be finitary (i.e. have only a finite number of arguments). We will assume that the collection of sorts of each language is closed under taking finite sequences. This is a convention from which we loose no generality, but which will simplify the presentation as it will allow us to treat a finite set of sorts as a single sort. In particular we will be able to assume, without loss of generality, that all functions and relations have a single argument. If S is a sort we denote the sequence consisting of n copies of S by S^n. Given a sequence of sorts, the interpretation in any model of the sequence is always the product of the interpretations of the sorts in the sequence. In this paper L and its variants will be (first order) languages.

If $\mathcal{M} = (M, d_M)$ is a metric space and $C \subseteq M$ then we let $\overline{C}$ denote the closure of C (in M). If $x \in M$ and $q \in \mathbb{Q}^{>0}$ we define the **open ball** around x of radius q to be $B^{\mathcal{M}}(x, r) := \{y \in \mathcal{M} : d(x, y) < q\}$ (and we will omit the superscript when it is clear from the context). If $\leq$ is a linear order we let "$z = \max\{x_1, \ldots, x_n\}$" be the quantifier free 1st order formula which says z is the greatest element of $\{x_1, \ldots, x_n\}$.

For theorems or definitions not explicitly mentioned in this paper we refer the reader to such standard texts as [5] for model theory and [6] for set theory.

1.2. **Related Work.** In addition to studying absoluteness of metric spaces we could also consider which properties are absolute when we consider them as topological spaces. One particularly fruitful way to study absoluteness in this context is by taking an elementary substructure of SET containing our space and considering the differences between our space and the corresponding object in the elementary substructure. This approach differs from ours in several ways. For example it uses non-transitive models of set theory, it only looks at the topological structure of the spaces and not their metric structure, and it requires a tight relationship between the two models of set theory which are considered. For more on this approach to absoluteness of topological spaces we refer the reader to [3] or [4].

2. Absoluteness

2.1. **Relativization.** Suppose we have an L-structure $\mathcal{M}_0$ in V_0 which satisfies some, possibly higher order, property $\mathcal{P}$. In V_1, while $\mathcal{M}_0$ is still an L-structure, $\mathcal{M}_0$ may no longer satisfy $\mathcal{P}$. However, it may be the case that there is an L-structure $\mathcal{M}_1$ in V_1 which satisfies $\mathcal{P}$, contains $\mathcal{M}_0$, and is the *smallest* structure in V_1 with these two properties. In this case we call $\mathcal{M}_1$ a *relativization* of $\mathcal{M}_0$ to V_1 for $\mathcal{P}$.

We now make this precise. For notational convenience we will restrict attention to first order languages, i.e. languages where relations and functions only take elements and not subsets as arguments. $\mathcal{P}$ and its variants will always be an *abstract* property of L structures, i.e. some class $\mathcal{P}(x)$ of L-structures definable in the language of set theory with parameters. We will write $\mathcal{M} \models \mathcal{P}$ for SET $\models \mathcal{P}(\mathcal{M})$.

For a more thorough discussion of the notion of relativization, which includes higher order relations, see Section 3.4 of [] where the concept was originally introduced.

Definition 2.1. *Let Mod_{L} be the category whose objects are* L*-structures and whose maps are homomorphisms. We let $Mod_{\mathrm{L}}(\mathcal{P})$ be the full subcategory of Mod_{L} consisting of those models which satisfy $\mathcal{P}$.*

Definition 2.2. *Let $\mathcal{M}$ be an* L*-structure. We define $\mathrm{Ext}^{\mathcal{P}}\mathcal{M}$ to be the category whose objects are* L*-structures $\mathcal{N}$ such that:*

- $\mathcal{M} \subseteq \mathcal{N}$.
- *The inclusion map of $\mathcal{M}$ into $\mathcal{N}$ is a homomorphism.*
- $\mathcal{N} \models \mathcal{P}$.

and whose morphism are those homomorphisms $f : \mathcal{N} \to \mathbb{S}$ such that $f(m) = m$ for all $m \in \mathcal{M}$.

We can think of $\mathrm{Ext}^{\mathcal{P}}\mathcal{M}$ as the category of extensions of $\mathcal{M}$ to models which satisfy $\mathcal{P}$. It is worth noting that $\mathrm{Ext}^{\mathcal{P}}\mathcal{M}$ can be described by a formula in the language of set theory and as such, for any standard model of set theory V containing $\mathcal{M}$, it makes sense to talk about $(\mathrm{Ext}^{\mathcal{P}}\mathcal{M})^V$. In particular $(\mathrm{Ext}^{\mathcal{P}}\mathcal{M})^V$ is the category whose objects are those L-structures $\mathcal{N} \in V$ which contain $\mathcal{M}$ and where $(\mathcal{N} \models \mathcal{P})^V$.

Definition 2.3. *Suppose* $\mathcal{M}$ *is an* L*-structure such that:*

- $\mathcal{M} \in V_0$.
- $(\mathcal{M} \models \mathcal{P})^{V_0}$.

Suppose $\mathcal{N}$ *is an object of* $(\mathrm{Ext}^{\mathcal{P}}\, \mathcal{M})^{V_1}$ *such that:*

(1) *For every object* $\mathbb{S}$ *of* $(\mathrm{Ext}^{\mathcal{P}}\, \mathcal{M})^{V_1}$ *there is a map* $i : \mathcal{N} \to \mathbb{S}$ *in* $(\mathrm{Ext}^{\mathcal{P}}\, \mathcal{M})^{V_1}$.
(2) *Every endomorphism of* $\mathcal{N}$ *in* $(\mathrm{Ext}^{\mathcal{P}}\, \mathcal{M})^{V_1}$ *is an automorphism.*

We then say that $\mathcal{N}$ *is a* **relativization** *of* $\mathcal{M}$ *to* V_1 *for* $\mathcal{P}$.

If $\mathcal{M} \in V_0$ and $\mathcal{N}$ is a relativization of $\mathcal{M}$ to V_1 for $\mathcal{P}$ then (1) ensures that when $\mathbb{S}$ is any model of $\mathcal{P}$ in V_1 which contains $\mathcal{M}$ then $\mathbb{S}$ must also contain a copy of $\mathcal{N}$. Further, condition (2) ensures that if we have two distinct relativizations then there must be an isomorphism between them which is the identity on $\mathcal{M}$. It therefore makes sense to think of the relativization as the *smallest* extension of $\mathcal{M}$ to a model of $\mathcal{P}$ in V_1.

Lemma 2.4. *Suppose* $Th \in \mathcal{L}_{\infty,\omega}(L)$ *and* $(\mathcal{M} \models Th)^{V_0}$. *Then* $\mathcal{M}$ *is a relativization of* $\mathcal{M}$ *to* V_1 *for the formula "*$- \models Th$*".*

Proof. This is because the satisfaction relation for $\mathcal{L}_{\infty,\omega}(\mathrm{L})$ is absolute and hence $(\mathcal{M} \models Th)^{V_1}$ as well. □

As we will see, Lemma 2.4 will allow us to conclude that the relativization of any metric structure, as a metric structure, is itself. However, the case of complete metric structures will be slightly more complicated.

2.2. **Absoluteness.** Now that we have a notion of a relativization we can make precise what it means for a property, relation, or function to be absolute.

Definition 2.5. *Suppose* $\mathcal{P}$ *and* $\mathcal{P}^*$ *are properties of* L*-structures. We say* $\mathcal{P}^*$ *is* **upward absolute** *between* V_0 *and* V_1 *for* $\mathcal{P}$ *if for all* L*-structures* $\mathcal{M}_0 \in V_0$ *with* $(\mathcal{M}_0 \models \mathcal{P})^{V_0}$ *and* $\mathcal{M}_1 \in V_1$ *with* $\mathcal{M}_1$ *the relativization of* $\mathcal{M}_0$ *to* V_1 *for* $\mathcal{P}$ *we have*

$$(\mathcal{M}_0 \models \mathcal{P}^*)^{V_0} \text{ implies } (\mathcal{M}_1 \models \mathcal{P}^*)^{V_1}$$

We say $\mathcal{P}^*$ *is* **downward absolute** *between* V_0 *and* V_1 *for* $\mathcal{P}$ *if* $\neg\mathcal{P}^*$ *is upward absolute between* V_0 *and* V_1 *for* $\mathcal{P}$. *We say* $\mathcal{P}^*$ *is* **absolute** *between* V_0 *and* V_1 *for* $\mathcal{P}$ *if it is both upwards and downwards absolute between* V_0 *and* V_1 *for* $\mathcal{P}$.

In other words a property $\mathcal{P}^*$ is upwards absolute between V_0 and V_1 for $\mathcal{P}$ if, whenever (an appropriate) L-structure satisfies $\mathcal{P}^*$ in V_0, its relativization (to V_1 for $\mathcal{P}$) also satisfies $\mathcal{P}^*$ in V_1. Similarly $\mathcal{P}^*$ is downwards absolute between V_0 and V_1 for $\mathcal{P}$ if whenever the relativization (to V_1 for $\mathcal{P}$) of an (appropriate) L-structure satisfies $\mathcal{P}^*$ in V_1 the original must also have satisfied $\mathcal{P}^*$ in V_0.

We now give an a collection of sentences which are always upwards absolute.

Definition 2.6. *Let $\mathcal{CD}_{\alpha,\beta}(\mathrm{L})$ be the smallest collection of quantifier free formulas of $\mathcal{L}_{\infty,\infty}(\mathrm{L})$ such that*

- *$\mathcal{CD}_{\alpha,\beta}(\mathrm{L})$ contains all atomic and negation of atomic formulas in* L.
- *$\mathcal{CD}_{\alpha,\beta}(\mathrm{L})$ is closed under conjunctions of size $< \alpha$.*
- *$\mathcal{CD}_{\alpha,\beta}(\mathrm{L})$ is closed under disjunctions of size $< \beta$.*

We use ∞ in place of α or β if we allow arbitrary set sized conjunctions or disjunctions respectively. We say a formula $\varphi \in \mathcal{CD}_{\alpha,\beta}(\mathrm{L})$ is **positive** *if it doesn't contain the negation of any atomic formula as a subformula.*

Notice that every quantifier free formulas of $\mathcal{L}_{\infty,\infty}(\mathrm{L})$ is equivalent to a formula in $\mathcal{CD}_{\infty,\infty}(\mathrm{L})$.

Definition 2.7. *We say a sentence is (positive) $\Sigma_1^{\alpha,\beta}(\mathrm{L})$ or $\Pi_1^{\alpha,\beta}(\mathrm{L})$ if it is of the form $(\exists X)\varphi$ or $(\forall X)\varphi$ respectively where φ is (positive and) in $\mathcal{CD}_{\alpha,\beta}(\mathrm{L})$, and X contains all variables which are free in φ. We define $\Sigma_2^{\alpha,\beta}(\mathrm{L})$ similarly.*

We will omit mention of the language L in $\mathcal{CD}_{\alpha,\beta}(\mathrm{L})$, $\Sigma_1^{\alpha,\beta}(\mathrm{L})$, $\Pi_1^{\alpha,\beta}(\mathrm{L})$, and $\Sigma_2^{\alpha,\beta}(\mathrm{L})$ when it is clear from context.

Note that we don't give any bound on the number of free variables in formulas in $\mathcal{CD}_{\alpha,\beta}$ and hence we do not have any bound on the size of the quantifiers needed in $\Sigma_1^{\alpha,\beta}$, $\Pi_1^{\alpha,\beta}$ or $\Sigma_2^{\alpha,\beta}$ sentences.

Lemma 2.8. *Suppose $(\exists X)\varphi$ is any positive $\Sigma_1^{\infty,\infty}$ sentence in V_0. Then φ is upwards absolute between V_0 and V_1 for $\mathcal{P}$ (for any property $\mathcal{P}$).*

Proof. Let $\mathcal{P}$ be any property. Suppose $\mathcal{M}_0 \in V_0$ is any L-structure satisfying $\mathcal{P}$ in V_0 with $\mathcal{M}_1$ its relativization to V_1 for $\mathcal{P}$. If $(\mathcal{M}_0 \models (\exists X)\varphi)^{V_0}$ then there is an assignment $\overline{a} \in V_0$ of the variables of X to elements of $\mathcal{M}_0$ such that $(\mathcal{M}_0 \models \varphi[\overline{a}])^{V_0}$. But then as $\mathcal{M}_0 \subseteq \mathcal{M}_1$ with the inclusion a homomorphism, and as $\varphi(X)$ is positive, we must also have $(\mathcal{M}_1 \models \varphi[\overline{a}])^{V_1}$. But this implies $(\mathcal{M}_1 \models (\exists X)\varphi(X))^{V_1}$ and hence $(\exists X)\varphi$ is upwards absolute between V_0 and V_1 for $\mathcal{P}$. $\square$

2.2.1. *General Negative Results.* We now give a few general situations where we can easily show that absoluteness doesn't hold. In particular it is not the case that all $\Sigma_1^{\infty,\infty}$-sentences are downward absolute.

Example 2.9. *Suppose $\mathcal{M}_0 \in V_0$ is any infinite* L*-structure satisfying $\mathcal{P}$ in V_0 with $\mathcal{M}_1$ its relativization to V_1 for $\mathcal{P}$. Let $\varphi(X) = \bigwedge_{x \neq y \in X} x \neq y$ where $V_0 \models |X| => |\mathcal{M}_0|$. Because $V_0 \models |X| > |\mathcal{M}_0|$ we must have $(\mathcal{M}_0 \not\models (\exists X)\varphi(X))^{V_0}$.*

Now assume V_1 is such that $V_1 \models |X| = |\mathcal{M}_0|$. Then, because $V_1 \models |X| = |\mathcal{M}_0| \leq |\mathcal{M}_1|$, we have $(\mathcal{M}_1 \models (\exists X)\varphi(X))^{V_1}$.

In the example above our $\Sigma_1^{\infty,\infty}$-sentence was not downwards absolute because it allowed us to say that the size of $\mathcal{M}_0$ was at least the size of X, a fact which is not downward absolute. In order to say this, our formula needed a conjunction of size $|X|$. We will see in Theorem 4.1 that in the case of complete metric structures (with closed relations) this large conjunction is the main type obstacle to overcome, i.e. if we restrict ourselves to countable conjunctions then in fact $\Sigma_1^{\omega_1,\infty}$ sentences are absolute.

Building on Theorem 4.1 we will see, in Lemma 4.4, that in the case of complete metric structures $\Sigma_2^{\infty,\omega_1}$ sentences are upward absolute. In general though we can not hope even for all Σ_2^{0,ω_1} sentences to be downward absolute.

Example 2.10. *Suppose $\mathcal{M}$ is an* L*-structure satisfying $\mathcal{P}$ in V_0 such that $\mathcal{M}$ is its own relativization to V_1 for $\mathcal{P}$. Further suppose there is a set of variables X with $(|X| < |\mathcal{M}|)^{V_0}$ but $(|X| = |\mathcal{M}|)^{V_1}$. Let $\varphi(X, y) = \bigvee_{x \in X} x = y$. It is then clear that $(\mathcal{M} \models (\exists X)(\forall y)\varphi(X, y))^{V_1}$ but $(\mathcal{M} \not\models (\exists X)(\forall y)\varphi(X, y))^{V_0}$. Hence $(\exists X)(\forall y)\varphi(X, y)$ is not downwards absolute for $\mathcal{P}$.*

2.3. **Definable Expansions.** In what follows we will want to say when the definition of a relation or function is absolute. What this means, intuitively, is that if we start with the definition of a relation/function and then relativize the whole structure, the resulting relativized relation/function satisfies the same definition (in the larger model of set theory). We now make this precise.

Definition 2.11. *Suppose $\mathrm{L}_0 \subseteq \mathrm{L}_1$ are languages. We say a property $\mathbb{D}$* **defines** *$\mathrm{L}_1 - \mathrm{L}_0$ with respect to $\mathcal{P}$ if every L_0-structure satisfying $\mathcal{P}$ has a unique expansion to an L_1 structure satisfying $\mathcal{P}$ and $\mathbb{D}$. We call that unique expansion the* **expansion by** *$\mathbb{D}$.*

Definition 2.12. *Suppose $\mathrm{L}_0 \subseteq \mathrm{L}_1$ are languages and $\mathbb{D}$ is a property which defines $\mathrm{L}_1 - \mathrm{L}_0$ with respect to $\mathcal{P}$ in both V_0 and V_1. Further suppose for all $\mathcal{M}_0, \mathcal{M}_0^{\mathbb{D}} \in V_0$ and $\mathcal{M}_1, \mathcal{M}_1^{\mathbb{D}}, \mathcal{M}_1^* \in V_1$ such that*

- *$\mathcal{M}_1$ is the relativization of $\mathcal{M}_0$ to V_1 for $\mathcal{P}$,*
- *$\mathcal{M}_0^{\mathbb{D}}$, $\mathcal{M}_1^{\mathbb{D}}$ are the expansions by $\mathbb{D}$ of $\mathcal{M}_0$ (in V_0) and $\mathcal{M}_1$ (in V_1) respectively, and*
- *$\mathcal{M}_1^*$ is the relativization of $\mathcal{M}_0^{\mathbb{D}}$ to V_1 for $\mathcal{P}$*

we have $\mathcal{M}_1^* = \mathcal{M}_1^{\mathbb{D}}$. Then we say that $\mathbb{D}$ is **absolute** *between V_0 and V_1 for $\mathcal{P}$.*

At this point it may be helpful to give an example of an absolute definition and a non-absolute definition.

Example 2.13. *Let $\mathcal{P}$ be the property that says our structure $\mathcal{M}$ consists of a complete metric space (M, d_M) along with a continuous function $f : M \times M \to \mathbb{R}$. Now consider the definition $\mathbb{D}$ of $g : M \to \mathbb{R}$ which says $g(x) = inf_y\{f(x, y) : x \in M\}$.*

We will see in Proposition 3.21 that any such structure $\mathcal{M}_0$ in V_0 satisfying $\mathcal{P}$ has a relativization $\mathcal{M}_1$ to V_1 for $\mathcal{P}$ were $\mathcal{M}_1 = \mathbf{cc}(\mathcal{M}_0)^{V_1}$ and $f^{\mathcal{M}_1} = \mathbf{cc}(f^{\mathcal{M}_0})^{V_1}$ (recall **cc** *is the Cauchy completion functor). The statement that $\mathbb{D}$ is absolute between V_0 and V_1 for $\mathcal{P}$ amounts to saying that in V_1 we have $\mathbf{cc}(g^{\mathcal{M}_0}) = inf_y\{f^{\mathcal{M}_1}(x, y) : x \in M\}$, i.e. to determine $inf_y\{f(x, y) : x \in m\}$ (with $x \in V_0$) it doesn't matter if we do the calculation in V_0 or in V_1. We will see in Lemma 5.2 that in fact this is the case.*

Now for an example of a non-absolute definition.

Example 2.14. *Let $\mathcal{P}$ be the property which says our structure is a subset $E \subseteq \mathbb{R}$. Now consider the definition $\mathbb{D}$ which says "E is a closed subset containing all the rationals".*

Note that if $(E \subseteq \mathbb{R})^{V_0}$ then $(E \subseteq \mathbb{R})^{V_1}$ as $\mathbb{R}^{V_0} \subseteq \mathbb{R}^{V_1}$. Hence any structure satisfying $\mathcal{P}$ in V_0 satisfies $\mathcal{P}$ in V_1 and so is the relativization of itself to V_1 for $\mathcal{P}$.

Now if we start in V_0 with a closed set E containing all the rationals then $(E = \mathbb{R})^{V_0}$. In this case the relativization of E to V_1 for $\mathcal{P}$ is the set $\mathbb{R}^{V_0}$. However if $\mathbb{R}^{V_0} \neq \mathbb{R}^{V_1}$, then E does not satisfy $\mathbb{D}$ (i.e. is not closed) in V_1. Hence our definition is not absolute between V_0 and V_1 for $\mathcal{P}$ (if $\mathbb{R}^{V_0} \neq \mathbb{R}^{V_1}$).

3. Languages and Models

In this section we give our representation of complete metric structures.

3.1. Basic Definitions.

We begin by defining a theory whose models are meant to represent the non-negative reals.

Definition 3.1. *Let $\mathrm{L}_{\mathbb{R}}$ be the language where:*

- *The sorts are finite products of a single sort R.*
- *There is a (unique binary) relation symbol $\leq$* [1] *: $R \times R$.*
- *The functions of $\mathrm{L}_{\mathbb{R}}$ are $\{d_{R^n} : n \in \omega\}$ where $d_{R^n} : R^n \times R^n \to R$.*
- *The constants are $\{\widehat{q} : q \in \mathbb{Q}^{\geq 0}\}$ all of which are of type R.*

Let $\mathrm{Th}_{\mathbb{R}} \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L}_{\mathbb{R}})$ be the conjunction of:

- *$\leq$ is a linear order.*
- *$(\forall x : R)\ x \geq \widehat{0}$ and $(\forall x : R) \bigvee_{q \in \mathbb{Q}^{\geq 0}} x \leq \widehat{q}$.*
- *$(\forall x) \left[\bigwedge_{q \in \mathbb{Q}^{>0}} x \leq \widehat{q}\right] \to x = \widehat{0}$.*
- *$\bigwedge\{\widehat{p} \leq \widehat{q} : p \leq q, p, q \in \mathbb{Q}^{\geq 0}\}$.*
- *$\bigwedge\{d_R(\widehat{p}, \widehat{q}) = \widehat{|p - q|}, p, q \in \mathbb{Q}^{\geq 0}\}$.*
- *$(\forall r_0, r_1, r_2, r_3 : R)\ r_0 \leq r_1 \leq r_2 \leq r_3 \to d_R(r_0, r_3) \geq d_R(r_1, r_2)$.*
- *$(\forall x, y : R)\ [d_R(x, y) = \widehat{0}] \leftrightarrow x = y$.*
- *$(\forall x, y : R)\ d_R(x, y) = d_R(y, x)$.*
- *$(\forall x, y, z : R) \bigwedge_{p,q \in \mathbb{Q}^{\geq 0}} [d_R(x, y) \leq \widehat{p} \wedge d_R(y, z) \leq \widehat{q}] \to d_R(x, z) \leq \widehat{p + q}$.*
- *$(\forall \langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_n\rangle : R^n)\ d_{R^n}(\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_n\rangle) = \max\{d_R(x_i, y_i) : i \leq n\}$*

It is easy to see that for any model $\mathcal{M} \models \mathrm{Th}_{\mathbb{R}}$, the map $i^{\mathcal{M}} : \{\widehat{q}^{\mathcal{M}} : q \in \mathbb{Q}^{\geq 0}\} \to \mathbb{R}^{\geq 0}$ with $i^{\mathcal{M}}(\widehat{q}^{\mathcal{M}}) = q$ for all $q \in \mathbb{Q}^{\geq 0}$ has a unique order preserving extension to a map $i^{\mathcal{M}}_{\mathbb{R}} : R^{\mathcal{M}} \to \mathbb{R}^{\geq 0}$. Further, if $\mathbb{Q}^{\geq 0} \subseteq X \subseteq \mathbb{R}^{\geq 0}$ with X closed under subtraction, then there is a unique model $\mathcal{M}_X$ of $\mathrm{Th}_{\mathbb{R}}$ with $R^{\mathcal{M}_X} = X$, $\leq^{\mathbb{R}} = \leq^{\mathcal{M}_X}$ and $(\forall x, y \in X)\mathcal{M}_X \models d_R^{\mathcal{M}_X}(x, y) = |x - y|$. Hence, up to isomorphism, models of $\mathrm{Th}_{\mathbb{R}}$ are subsets of the non-negative reals which contain all non-negative rationals and are closed under subtraction. In particular this implies that if a model of $\mathrm{Th}_{\mathbb{R}}$ is complete (as a metric space or as a linear order) then that model must be isomorphic to $(\mathbb{R}^{\geq 0}, \mathbb{Q}^{\geq 0}, \leq)$. We now give our definition of a metric space.

Definition 3.2. *Let $\mathrm{L}_{\mathrm{Met}}(S)$ be the language such that:*

- *$\mathrm{L}_{\mathbb{R}} \subseteq \mathrm{L}_{\mathrm{Met}}(S)$.*
- *The sorts of $\mathrm{L}_{\mathrm{Met}}(S)$ are finite sequences of $\{R, S\}$.*
- *The functions of $\mathrm{L}_{\mathrm{Met}}(S)$ are $\{d_{S^*} : S^*$ a sort$\}$ where $d_{S^*} : S^* \times S^* \to R$.*

Let $\mathrm{Th}_{\mathrm{Met}}(S)$ be the conjunction of:

- *$(\forall x, y : S)\ d_S(x, y) = \widehat{0} \leftrightarrow x = y$.*
- *$(\forall x, y : S)\ d_S(x, y) = d_S(y, x)$.*

[1] We will abuse notation and use $a \leq b$ for $\leq (a, b)$.

- $(\forall x, y, z : S) \bigwedge_{p,q \in \mathbb{Q}^{\geq 0}} d_S(x, y) \leq \widehat{p} \wedge d_S(y, z) \leq \widehat{q} \to d_S(x, z) \leq \widehat{p + q}$.
- *If* $S^* = (S_1, \ldots, S_n)$ *then*

$$(\forall \langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle : S^*)\ d_{S^*}(\langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle) = \max\{d_{S_i}(x_i, y_i) : i \leq n\}.$$

If $\mathcal{M}$ is a model of $\mathrm{Th}_{\mathrm{Met}}(S)$ then $(S^{\mathcal{M}}, i_{\mathbb{R}}^{\mathcal{M}} \circ d_S^{\mathcal{M}})$ is a metric space. In this way any model $\mathcal{M} \models \mathrm{Th}_{\mathrm{Met}}(S)$ *represents* a metric space.

It is worth mentioning that while every model of $\mathrm{Th}_{\mathrm{Met}}(S)$ represents a metric spaces this representation need not be unique, i.e. there may be two models $\mathcal{M}, \mathcal{M}' \models \mathrm{Th}_{\mathrm{Met}}(S)$ such that $(S^{\mathcal{M}}, i_{\mathbb{R}}^{\mathcal{M}} \circ d_S^{\mathcal{M}}) = (S^{\mathcal{M}'}, i_{\mathbb{R}}^{\mathcal{M}'} \circ d_S^{\mathcal{M}'})$ but $\mathcal{M} \not\cong \mathcal{M}'$. The reason is that the sort R, which is intended to represent the reals, need not contain a representative for each real number. Hence it is possible to have two models of $\mathrm{Th}_{\mathrm{Met}}(S)$ which represent the same metric spaces, but where the models each have elements of R which represent reals not in the other model (just so long as those reals never occur as distances in the metric spaces $\mathcal{M}$ and $\mathcal{M}'$ represent). Despite this we will still refer to a model of $\mathrm{Th}_{\mathrm{Met}}(S)$ as a **metric space** when no confusion can arise. We also will write $(S^{\mathcal{M}}, d_S^{\mathcal{M}})$ fro $(S^M, i_{\mathbb{R}}^{\mathcal{M}} \circ d_S^{\mathcal{M}})$.

The following lemma then follows immediately from the fact that homomorphisms preserve relations but not the negations of relations.

Lemma 3.3. *If* $\mathcal{N}_0, \mathcal{N}_1 \models \mathrm{Th}_{\mathrm{Met}}(S)$ *and* $\alpha : \mathcal{N}_0 \to \mathcal{N}_1$ *is a homomorphism then* $\alpha_S : (S, i_{\mathbb{R}}^{\mathcal{N}_0} \circ d_S^{\mathcal{N}_0}) \to (S, i_{\mathbb{R}}^{\mathcal{N}_1} \circ d_S^{\mathcal{N}_1})$ *is a Lipschitz function with Lipschitz constant 1 (where* α_S *is the component of the homomorphism corresponding to the sort* S*).*

The following is easily checked from Definition 3.1, Definition 3.2 and the fact $\mathrm{Th}_{\mathbb{R}}$ implies $\mathrm{Th}_{\mathrm{Met}}(R)$.

Lemma 3.4. *For any sort* $S^* \in \mathrm{L}_{\mathrm{Met}}(S)$, $\mathrm{Th}_{\mathrm{Met}}(S)$ *implies* $\mathrm{Th}_{\mathrm{Met}}(S^*)$.

In particular, as $\mathrm{Th}_{\mathbb{R}}$ is equivalent to $\mathrm{Th}_{\mathrm{Met}}(R)$, $\mathrm{Th}_{\mathbb{R}}$ implies that the underlying sort R can be made into a metric space.

Definition 3.5. *If* $\mathcal{M} \models \mathrm{Th}_{\mathrm{Met}}(S)$ *we define a* **Cauchy Sequence** *in* $S^{\mathcal{M}}$ *to be a function* $cs : \mathbb{N} \to S^{\mathcal{M}}$ *such that:*

- $(\forall m, n \in \mathbb{N})\ \mathcal{M} \models d_S(cs(m), cs(n)) \leq 2^{\widehat{-\min\{m,n\}}}$.

We say a Cauchy sequence cs **converges** *to a point* x *if*

$$(\forall n \in \mathbb{N}) \mathcal{M} \models d_S(cs(n), x) \leq \widehat{2^{-n}}.$$

We write $\mathcal{M} \models CS_S(cs)$ *if* cs *is a Cauchy sequence in* $S^{\mathcal{M}}$ *and* $\mathcal{M} \models CS_S(cs, x)$ *if* cs *is a Cauchy sequence in* $S^{\mathcal{M}}$ *which converges to* x*.*

We now can define the second order theory of Cauchy complete metric spaces.

Definition 3.6. *Let* $\mathrm{Th}_{\mathrm{CMet}}(S)$ *be the second order* $\mathrm{L}_{\mathrm{Met}}(S)$*-sentence which says:*

- $\mathrm{Th}_{\mathrm{Met}}(S)$.
- $(\forall cs : \mathbb{N} \to R) CS_R(cs) \to (\exists x \in S) CS_R(cs, x)$.
- $(\forall cs : \mathbb{N} \to S) CS_S(cs) \to (\exists x \in S) CS_S(cs, x)$.

We call a model of $\mathrm{Th}_{\mathrm{CMet}}(S)$ *a* **(Cauchy) complete metric space***.*

Note the following is easily checked.

Lemma 3.7. *If* S^* *is any sort in* $\mathrm{L}_{\mathrm{Met}}(S)$ *then* $\mathrm{Th}_{\mathrm{CMet}}(S)$ *implies* $\mathrm{Th}_{\mathrm{CMet}}(S^*)$.

The difficulty of multiple models of $\mathrm{Th}_{\mathrm{Met}}(S)$ representing the same metric space ceases to be an issue if the sort R is a complete metric space (and hence isomorphic to $\mathbb{R}$). Hence two models of $\mathrm{Th}_{\mathrm{CMet}}(S)$ represent the same complete metric space if and only if they are actually isomorphic.

Now that we have formal definitions of metric spaces and complete metric spaces we give a formal definition of a continuous map between metric spaces.

Definition 3.8. *Suppose* L *is a language with* $\mathrm{L}_{\mathrm{Met}}(S) \subseteq \mathrm{L}$ *for all sorts in* S *in* L*. If* t *is a* L*-term then let* $\mathrm{Th}_{\mathrm{Fun}}(t)$ *be the conjunction of:*

- $\mathrm{Th}_{\mathrm{Met}}(U)$ *for any sort* U *in* L.
- $(\forall x : S) \ \bigwedge_{e \in \mathbb{Q}^{>0}} \bigvee_{d \in \mathbb{Q}^{>0}} (\forall y : S) d_S(x, y) \leq \widehat{d} \to d_T(t(x), t(y)) \leq \widehat{e}$.

In particular the following two lemmas are immediate from the definition.

Lemma 3.9. *For any* L*-structure* $\mathcal{M}$ *we have* t *is a ted from* S *to* T *and* $\mathcal{M} \models \mathrm{Th}_{\mathrm{Fun}}(t)$ *if and only if* $t^{\mathcal{M}} : (S^{\mathcal{M}}, i_{\mathbb{R}}^{\mathcal{M}} \circ d_S^{\mathcal{M}}) \to (T^{\mathcal{M}}, i_{\mathbb{R}}^{\mathcal{M}} \circ d_T^{\mathcal{M}})$ *is a continuous function between metric spaces.*

Lemma 3.10. $\mathrm{Th}_{\mathrm{Met}}(S)$ *implies* $\mathrm{Th}_{\mathrm{Fun}}(d_S)$.

While most of the time we will only require our maps to be continuous there are certain results, like Proposition 4.5, where it will be important that our maps are uniformly continuous and that we have a handle on the modulus of convergence.

Definition 3.11. *Suppose* L *is a language with* $\mathrm{L}_{\mathrm{Met}}(S) \subseteq \mathrm{L}$ *for all sorts* S *in* L*. If* t *and* $\mathfrak{m}_t$ *are* L*-terms with* $\mathfrak{m}_t : R \to R$ *then let* $\mathrm{Th}_{\mathrm{UCon}}(t, \mathfrak{m}_t)$ *be conjunction of:*

- $\mathrm{Th}_{\mathrm{Fun}}(t)$ *and* $\mathrm{Th}_{\mathrm{Fun}}(\mathfrak{m}_f)$.
- $(\forall x, y : R)\ x \leq y \to \mathfrak{m}_t(x) \leq \mathfrak{m}_t(y)$.
- $\mathfrak{m}_t(\widehat{0}) = \widehat{0}$.
- $(\forall x, y : S) d_T(t(x), t(y)) \leq \mathfrak{m}_f(d_S(x, y))$.

Notice that $\mathcal{M} \models \mathrm{Th}_{\mathrm{UCon}}(t, \mathfrak{m}_t)$ if and only if $t^{\mathcal{M}} : (S, i_{\mathbb{R}}^{\mathcal{M}} \circ d_S^{\mathcal{M}}) \to (T, i_{\mathbb{R}}^{\mathcal{M}} \circ d_T^{\mathcal{M}})$ is uniformly continuous, $i_{\mathbb{R}}^{\mathcal{M}} \circ \mathfrak{m}_t^{\mathcal{M}} \circ (i_{\mathbb{R}}^{\mathcal{M}})^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous, and $i_{\mathbb{R}}^{\mathcal{M}} \circ \mathfrak{m}_t^{\mathcal{M}} \circ (i_{\mathbb{R}}^{\mathcal{M}})^{-1}$ is a modulus of continuity for $i_{\mathbb{R}}^{\mathcal{M}} \circ t^{\mathcal{M}}$.

Lemma 3.12. *For each* L*-term* t *and* L*-structure* $\mathcal{M}$ *such that* $\mathcal{M} \models \mathrm{Th}_{\mathrm{Fun}}(t)$, $t^{\mathcal{M}} : S^{\mathcal{M}} \to T^{\mathcal{M}}$ *is uniformly continuous if and only if there is an expansion of* $\mathcal{M}$ *to a structure* $\mathcal{M}'$ *where* $\mathcal{M}' \models \mathrm{Th}_{\mathrm{UCon}}(t, \mathfrak{m}_t)$ *for some term* $\mathfrak{m}_t$.

Proof. This follows immediately from the fact that every uniformly continuous map has a continuous modulus of convergence. □

If $f : S \to T$ is a function symbol, it will also be useful to define $\mathrm{L}_{\mathrm{Fun}}(f)$ to be the smallest language containing $\mathrm{L}_{\mathrm{Met}}(S) \cup \mathrm{L}_{\mathrm{Met}}(T)$ along with a function symbol $f : S \to T$. Similarly if $\mathfrak{m}_f : R \to R$ is a function symbol it will be useful to define $\mathrm{L}_{\mathrm{UCon}}(f, \mathfrak{m}_f) := \mathrm{L}_{\mathrm{Fun}}(f) \cup \{\mathfrak{m}_f\}$.

3.2. **Metric Structures.** We now have all of the components we need to define a metric structure.

Definition 3.13. *We say* L_{MS} *is a* **metric language** *if*

- *For all sorts* S *in* L_{MS}, $\mathrm{L}_{\mathrm{Met}}(S) \subseteq \mathrm{L}_{\mathrm{MS}}$.

The **theory of metric structures** *for* L_{MS}, *denoted* $\mathrm{Th}_{\mathrm{MS}}(\mathrm{L}_{\mathrm{MS}})$ *is the conjunction of:*

(1) $\mathrm{Th}_{\mathrm{Met}}(S)$, *for each sort* S *in* L_{MS}.
(2) $\mathrm{Th}_{\mathrm{Fun}}(f)$, *for any function* $f : S \to T$ *in* L_{MS}.

We call a model of $\mathrm{Th}_{\mathrm{MS}}$ *a* **metric** L_{MS}**-structure**.

Condition (1) guarantees that each sort in L_{MS} is a metric space while (2) guarantees that that each function f is continuous. Note that, at this point, we have not placed any restrictions on the relations in the language. In what follows L_{MS} and its variants will always be metric languages.

The following definition will be important later.

Definition 3.14. *Suppose* L_{MS} *is a metric language and* $\mathcal{M} \models \mathrm{Th}_{\mathrm{MS}}(\mathrm{L}_{\mathrm{MS}})$. *We say* $\langle D_{S,\mathcal{M}} : S$ *a sort in* $\mathrm{L}_{\mathrm{MS}}\rangle$ *is a* **dense subset** *of* $\mathcal{M}$ *if* $D_{S,\mathcal{M}} \subseteq S^{\mathcal{M}}$ *for each sort and* $D_{S,\mathcal{M}}$ *is a dense subset of* $(S^{\mathcal{M}}, d_S^{\mathcal{M}})$.

We now have the following easy lemma.

Lemma 3.15. *If $\mathcal{M}$ is a metric L_{MS}-structure in V_0 then it is its own relativization to V_1 as a metric L_{MS}-structure.*

Proof. This follows from the fact that being a metric structure is describable by a sentence of $\mathcal{L}_{\infty,\omega}(\mathrm{L}_{\mathrm{MS}})$ and Lemma 2.4. □

In particular, questions about the absoluteness of properties for metric structures reduce to questions about the corresponding first order structures. While metric structures relativize in a straightforward way, complete metric structures relativize in a (slightly) more complicated manner.

Definition 3.16. *Suppose L_{MS} is a metric language and $\mathcal{M}$ is a L_{MS}-structure. We say $\mathcal{M}$ is a* **(Cauchy) complete metric L_{MS}-structure** *if $\mathcal{M} \models \mathrm{Th}_{\mathrm{CMS}}(\mathrm{L}_{\mathrm{MS}})$ where $\mathrm{Th}_{\mathrm{CMS}}(\mathrm{L}_{\mathrm{MS}})$ is the conjunction of*

- $\mathrm{Th}_{\mathrm{MS}}(\mathrm{L}_{\mathrm{MS}})$.
- $\mathrm{Th}_{\mathrm{CMet}}(S)$*, for each sort S in L_{MS}.*

We will show that complete metric structures relativize and in particular the relativization of a complete metric structure $\mathcal{M}$ in V_0 to V_1 is the structure obtained by applying the Cauchy completion functor $\mathbf{cc}(\cdot)$ to each sort and each function.

Definition 3.17. *We say a map $f : S_0 \to S_1$ of metric spaces is* **Cauchy continuous** *if it is continuous and whenever $\langle x_i \rangle_{i\in\mathbb{N}}$ is a Cauchy sequence in S_0 then $\langle f(x_i)\rangle_{i\in\mathbb{N}}$ is a Cauchy sequence in S_1.*

In particular if a map $f : S_0 \to S_1$ of metric spaces is Cauchy continuous then there is a unique map $f' : \mathbf{cc}(S_0) \to \mathbf{cc}(S_1)$ which agrees with f on S_0.

Lemma 3.18. *Suppose $f : S_0 \to S_1$ is a function symbol. The property "f is Cauchy-continuous" is absolute between V_0 and V_1 for $\mathrm{Th}_{\mathrm{Fun}}(f)$.*

Proof. Let $\mathcal{M} \in V_0$ with $\mathcal{M} \models \mathrm{Th}_{\mathrm{Fun}}(f)$. So in particular $f^{\mathcal{M}}$ is continuous. We will construct a tree $(T, \leq_T)$, contained in V_0, such that $f^{\mathcal{M}}$ is not Cauchy-continuous (in V_0 or V_1) if and only if $(T, \leq_T)$ is ill-founded.

Let T be the set consisting of those pairs $\langle q, \langle x_1, \dots, x_{2n+2}\rangle\rangle$ where:

- $q \in \mathbb{Q}^{>0}$.
- $x_1, \dots, x_{2n+2} \in S_0^{\mathcal{M}}$ and for all $i, j \leq 2n+2$, $d^{\mathcal{M}}_{S_0}(x_i, x_j) \leq 2^{-2\cdot\min\{i,j\}}$.
- $d^{\mathcal{M}}_{S_1}(f^{\mathcal{M}}(x_{2n+1}), f^{\mathcal{M}}(x_{2n+2})) \geq \widehat{q}$.

We then let $\langle q, \langle x_1, \dots, x_{2n}\rangle\rangle \leq_T \langle q', \langle x'_1, \dots, x'_{2n'}\rangle\rangle$ if and only if $q = q'$, $n \geq n'$ and $x_i = x'_i$ for all $1 \leq i \leq 2n'$. It is immediate that T is a tree and $(T, \leq_T)^{V_0} = (T, \leq_T)^{V_1}$.

Claim 3.19. *$f^{\mathcal{M}}$ is not Cauchy continuous if and only if $(T, \leq_T)$ is ill-founded.*

Proof. Notice $f^{\mathcal{M}}$ is not Cauchy-continuous if and only if there is a Cauchy sequence $\langle x_i : i \in \mathbb{N}\rangle \subseteq S_0^{\mathcal{M}}$ and $q \in \mathbb{Q}^{>0}$ such that for each such n there are $n_0, n_1 \geq n$ with $d_{S_1}^{\mathcal{M}}(f^{\mathcal{M}}(x_{n_0}), f^{\mathcal{M}}(x_{n_1})) \geq \widehat{q}$.

Now if $f^{\mathcal{M}}$ is not Cauchy continuous then from the sequence described in the previous paragraph we can easily construct an infinite branch through $(T, \leq_T)$.

In the other direction, if there is an infinite branch through $(T, \leq_T)$ we can construct a pair $\langle q, \langle x_i : i \in \mathbb{N}\rangle\rangle$ where $\langle q, \langle x_i : i \leq 2n\rangle$ are the elements of the infinite branch. But then $\langle x_i : i \in \mathbb{N}\rangle$ is a Cauchy sequence and for all n, $d_{S_1}^{\mathcal{M}}(f^{\mathcal{M}}(x_{2n+1}), f^{\mathcal{M}}(x_{2n+2})) \geq \widehat{q}$. Hence $f^{\mathcal{M}}$ is not Cauchy-continuous. □

Finally, as $(T, \leq_T)^{V_0} = (T, \leq_T)^{V_1}$, as the relativization of a $\mathcal{M}$ to V_1 for $\mathrm{Th}_{\mathsf{Fun}}(f)$ is itself by Lemma 3.15, and by Claim 3.19 we have that $f^{\mathcal{M}}$ being Cauchy-continuous is absolute for $\mathrm{Th}_{\mathsf{Fun}}(f)$. □

As a consequence we have the following immediate but important corollary.

Corollary 3.20. *Suppose S_0, S_1 are complete metric spaces in V_0 and suppose $f : S_0 \to S_1$ is a continuous function (in V_0). Then, in V_1, there is a unique continuous function $\mathbf{cc}(f) : \mathbf{cc}(S_0) \to \mathbf{cc}(S_1)$ which agrees with with f on S_0.*

Proof. Note f is Cauchy-continuous in V_0 and hence is also Cauchy-continuous in V_1. Hence there is a unique continuous map $\mathbf{cc}(f)$: $\mathbf{cc}(S_0) \to \mathbf{cc}(S_1)$ which agrees with f on S_0. □

We can now show that Cauchy complete metric structures relativize.

Proposition 3.21. *If $\mathcal{M}_0$ is a Cauchy complete metric structure in V_0 and $\mathcal{M}_1$ is the metric structure in V_1 where:*

- *For each sort S, $(S^{\mathcal{M}_1}, d_S^{\mathcal{M}_1}) = \mathbf{cc}(S^{\mathcal{M}_0}, d_S^{\mathcal{M}_0})$.*
- *For every function $f^{\mathcal{M}_0} : S_0^{\mathcal{M}_0} \to S_0^{\mathcal{M}_0}$, $f^{\mathcal{M}_1}$ is the map $\mathbf{cc}(f^{\mathcal{M}_0})$.*
- *If E is a relation of sort S then $E^{\mathcal{M}_1} = E^{\mathcal{M}_0}$.*

then $\mathcal{M}_1$ is the relativization of $\mathcal{M}_0$ to V_1 for being a complete metric structure.

Proof. First note $\mathcal{M}_1$ is well defined follows from Corollary 3.20. That $\mathcal{M}_1$ is a compete metric structure follows immediately from the definition. To see that $\mathcal{M}_1$ is the relativization of $\mathcal{M}_0$ first note that $\mathcal{M}_0 \subseteq \mathcal{M}_1$ and the inclusion map is a homomorphism. Further, because for any relation E of sort S and $s \in S^{\mathcal{M}_0}$, $\mathcal{M}_0 \models E(s)$ if and only if $\mathcal{M}_1 \models E(s)$ and because $\mathbf{cc}(\cdot)$ is the Cauchy completion functor, if $\mathcal{N} \in V_1$ is a Cauchy compete metric structure with $(\mathcal{M}_0 \subseteq \mathcal{N})^{V_1}$ and the inclusion map a homomorphism, then the inclusion map must also be uniquely extendable in V_1 to a map $\mathcal{M}_1 \subseteq \mathcal{N}$. □

There are two points regarding Proposition 3.21 worth mentioning. First, the fact that taking the Cauchy completion of $\mathcal{M}_0$ in V_1 is well-defined made fundamental use of the fact that $\mathcal{M}_0$ was complete in V_0. It is not the case that the Cauchy completion of an arbitrary metric structure is well-defined. The reason for this is because being a metric structure only requires the functions to be continuous and not Cauchy-continuous and hence there are metric structures with functions that don't have a unique extension to the Cauchy completions of their domain and codomain.

Second, it is worth stressing that the relativization of a complete metric structure as a complete metric structure may be different than its relativization as a metric structure. The quintessential example of this is the reals $\mathbb{R}$. In this case the relativization of $\mathbb{R}^{V_0}$ to V_1 with respect to the theory of metric spaces is just the metric space $\mathbb{R}^{V_0}$. However the relativization with respect to the theory of complete metric spaces is $\mathbb{R}^{V_1}$. Hence, if $\mathbb{R}^{V_0} \neq \mathbb{R}^{V_1}$ then these relativization are not the same.

We will end this section with the observation that being uniformly continuous is absolute for being a complete metric structure.

Corollary 3.22. $\mathrm{Th}_{\mathrm{UCon}}(f, \mathfrak{m}_f)$ *is absolute for* $\mathrm{Th}_{\mathrm{CMS}}(\mathrm{L}_{\mathrm{UCon}}(f, \mathfrak{m}_f))$.

Proof. This follows from the fact that if $f : (M, d_M) \to (N, d_N)$ is any uniformly continuous map between metric spaces such that

- $X = im(d_M) \cup im(d_N)$, and
- $\mathfrak{m}_f : X \to X$ is a modulus of convergence for f

then $\mathbf{cc}(\mathfrak{m}_f)$ is a modulus of convergence for $\mathbf{cc}(f)$. □

3.3. **Relativizations of Relations.** Now that we have shown complete metric structures relativize, we can ask which properties are absolute for complete metric structures. Unfortunately though, we quickly see that even first order Π_1-sentences need not be absolute.

Lemma 3.23. *Suppose* L_{MS} *is a metric language containing sort* S *and relation* E *of type* S. *Then the sentence* $(\forall x : S)E(x)$ *is not (necessarily) upwards absolute for the theory of complete metric structures.*

Proof. Suppose $\mathcal{M}$ is a complete metric structure in V_0 and $(\mathcal{M} \models (\forall x : S)E(x))^{V_0}$. Further suppose $(S^{\mathcal{M}_0} \subsetneq \mathbf{cc}(S^{\mathcal{M}_0}))^{V_1}$. If $\mathcal{M}_1$ is a relativization of $\mathcal{M}_0$ to V_1 for $\mathrm{Th}_{\mathrm{CMS}}(\mathrm{L}_{\mathrm{MS}})$ then $(\mathcal{M}_1 \models \neg(\forall x : S)E(x))^{V_1}$. □

The reason why this Π_1 first order sentence isn't absolute is that when we move from a complete metric structure in V_0 to its relativization in V_1 our underlying sorts can gain structure (in the sense that potentially new Cauchy sequences are being forced to converge). However in V_1, even though there may be new elements in the structure, there are no new elements which satisfy our relation. Hence, in V_1 we can loose a connection between the sort and the relation.

This suggests that if we require our relations to have more structure than just being sets, the relativizations might preserve more properties. In this section we consider three possible types of structure we could require of relations, and consider when requiring relations to have this extra structure will allow there to be a relativization of the complete metric structure. The three types of structure we consider requiring a relation to have are: being open, being closed, and being Borel.

For a metric language L_{MS} we consider the following three theories:

- $\mathrm{Th}_{\mathrm{Opn}} := \mathrm{Th}_{\mathrm{CMS}}(\mathrm{L}_{\mathrm{MS}}) \cup \{$"$\{x : E(x)\}$ is an open set":E is a non-equality relation in $\mathrm{L}_{\mathrm{MS}}\}$.
- $\mathrm{Th}_{\mathrm{Cls}} := \mathrm{Th}_{\mathrm{CMS}}(\mathrm{L}_{\mathrm{MS}}) \cup \{$"$\{x : E(x)\}$ is an closed set":E is a non-equality relation in $\mathrm{L}_{\mathrm{MS}}\}$.
- $\mathrm{Th}_{\mathrm{Bor}} := \mathrm{Th}_{\mathrm{CMS}}(\mathrm{L}_{\mathrm{MS}}) \cup \{$"$\{x : E(x)\}$ is an Borel set":E is a non-equality relation in $\mathrm{L}_{\mathrm{MS}}\}$.

We now consider when the above theories relativizes.

Lemma 3.24. *Suppose $\mathcal{M}_0$ satisfies $\mathrm{Th}_{\mathrm{Opn}}(\mathrm{L}_{\mathrm{MS}})$ in V_0. Then the following are equivalent*

(1) *$\mathcal{M}_0$ has a relativization to V_1 for $\mathrm{Th}_{\mathrm{Opn}}(\mathrm{L}_{\mathrm{MS}})$,*
(2) *For each non-equality relation E of sort S, $\{x : \mathcal{M}_0 \models E(x)\}$ is an open subset of $\mathbf{cc}(S^{\mathcal{M}_0})^{V_1}$.*

Proof. First if (2) is satisfied and $\mathcal{M}_1$ is the relativization of $\mathcal{M}_0$ to V_1 for $\mathrm{Th}_{\mathrm{CMS}}$ then every relation in $\mathcal{M}_1$ is open and hence $\mathcal{M}_1$ satisfies $\mathrm{Th}_{\mathrm{Opn}}$. In particular as $\mathrm{Th}_{\mathrm{Opn}}$ implies $\mathrm{Th}_{\mathrm{CMS}}$ we have

$$(\mathrm{Ext}^{\mathrm{Th}_{\mathrm{Opn}}(\mathrm{L}_{\mathrm{MS}})}(\mathcal{M}_0))^{V_0} \subseteq (\mathrm{Ext}^{\mathrm{Th}_{\mathrm{Opn}}(\mathrm{L}_{\mathrm{MS}})}(\mathcal{M}_0))^{V_1}$$

and so $\mathcal{M}_1$ must be the relativization of $\mathcal{M}_0$ to V_1 for $\mathrm{Th}_{\mathrm{Opn}}(\mathrm{L}_{\mathrm{MS}})$ as well.

Now suppose (1) holds and $\mathcal{M}_1$ is the relativization of $\mathcal{M}_0$ to V_1 for $\mathrm{Th}_{\mathrm{Opn}}$. Let $\mathcal{M}_1^*$ be the Cauchy complete metric structure in V_1 where for every sort S, $S^{\mathcal{M}_1^*} = \mathbf{cc}(S^{\mathcal{M}_0})^{V_1}$, for every function f, $f^{\mathcal{M}_1^*} = \mathbf{cc}(f^{\mathcal{M}_0})^{V_0}$ and for every non-equality relation E of sort S, $\mathcal{M}_1^* \models$

$(\forall x : S)E(x)$. It is then immediate that $(\mathcal{M}_1^* \models \mathrm{Th}_{\mathrm{Opn}})^{V_1}$ and that $\mathcal{M}_0 \subseteq \mathcal{M}_1^*$ with the inclusion map a homomorphism. There then must be a homomorphism $\mathcal{M}_1$ into $\mathcal{M}_1^*$ which is constant on $\mathcal{M}_0$. But as the sorts of $\mathcal{M}_1$ are complete metric spaces and the sorts of $\mathcal{M}_1^*$ are the completions of the sorts of $\mathcal{M}_0$, this implies that the underlying sorts and functions of $\mathcal{M}_1$ are the same as those of $\mathcal{M}_1^*$.

Now assume to get a contradiction that E is a non-equality relation of sort S such that $\{x : \mathcal{M}_0 \models E(x)\}$ is not an open subset of $S^{\mathcal{M}_1}$ in V_1.

Then, as $\{x : \mathcal{M}_0 \models E(x)\} \subseteq \{x : \mathcal{M}_1 \models E(x)\}$ and $\{x : \mathcal{M}_1 \models E(x)\}$ is open there must be some $y \in \{x : \mathcal{M}_1 \models E(x)\} - \{x : \mathcal{M}_0 \models E(x)\}$. But then we also have $\{x : \mathcal{M}_0 \models E(x)\} \subseteq \{x : \mathcal{M}_1 \models E(x)\} - \{y\}$ and $\{x : \mathcal{M}_1 \models E(x)\} - \{y\}$ is open as $\{x : \mathcal{M}_1 \models E(x)\}$ is open. Hence if we let $\mathcal{M}_1'$ be the same structure as $\mathcal{M}_1$ with the sole exception that $\mathcal{M}_1' \models \neg E(y)$ we have $\mathcal{M}_0 \subseteq \mathcal{M}_1'$ with the inclusion map a homomorphism and $\mathcal{M}_1' \models \mathrm{Th}_{\mathrm{Opn}}(\mathrm{L}_{\mathrm{MS}})$. But this contradicts our assumption that $\mathcal{M}_1$ is the relativization of $\mathcal{M}_0$ to V_1 for $\mathrm{Th}_{\mathrm{Opn}}(\mathrm{L}_{\mathrm{MS}})$ as any homomorphism between $\mathcal{M}_1$ and $\mathcal{M}_1'$ which is the identity on $\mathcal{M}_0$ must itself be the identity (and hence can't be a homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_1'$). □

Lemma 3.25. *Suppose $\mathcal{M}_0$ satisfies $\mathrm{Th}_{\mathrm{Bor}}(\mathrm{L}_{\mathrm{MS}})$ in V_0. Then $\mathcal{M}_0$ has a relativization to V_1 for $\mathrm{Th}_{\mathrm{Bor}}$ if and only if for each non-equality relation E of sort S, $\{x : \mathcal{M}_0 \models E(x)\}$ is a Borel subset of $\mathbf{cc}(S^{\mathcal{M}_0})^{V_1}$.*

Proof. The proof is identical to the proof of Lemma 3.24. □

In particular adding the requirement that our relations are open or Borel doesn't (in general) relativize. Now we consider what happens if we require our relations to be closed sets.

Lemma 3.26. *Every model in V_0 of $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$ has a relativization to V_1 for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$.*

Proof. Let $\mathcal{M}_0 \models \mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$ and let $\mathcal{M}_1 \in V_1$ be the structure where:

- For each sort S, $S^{\mathcal{M}_1} = \mathbf{cc}(S^{\mathcal{M}_0})^{V_1}$.
- For each function f, $f^{\mathcal{M}_1} = \mathbf{cc}(f^{\mathcal{M}_0})^{V_1}$.
- For each relation E, $\{x : \mathcal{M}_1 \models E(x)\} = (\overline{\{x : \mathcal{M}_0 \models E(x)\}})^{V_1}$.

That $\mathcal{M}_1$ is the relativization of $\mathcal{M}_0$ to V_1 for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$ then follows from the fact that any closed set in V_1 containing $\{x : \mathcal{M}_0 \models E(x)\}$ must also contain $\{x : \mathcal{M}_1 \models E(x)\}$. □

The following is immediate.

Corollary 3.27. *Suppose $\mathcal{M}_0$ is a model of* $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$ *in V_0 and C a relation in* L_{MS} *of sort S. Further suppose $\mathcal{M}_1$ is the relativization of $\mathcal{M}_0$ to V_1 for* $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$*. Suppose C_0 is a complete metric space in V_0 which is isomorphic (in V_0) to the subspace $\{x : \mathcal{M}_0 \models C(x)\}$ of $(S^{\mathcal{M}_0}, d_S^{\mathcal{M}_0})$. Suppose C_1 is the relativization of C_0 to V_1 for* $\mathrm{Th}_{\mathrm{CMet}}$*. Then (in V_1) C_1 is isomorphic to the subspace $\{x : \mathcal{M}_1 \models C(x)\}$ of $(S^{\mathcal{M}_0}, d_S^{\mathcal{M}_0})$.*

In particular this tells us that relativizing a closed set of a complete metric space as a closed set gives the same thing as relativizing it as a complete metric space.

Suppose E is a closed subset of a metric space S and $d(x, E) := \inf\{y \in E : d(x, y)\}$ is the continuous function giving the distance to E. When E is closed, we have for all $x \in S$, $x \in E$ if and only if $d(x, E) = 0$. This property of closed sets allows us to use functions to define relations which are closed. Further, if we are only dealing with relations which are closed (and theories which guarantee this fact) then there is no harm in restricting our attention to formulas which only use functions, a fact which will be important in Theorem 4.1.

Definition 3.28. *Suppose $\varphi \in \mathcal{L}_{\infty,\infty}(\mathrm{L}_{\mathrm{MS}})$ is a formula. We say φ is* **continuous** *if it is positive and the only relations which are subformulas are equalities on the sort R.*

Lemma 3.29. *For each metric language* $\mathrm{L}_{\mathrm{MS0}}$ *there is a metric language* $\mathrm{L}_{\mathrm{MS1}}$ *where* $\mathrm{L}_{\mathrm{MS1}} \setminus \mathrm{L}_{\mathrm{MS0}} = \{\delta_E : S \to R \text{ s.t. } E \in \mathrm{L}_{\mathrm{MS0}}$ *is a relation of sort* $S\} \cup \{\mathfrak{op}_-, \mathfrak{op}_+, \mathfrak{op}_\times : R \times R \to R\}$ *and there is a definition $\mathbb{D}$ of* $\mathrm{L}_{\mathrm{MS1}} \setminus \mathrm{L}_{\mathrm{MS0}}$ *such that:*

(1) $\mathbb{D} \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L}_{\mathrm{MS1}})$.
(2) *For every positive finitary Boolean combination of atomic formula $\varphi(\overline{x})$ there is a term $t_\varphi(\overline{x})$ such that $\mathbb{D} \models (\forall \overline{x})\varphi(\overline{x}) \leftrightarrow [t_\varphi(\overline{x}) = 0]$. Further if for each term t in φ we have $\mathcal{M} \models \mathrm{Th}_{\mathrm{UCon}}(t, \mathfrak{m}_t)$ for some term $\mathfrak{m}_t$ then there is a term $\mathfrak{m}_\varphi$, such that $\mathcal{M} \models \mathrm{Th}_{\mathrm{UCon}}(t_\varphi, \mathfrak{m}_\varphi)$.*
(3) *For every negation of an atomic formula $\varphi(\overline{x})$ there is a formula $\varphi^*(\overline{x})$ which is a countable disjunction of formulas of the form $t(\overline{y}) = 0$ our definition implies $\mathbb{D} \models (\forall \overline{x})\varphi(\overline{x}) \leftrightarrow \varphi^*(\overline{x})$.*
(4) *The definition is absolute for* $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$.

Proof. The $\mathbb{D}$ be the conjunction of the following:

- For each non-equality relation E of sort S:

$$(\forall x) \bigwedge_{q\in\mathbb{Q}^{\geq 0}} \left[\delta_E(x) \leq \widehat{q} \leftrightarrow \left[\bigwedge_{q<p\in\mathbb{Q}^{>0}} (\exists y)E(y) \wedge d_S(x,y) \leq \widehat{p}\right]\right]$$

- For each $p \leq q \in \mathbb{Q}^{\geq 0}$, $\mathfrak{op}_-(\widehat{q},\widehat{p}) = \widehat{q-p} \wedge \mathfrak{op}_-(\widehat{p},\widehat{q}) = \widehat{0}$.
- For each $p, q \in \mathbb{Q}^{\geq 0}$, $\mathfrak{op}_+(\widehat{p},\widehat{q}) = \widehat{p+q}$.
- For each $p, q \in \mathbb{Q}^{\geq 0}$, $\mathfrak{op}_\times(\widehat{p},\widehat{q}) = \widehat{p \times q}$ if $p \times q \leq 1$ and $\mathfrak{op}_\times(\widehat{p},\widehat{q}) = 1$ otherwise.

Because every function in a metric structure must be continuous and $\mathfrak{op}_+, \mathfrak{op}_-$, and $\mathfrak{op}_\times$ are completely defined on $\{\widehat{q} : q \in \mathbb{Q}^{\geq 0}\}$, it is easily checked that for any $\mathrm{L}_{\mathrm{MS0}}$-structure $\mathcal{M}$ which satisfies $\mathrm{Th}_{\mathrm{Cls}}$, there is a unique expansion of $\mathcal{M}$ to an $\mathrm{L}_{\mathrm{MS1}}$-structure $\mathcal{M}'$ with $\mathcal{M}' \models \mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS1}}) \cup \{\mathbb{D}\}$. Note intuitively $\mathfrak{op}_-$ is subtraction, $\mathfrak{op}_+$ is addition, and $\mathfrak{op}_\times$ is bounded multiplication.

We now show (2) holds. Notice that $\mathrm{Th}_{\mathrm{Cls}} \cup \mathbb{D}$ implies that $\delta_E(x) = \inf\{d_S(x,y) : E(y)\}$, i.e. $\delta_E(x)$ is the distance from x to the set $\{y : E^{\mathcal{M}}(y)\}$. But then as $\{y : E^{\mathcal{M}}(y)\}$ is closed we know that $\delta_E(x) = \widehat{0}$ if and only if $E(x)$ holds. Hence our definition implies $(\forall x)[E(x) \leftrightarrow \delta_E(x) = \widehat{0}]$.

Next consider the equality relation $=_S$ on a sort S. Note we have $(\forall x, y : S)x =_S y \leftrightarrow d_S(x,y) = \widehat{0}$. In particular this implies (2) holds of all atomic formulas. Finally, notice that $\mathfrak{op}_+(x,y) = \widehat{0}$ if and only if "$x = \widehat{0}$ or $y = \widehat{0}$" and $\mathfrak{op}_\times(x,y) = 0$ if and only if "$x = \widehat{0}$ and $y = \widehat{0}$". Hence (2) holds for all positive finitary Boolean combinations of atomic formulas.

Further, as $\mathfrak{op}_+$ and $\mathfrak{op}_\times$ are uniformly continuous and δ_E is uniformly continuous for each E, we have whenever φ is a positive finitary Boolean combination of atomic formulas such that φ only contains terms whose interpretations are uniformly continuous then $\varphi(\overline{x})$ is equivalent to $t_\varphi(\overline{x}) = 0$ where t_φ is interpreted by a uniformly continuous function with modulus of convergence the supremum of the modulus of convergence of its subterms.

To see that (3) holds, notice from (2) every atomic formula $E(x)$ is equivalent to a formula $t_E(x) = 0$ for a term $t_E(x)$. Further, as $\leq$ is a linear order, $\neg E(x)$ is equivalent to $t_E(x) > 0$. In particular it suffices to restrict to the relation $>$ (i.e. $\neg \leq$). Next notice that $(\forall x, y : R)x > y \leftrightarrow \bigvee_{p<q\in\mathbb{Q}^{\geq 0}} y \leq \widehat{p} \wedge \widehat{q} \leq x$ and hence $\mathbb{D} \models (\forall x, y : R)x > y \leftrightarrow \bigvee_{p<q\in\mathbb{Q}^{\geq 0}} \mathfrak{op}_-(\widehat{p}, y) = \widehat{0} \wedge \mathfrak{op}_-(\widehat{q}, x) = \widehat{0}$. But $\mathbb{D} \models (\forall x, y :$

$R)x = \widehat{0} \wedge y = \widehat{0} \leftrightarrow \mathfrak{op}_+(x,y) = \widehat{0}$. Hence $\mathbb{D} \models (\forall x, y : R) x > y \leftrightarrow \bigvee_{p<q\in\mathbb{Q}^{\geq 0}} \mathfrak{op}_+(\mathfrak{op}_-(\widehat{p}, y), \mathfrak{op}_-(\widehat{q}, x)) = \widehat{0}$.

Finally, to see (4), note that it is immediate that the definitions of $\mathfrak{op}_+, \mathfrak{op}_-$ and $\mathfrak{op}_\times$ are absolute for $\mathrm{Th}_{\mathrm{CMet}}(\mathrm{L}_{\mathrm{MS}1})$ and hence also absolute for $\mathrm{Th}_{\mathrm{Cls}}$. Further, in any complete metric $\mathrm{L}_{\mathrm{MS}1}$-structure $\mathcal{M}_0 \in V_0$ which satisfies $\mathbb{D}$ and any relation E of sort S we have $\delta_E^{\mathcal{M}_0}(x) = \inf\{d_S^{\mathcal{M}_0}(x,y) : y \in \{z : \mathcal{M}_0 \models E(z)\}\} = \inf\{d_S^{\mathcal{M}}(x,y) : y \in \overline{\{z : \mathcal{M} \models E(z)\}}\}$. Now suppose $(\mathcal{M}_0 \models \mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}1}))^{V_0}$ and $\mathcal{M}_1$ is the relativization of $\mathcal{M}_0$ to V_1 for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}1})$. Then $(\{z : \mathcal{M}_1 \models E(z)\})^{V_1} = (\overline{\{z : \mathcal{M}_0 \models E(z)\}})^{V_1}$. Hence for any $x \in \mathcal{M}_0$, $\delta_E^{\mathcal{M}_0}(x) = \delta_E^{\mathcal{M}_1}(x)$ and therefore the definition of δ_E is absolute for $\mathrm{Th}_{\mathrm{Cls}}$. □

Corollary 3.30. *Given the same set up as Lemma 3.29, if $\beta > \omega$ then for every $\mathcal{CD}^{V_0}_{\alpha,\beta}$-sentence φ there is a positive continuous $\mathcal{CD}^{V_0}_{\alpha,\beta}$-formula $\varphi^*(\overline{x})$ such that* $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}1}) \cup \mathbb{D} \models (\forall \overline{x})\varphi(\overline{x}) \leftrightarrow \varphi^*(\overline{x})$ *(in both V_0 and V_1). The same holds if $\mathcal{CD}_{\alpha,\beta}$ is replaced by $\Sigma_1^{\alpha,\beta}, \Pi_1^{\alpha,\beta}$ or $\Sigma_2^{\alpha,\beta}$.*

In particular Lemma 3.29 and Corollary 3.30 imply that when considering models of $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}1})$ there is no harm in restricting our attention to continuous formulas provided we allow countably infinite disjunctions. This will be important as we will show in Theorem 4.1 that such continuous $\Sigma_1^{\omega_1,\infty}$ sentences are absolute for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}1})$. This will then allow us to conclude that, unlike for the theory $\mathrm{Th}_{\mathrm{CMS}}(\mathrm{L}_{\mathrm{MS}1})$, all $\Sigma_1^{\omega_1,\infty}$ sentences are absolute for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}1})$.

4. Infinitary Formulas

In this section we consider which classes of infinitary sentences are absolute for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$. In what follows we will assume $\mathrm{L}_{\mathrm{MS}} \in V_0$ is a metric language, $\mathcal{M}_0 \in V_0$ is a model of $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$ and $\mathcal{M}_1 \in V_1$ is the relativization of $\mathcal{M}_0$ to V_1 for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$.

4.1. Σ_1-Sentences.

First recall from Lemma 2.8 and Example 2.9 that all positive $\Sigma_1^{\infty,\infty}$ sentences are upwards absolute, but in general, for cardinality reasons, not all $\Sigma_1^{\infty,\infty}$ sentences are downward absolute. In particular there are $\Sigma_1^{\omega_2,0}$ sentences which are not downward absolute. The problem with downward absoluteness of $\Sigma_1^{\infty,\infty}$ sentences stems from fact that we are able, with uncountable conjunctions, to say something about the cardinality of the model we are considering. As we will see in the next theorem, this is the only obstacle to the downward absoluteness of $\Sigma_1^{\alpha,\beta}$ sentences.

Theorem 4.1. *Suppose* $(\exists X)\varphi$ *is a* $\Sigma_1^{\omega_1,\infty}$ *is a sentence. Then* $(\exists X)\varphi$ *is absolute for* $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$.

Proof. First notice, by Lemma 3.29, it suffices to assume that φ is continuous. Now, as the only conjunctions in φ are countable, every such φ is equivalent to one where the subformulas of each conjunction are indexed by ω (this may entail enlarging some conjunctions by adding the equation $x = x$).

Further, as we can combine successive conjunctions as well as successive disjunctions together, every continuous quantifier free formula is equivalent to one in which each immediate subformula of a disjunction is always a conjunction or atomic formula and each immediate subformula of a conjunction is either a disjunction or an atomic formula. In particular we can assume without loss of generality that φ is of this form.

We will prove this theorem by showing that there is a tree in V_0 such that in both V_0 and V_1, $(\exists X)\varphi$ holds if and only if the tree is ill-founded. The first step in constructing our tree will be to define what we call a *path* through a quantifier free formula φ. Intuitively such a path is a tree labeled by subformulas of φ where leaves are labeled by atomic formulas, such that if a realization $\overline{a}$ of the variables $\overline{x}$ causes every atomic formula on all the leaves to be satisfied then $\varphi[\overline{a}]$ is true. We now make this precise.

We define $\mathfrak{pp}(\varphi)$, the collection of **partial paths** through φ, by induction on φ. Note each element of $\mathfrak{pp}(\varphi)$ is a partial function with domain $\omega^{<\omega}$.

- If A is an atomic formula then $\mathfrak{p} \in \mathfrak{pp}(A)$ if:
  - $\mathfrak{p}(\langle\rangle) = A$ if $\langle\rangle$ is in the domain.
  - $\mathfrak{p}(s) = \top$ if s is in the domain and $s \neq \langle\rangle$.
- If $\varphi = \bigwedge_{i\in\omega} \psi_i$ then $\mathfrak{p} \in \mathfrak{pp}(\varphi)$ if:
  - $\mathfrak{p}(\langle\rangle) = \varphi$.
  - If $\mathfrak{p}_i(s) := \mathfrak{p}(i^\wedge\langle s\rangle)$ then $\mathfrak{p}_i \in \mathfrak{pp}(\psi_i)$.
- If $\varphi = \bigvee_{i\in I} \psi_i$ then $\mathfrak{p} \in \mathfrak{pp}(\varphi)$ if:
  - $\mathfrak{p}(\langle\rangle) = \varphi$
  - $\mathfrak{p}(\langle n+1\rangle^\wedge s) = \top$ if $\langle n+1\rangle^\wedge s$ is in the domain.
  - $\mathfrak{p}(\langle 0\rangle) = \psi_i$ for some $i \in I$ if $\langle 0\rangle$ is in the domain.
  - If $\mathfrak{p}(\langle 0\rangle) = \psi_i$ and $\mathfrak{p}^*(s) := \mathfrak{p}(0^\wedge\langle s\rangle)$ then $\mathfrak{p}^* \in \mathfrak{pp}(\psi_i)$.

By a **path** through φ we mean a map with domain $\omega^{<\omega}$ such that when we restrict the map to any finite subtree of $\omega^{<\omega}$ then the result is a partial path through φ. The intuition is that a path tells us for each disjunction which disjunct we want to satisfy and at the same time keeps track of all of the conjunctions which must be satisfied for φ to hold.

Next we define a **witness** to $(\exists X)\varphi$ in $\mathcal{M}$ for a dense set $\mathrm{DE}^{\mathcal{M}}$ to be a tuple $\langle \alpha, \alpha_{cs}, Y, \mathfrak{p}\rangle$ such that:

- $\mathfrak{p}$ is a path for φ, and Y is the collection of all free variables in all atomic formula in the range of $\mathfrak{p}$.
- α is an assignment of the variables of Y to $\mathcal{M}$ such that every atomic formula in the range of $\mathfrak{p}$ is true. For every y, $\alpha_{cs}(y)$ is a Cauchy sequence of elements from $\mathrm{DE}^{\mathcal{M}}$ which converges to $\alpha(y)$.

Note we need both $\alpha(y)$ and $\alpha_{cs}(y)$ because we are not assuming the axiom of choice. Also note that $\alpha_{cs}(y)$ is well-defined because of our set theoretic assumption $(+_{\mathbf{cc}})$.

Claim 4.2. *If $DE^{\mathcal{M}}$ is a dense subset of $\mathcal{M}$ then $(\exists X)\varphi$ holds in $\mathcal{M}$ if and only if there is a witness to $(\exists X)\varphi$ in $\mathcal{M}$ for $DE^{\mathcal{M}}$.*

Proof. It is immediate from the definition of a path, that if a witness $\langle \alpha, \alpha_{cs}, Y, \mathfrak{p}\rangle$ to $(\exists X)\varphi$ exists and α^* is any assignment of the variables X to elements of $\mathcal{M}_0$ which agrees with α on $Y \subseteq X$, then $\varphi(\alpha^*)$ holds.

Further, it is an easy induction on the complexity of φ (using the fact that φ can be represented as a well-founded tree) to see that if $\varphi(\alpha^*)$ holds for some assignment of variables α^* then there is a witness $\langle \alpha, \alpha_{cs} Y, \mathfrak{p}\rangle$ where α is the restriction of α^* to Y. ☐

We have now reduced the problem of showing that $(\exists X)\varphi$ holds to showing that there exists a witness. This is important as witnesses are fundamentally countable objects where as realizations of the variables X can be arbitrarily large objects (as X can be arbitrarily large). In particular witnesses can be approximated by finite objects.

Let $\mathrm{FSTr} := \{\text{finite subtrees of } \omega^{<\omega}\}$ and let $\iota : \omega \to \mathrm{FSTr}$ be a bijection in V_0 such that whenever $m \leq n$ we have $\iota(m) \subseteq \iota(n)$.

When $\mathrm{DE}^{\mathcal{M}} = \langle D_{S,\mathcal{M}} : S \text{ is a sort in Ł}\rangle$ is a dense subset of $\mathcal{M}$ define an **approximate witness** to φ for $\mathrm{DE}^{\mathcal{M}}$ to be a sequence $\langle n, Y, \langle a_1^*, \ldots, a_{k_n}^*\rangle, \mathfrak{p}\rangle$ where:

- $\mathfrak{p}$ is a partial path with domain $\iota(n)$.
- $Y = \langle y_1, \ldots, y_{k_n}\rangle$ is an enumeration of the free variables of those atomic formulas in the range of $\mathfrak{p}$.
- For each $1 \leq h \leq k_n$, $a_h^* = \langle a_h^1, \ldots, a_h^n\rangle \subseteq D_{S,\mathcal{M}_0}$ where y_h is of sort S_h and $d^{\mathcal{M}}_{S_h}(a_h^{j_0}, a_h^{j_1}) \leq 2^{-2\cdot\min\{j_0, j_1\}}$ for $1 \leq j_0, j_1 \leq n$.
- If $\alpha_n(y_h) = a_h^n$ for each $1 \leq h \leq k_n$ is an assignment of variables and "$f(\overline{x}) = g(\overline{y})$" is an atomic formula in the range of $\mathfrak{p}$,

then $d_R(f[\alpha_n], g[\alpha_n]) \leq \widehat{2^{-n}}$ (where $f[\alpha_n]$ is the value of f with variable assignments α_n).

We also say that $\langle n, Y_n, \langle a_1^*, \ldots, a_{k_n}^* \rangle, \mathfrak{p}_n \rangle \preceq \langle m, Y_m, \langle b_1^*, \ldots, b_{k_m}^* \rangle, \mathfrak{p}_m \rangle$ if and only if:

- $m \leq n$ and $k_m \leq k_n$.
- Y_m is an initial segment of Y_n.
- For each $1 \leq h \leq k_m$, b_h^* is an initial segment of a_h^*.

Let $\mathrm{PW}^{\mathrm{DE}^{\mathcal{M}_0}}\varphi$ be the collection of all approximate witnesses to φ for $\mathrm{DE}^{\mathcal{M}}$. As each approximate witness is a finite object $(\mathrm{PW}^{\mathrm{DE}^{\mathcal{M}}}\varphi, \preceq)^{V_1} = (\mathrm{PW}^{\mathrm{DE}^{\mathcal{M}}}\varphi, \preceq)^{V_0}$ whenever $\varphi \in V_0$, and $\mathrm{DE}^{\mathcal{M}} \in V_0$.

But we then also have the following in either V_0 or V_1.

Claim 4.3. *There is a witness (in V_i) to φ for $DE^{\mathcal{M}}$ if and only if $(PW^{DE^{\mathcal{M}}}\varphi, \preceq)$ is ill-founded (where $i \in \{0, 1\}$).*

Proof. First suppose there is such a witness $\langle \alpha, \alpha_{cs}, Y, \mathfrak{p} \rangle$. Because each function is continuous, we can find for each $f(\overline{x}) = g(\overline{y})$ in the range of the path and each n, an m_n such that if $\overline{a} = a_1, \ldots, a_{k_n}$ are the m_nth elements of $\alpha_{cs}(y_1), \ldots, \alpha_{cs}(y_{k_n})$ then $d_R^{\mathcal{M}}(f[\overline{a}], g[\overline{a}]) \leq 2^{-n}$. Hence we can use the witness to get an infinite path through $(\mathrm{PW}^{\mathrm{DE}^{\mathcal{M}}}\varphi, \preceq)$.

In the other direction, suppose we have an infinite path through $(\mathrm{PW}^{\mathrm{DE}^{\mathcal{M}}}\varphi, \preceq)$. Let $\mathfrak{p}$ be the union of all the approximate paths in the branch and Y be the collection of all free variables in the range of $\mathfrak{p}$. This branch then easily gives rise to a Cauchy sequence cs_y from $\mathrm{DE}^{\mathcal{M}_0}$ for each variable $y \in Y$. Let $\alpha_{cs}(y) = \mathrm{cs}_y$ and $\alpha(y)$ be the element the sequence converges to. Then $\langle \alpha, \alpha_{cs}, Y, \mathfrak{p} \rangle$ is the desired witness. $\square$

Let $\mathrm{DE}^{\mathcal{M}_0} := \langle D_{S,\mathcal{M}_0} : S \text{ a sort in } \mathrm{L}_{\mathrm{MS}} \rangle$ be a dense subset of $\mathcal{M}_0$ with $\mathrm{DE}^{\mathcal{M}_0} \in V_0$. Note in V_0 we have $(\mathcal{M}_0 \models (\exists X)\varphi)^{V_0}$ if and only $(\mathrm{PW}^{\mathrm{DE}^{\mathcal{M}_0}}\varphi, \preceq)$ is ill-founded in V_0. But $\mathrm{DE}^{\mathcal{M}_0}$ is a dense subset of $\mathcal{M}_1$ whenever it is a dense subset of $\mathcal{M}_0$ and so $(\mathcal{M}_1 \models (\exists X)\varphi)^{V_1}$ if and only if $(\mathrm{PW}^{\mathrm{DE}^{\mathcal{M}_0}}\varphi, \preceq)$ is ill-founded in (V_1). But $(\mathrm{PW}^{\mathrm{DE}^{\mathcal{M}_0}}\varphi, \preceq)$ is absolute between V_0 and V_1 and so, because being ill-founded is absolute, $(\mathcal{M}_0 \models (\exists X)\varphi)^{V_0}$ if and only if $(\mathcal{M}_1 \models (\exists X)\varphi)^{V_1}$. $\square$

It is worth mentioning that the above proof explicitly uses the fact that both our metric structure and its relativization are complete. It is not the case that if a metric structure $\mathcal{M}$ satisfies a $\Sigma_1^{\omega_1,\infty}$ sentence that its completion (when it exists) must also satisfy the same sentence. An easy example of this phenomenon is the case of $\mathbb{Q}$ treated as a metric space, which doesn't realize the sentence $(\exists x) \bigwedge\{\widehat{q} \leq x : q \leq \sqrt{2}\} \wedge \{x \leq \widehat{q} : q \geq \sqrt{2}\}$ (which says $\sqrt{2}$ exists). However the completion of

$\mathbb{Q}$ is $\mathbb{R}$ (treated as a metric space) and in $\mathbb{R}$, $\sqrt{2}$ does exist and so $\mathbb{R}$ satisfies the sentence.

4.2. **Σ_2-Sentences.** We now consider absoluteness properties of Σ_2-sentences.

4.2.1. *Upwards Absoluteness.*

Lemma 4.4. *Suppose φ is an (arbitrary) Boolean combination of $\Sigma_1^{\omega_1,\infty}$ formulas. Then $(\exists X)\varphi$ is upwards absolute for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$.*

Proof. Suppose $(\mathcal{M}_0 \models (\exists X)\varphi)^{V_0}$. Suppose $\overline{a} \in V_0$ is an assignment of variables in V_0 such that $(\mathcal{M}_0 \models \varphi[\overline{a}])^{V_0}$. Let $\mathrm{L}_{\mathrm{MS}}{}'$ be the enlargement of L_{MS} adding a constant for every element in $\overline{a}$ and let φ' be the sentence obtained by substituting elements of $\overline{a}$ in $\varphi[\overline{a}]$ for their corresponding constants. Let $\mathcal{M}_0'$ be the expansion of $\mathcal{M}_0$ to $\mathrm{L}_{\mathrm{MS}}{}'$ interpreting each new constant by its corresponding element. Finally let $\mathcal{M}_1'$ be the relativization of $\mathcal{M}_0'$ to V_1 for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}}{}')$. Then $\mathcal{M}_1'$ is the expansion of $\mathcal{M}_1$ obtained by interpreting the new constants by the same elements as in $\mathcal{M}_0$.

By Theorem 4.1 and the fact that any Boolean combination of absolute sentences is absolute, we have that $(\mathcal{M}_1' \models \varphi')^{V_1}$. But then we also have $\mathcal{M}_1 \models (\exists X)\varphi$.

□

Lemma 4.4 implies all $\Sigma_2^{\infty,\omega_1}$ sentences are upwards absolute for $\mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}})$. In general we couldn't hope for all $\Sigma_2^{\infty,\omega_1}$ sentences to be downward absolute as then all $\Sigma_1^{\infty,\omega_1}$ formulas would also be downwards absolute which we know by Example 2.9 is not the case.

4.2.2. *Downward Absoluteness.* In this section we will show that certain continuous $\Sigma_2^{\omega_1,\infty}$ formulas are downwards absolute.

Proposition 4.5. *Suppose $\varphi := \bigwedge_{i\in I} \gamma_i(\overline{z}_i) = \beta_i(\overline{w}_i)$ where:*
- *$\gamma_i : S_i \to R$ and $\beta_i : T_i \to R$ are terms,*
- *all free variables of φ are in $X \cup Y$,*
- *$V_0 \models |X \cup Y| \leq \omega$*

and suppose $\mathfrak{m} : R \to R$. Then $(\exists X)(\forall Y)\varphi(X, Y)$ is absolute for Th where $Th := \mathrm{Th}_{\mathrm{Cls}}(\mathrm{L}_{\mathrm{MS}}) \cup \{\bigwedge_{i\in I} \mathrm{Th}_{\mathrm{UCon}}(\gamma_i, \mathfrak{m}) \wedge \mathrm{Th}_{\mathrm{UCon}}(\beta_i, \mathfrak{m})\}$

Proof. First note that adding dummy variables to $X \cup Y$ does not change the truth value of $(\exists X)(\forall Y)\varphi(X, Y)$. Hence we can assume, without loss of generality that $V_0 \models |X| = |Y| = \omega$ and we can let $X = \langle x_i\rangle_{i\in\omega}$ with x_i of sort T_i for all $i \in \omega$ and $Y = \langle y_i\rangle_{i\in\omega}$ with y_i of sort S_i for all $i \in \omega$.

Next notice if $\gamma_i(\overline{z}_i)$ and $\beta_i(\overline{w}_i)$ are uniformly continuous with modulus of convergence $\mathfrak{m}$ then $d_R(\gamma_i(\overline{z}_i), \beta_i(\overline{w}_i))$ is uniformly continuous with modulus of convergence $2 \cdot \mathfrak{m}$. In particular as $\gamma_i(\overline{z}_i) = \beta_i(\overline{w}_i)$ is equivalent to $d_R(\gamma_i(\overline{z}_i), \beta_i(\overline{w}_i)) = 0$ this means we can assume, without loss of generality, that each $\beta_i(\overline{w}_i)$ is the constant function with value 0, i.e. $\varphi := \bigwedge_{i\in I} \gamma_i(\overline{z}_i) = 0$.

Now to simplify notation for each $\overline{x} \subseteq X \cup Y$ let $I_{\overline{x}} = \{i \in I : \overline{z}_i \subseteq \overline{x}\}$, i.e the collection of those formulas whose variables are contained in $\overline{x}$. We can then define for each ε and n an approximation

$$\varphi^\varepsilon_n := \bigwedge_{i\in I_{x_0\cdots x_n y_0 \cdots y_n}} \gamma_i(\overline{z}_i) \leq \varepsilon$$

to φ.

Now suppose $\mathcal{M}_0 \models Th$. Then by Corollary 3.22 we also have $\mathcal{M}_1 \models Th$.

We now let $\mathrm{DE}^{\mathcal{M}_0} = \langle D_{S,\mathcal{M}_0} : S \text{ a sort in } \mathrm{L}_{\mathrm{MS}}\rangle \in V_0$ be a dense subset of $\mathcal{M}_0$. We define a tree whose ill-foundedness will witness that $(\exists X)(\forall Y)\varphi(X, Y)$ holds. Let $P = \{\langle (a_1, \dots, a_n), \varepsilon, n\rangle : a_i \in D_{T_i,\mathcal{M}_0}$ for $1 \leq i \leq n$, $\varepsilon \in \mathbb{Q}^{>0}$ and $(\forall d_1 \in D_{T_1,\mathcal{M}_0}) \cdots (\forall d_n \in D_{T_n,\mathcal{M}_0})\varphi^\varepsilon_{|\overline{a}|+n}(a_1 \dots, a_n, d_1, \dots, d_n)$ holds$\}$.

For $\langle (a_1, \dots, a_{n_1})\varepsilon_1, n_1\rangle, \langle (b_1, \dots, b_{n_0}), \varepsilon_0, n_0\rangle \in P$ we now have that $\langle (a_1, \dots, a_{n_1}), \varepsilon_1, n_1\rangle \preceq_P \langle (b_1, \dots, b_{n_0}), \varepsilon_0, n_0\rangle$ if and only if

- $n_1 > n_0$.
- $\varepsilon_1 \leq \varepsilon_0/2$.
- For each $1 \leq i \leq n_0$, $d(a_i, b_i) \leq \varepsilon_0$.

We now have the following.

Claim 4.6. *For $i \in \{0, 1\}$, we have $(\mathcal{M}_i \models (\exists X)(\forall Y)\varphi(X, Y))^{V_i}$ if and only if $(P, \preceq_P)$ is ill-founded.*

Proof. Left implies Right:
Let $\langle e_n : n \in \omega\rangle$ be such that $\mathcal{M}_i \models (\forall Y)\varphi(\langle e_n : n \in \omega\rangle, Y)$ and let $\langle q_k : k \in \omega\rangle$ be a decreasing sequence of rationals such that $\mathfrak{m}(q_k) \leq 2^{-k-1}$ and $q_k \leq 2^{-k}$ for all $k \in \omega$. Now let $\langle\langle a^k_n : k \in \omega\rangle : n \in \omega\rangle$ be such that for each $n \in \omega$ $\langle a^k_n : k \in \omega\rangle$ is a Cauchy sequence which converges to e_n and $d(a^k_n, a^j_n) \leq \min\{q_{k+1}, q_{j+1}\}$.

Now suppose for $\ell \in I$, $\overline{z}_\ell = x_{j_1} \dots x_{j_s} y_{i_1} \dots y_{i_r}$ where $y_{i_1} \dots y_{i_r} \subseteq Y$ and $x_{j_1} \dots x_{j_s} \subseteq X$. Further suppose for $1 \leq \zeta \leq s$ we have $e_\zeta \in D_{T_{j_\zeta},\mathcal{M}_0}$. Then for each $\ell \in I$ and each $f_1, \dots, f_r$ such that for $1 \leq h \leq r$ we have $f_h \in D_{T_{i_h},\mathcal{M}_0}$ we have

$$\begin{aligned} d_R(\gamma_\ell(e_{j_1}, \dots, e_{j_s}, f_1, \dots, f_r), \gamma_i(a^k_{j_1}, \dots, a^k_{j_s}, f_1, \dots, f_r)) &\leq 2 \cdot \mathfrak{m}_{q_k} \\ &\leq 2 \cdot 2^{-k-1} = 2^{-k}. \end{aligned}$$

But as $\gamma_\ell(e_{j_1}, \dots, e_{j_s}, f_1, \dots, f_r) = 0$ we have $\gamma_i(a^k_{j_1}, \dots, a^k_{j_r}, f_1, \dots, f_r) \leq 2^{-k}$.

In particular as $f_1, \dots, f_r$ were arbitrary we have

$$(\forall y_0, \dots, y_r)\gamma_\ell(a^k_{j_1}, \dots, a^k_{j_r}, y_1, \dots, y_r) \leq 2^{-k}.$$

But as this holds for all $\ell \in I$ we have for all $m, n \in \omega$ that

$$(\forall y_0, \dots, y_r)\varphi_\ell^{2^{-k}}(a^k_{j_1}, \dots, a^k_{j_r}, y_1, \dots, y_r)$$

holds. Hence $\langle\langle a^k_0, \dots, a^k_k\rangle, 2^{-k}, k) : k \in \omega\rangle \subseteq P$ is the desired ill-founded branch.

Right implies Left:
Suppose $\langle(a^k_1, \dots, a^k_{n_k}), \varepsilon_k, n_k) : k \in \omega\rangle$ is an infinite descending chain in $(P, \preceq_P)$. Fix an $\ell \in I$ and let m be such that all variables in γ_ℓ are contained in $z_0 \dots z_m$. Our infinite path gives us for each $j \in \omega$ a Cauchy sequences $\langle a^n_j : n \in \omega\rangle$ and let b_i be the element it converges to.

For all $j \in \omega$ and $n \in \omega$ we have $d_{T_n}(a^j_n, b_n) \leq 2 \cdot \varepsilon_j$. Hence whenever we have $f_h \in D_{T^y_h}$ for all $1 \leq h \leq p$ then for all $j \in \omega$ we have $d_R(\gamma_\ell(a^j_0, \dots, a^j_m, f_1, \dots, f_p), \gamma_\ell(b_0, \dots, b_m, f_1, \dots f_p)) \leq \mathfrak{m}(\varepsilon_j)$. In particular this implies

$$\gamma_\ell(b_0, \dots, b_m, f_1, \dots, f_p) \leq \mathfrak{m}(\varepsilon_j) + \gamma_\ell(a^j_0, \dots, a^j_m, f_1, \dots, f_p) \leq \mathfrak{m}(\varepsilon_j) + \varepsilon_j.$$

But $\lim_{j\to\infty} \mathfrak{m}(\varepsilon_j) + \varepsilon_j = 0$ and so $\gamma_\ell(b_0, \dots, b_m, f_1 \dots, f_p) = 1$.

In particular, as ℓ and $d_1, \dots, d_p$ were arbitrary, we have $\mathcal{M}_i \models (\forall Y)\varphi(\langle b_n : n \in \omega\rangle, Y) = 0$. Hence $\mathcal{M}_i \models (\exists X)(\forall Y)\varphi(X, Y)$ as well.

□

Finally it is immediate from the definition that $(P, \preceq_P)^{V_0} = (P, \preceq_P)^{V_1}$ and so by the absoluteness of ill-foundedness we have $(\exists X)(\forall Y)\varphi(X, Y)$ is absolute.

□

Notice that this is in some sense optimal in that by Example 2.10 we can not replace the infinite conjunction with an infinite disjunction.

Corollary 4.7. *Suppose $\varphi \in \mathcal{CD}_{\omega,\omega}$ is a continuous formula. Then $(\exists X)(\forall Y)\varphi(X, Y)$ is absolute for $\mathrm{Th}_{\mathrm{Cls}} \cup \bigwedge_{i\in I} \mathrm{Th}_{\mathrm{UCon}}(\gamma_i, S_i, R, \alpha)$ where $\{\gamma_i : S_i \to R\}$ is the collection of atomic subformulas of φ.*

Proof. This is because $\mathfrak{op}_+, \mathfrak{op}_-, \mathfrak{op}_\times$ are uniformly continuous and so every $\varphi \in \mathcal{CD}_{\omega,\omega}$ is equivalent to one of the form $\beta(X, Y) = 0$ for a uniformly continuous β. □

5. Applications

In this section we will give several applications of our absoluteness results. In Section 5.1 we will show that inf operator is absolute. This will then allow us to explain in Section 5.1.1, why all continuous first order formulas (in the sense of [2]) are absolute. In Section 5.2 we will show how our absoluteness results can be used to give a version of Mostrowski absoluteness for κ^κ. Finally in Section 5.3 we will consider specific properties of complete metric spaces which are or are not absolute.

5.1. **Infimum.** The following will be important when proving the absoluteness of continuous formulas.

Definition 5.1. *Suppose* $f : S \times T \to R$. *Let* $\mathrm{Th}_{\mathrm{inf}}(S, T, f, g)$ *be the conjunction of:*

(1) $(\forall x : S)(\forall y : T) f(x, y) \geq g(y)$.
(2) $\bigwedge_{q \in \mathbb{Q}^{>0}} (\forall y : T)(\exists x : S) d_R(f(x, y), g(y)) < \widehat{q}$.

Lemma 5.2. *Suppose* $\{S, T, f, g\} \subseteq \mathrm{L_{MS}}$ *then* $\mathrm{Th}_{\mathrm{inf}}(S, T, f, g)$ *is a definition of* g *with respect to* $\mathrm{Th_{Cls}}(\mathrm{L_{MS}})$.

Proof. It is clear that if $\mathcal{M} \models \mathrm{Th}_{\mathrm{inf}}(S, T, f, g)$ then $(\forall y \in T^{\mathcal{M}}) g(y) = \inf\{f^{\mathcal{M}}(x, y) : x \in S^{\mathcal{M}}\}$ and so $\mathrm{Th}_{\mathrm{inf}}(S, T, f, g)$ is a definition of g with respect to $\mathrm{Th_{Cls}}(\mathrm{L_{MS}})$.

Further $\mathrm{Th}_{\mathrm{inf}}(S, T, f, g)$ is the conjunction of the negations of $\Sigma^{0,0}_2$ formulas and hence is downward absolute.

To see $\mathrm{Th}_{\mathrm{inf}}(S, T, f, g)$ is upwards absolute suppose $\mathcal{M}_0$ is a complete metric structure with $\mathcal{M}_1$ its relativization to V_1 for $\mathrm{Th_{Cls}}(\mathrm{L_{MS}})$ and $\mathcal{M}_0 \models \mathrm{Th_{Fun}}(f)$. Let g_i be the unique function such that $\mathcal{M}_i$ has an expansion $\mathcal{M}'_i$ (in V_i) with $\mathcal{M}'_i \models \mathrm{Th}_{\mathrm{inf}}(S, T, f, g)$. We need to show that $\mathbf{cc}(g_0) = g_1$ (in V_1).

First notice that $(\forall x : S^{\mathcal{M}_0})(\forall y : T^{\mathcal{M}_0}) f^{\mathcal{M}_1}(x, y) \geq g_0(y)$ as $f^{\mathcal{M}_1} = \mathbf{cc}(f^{\mathcal{M}_0})$. So $(\forall y : T^{\mathcal{M}_0})\mathbf{cc}(g_0)(y) \leq g_1(y)$ and hence we have $(\forall y \in T^{\mathcal{M}_1})\mathbf{cc}(g_0)(y) \leq g_1(y)$.

Now suppose to get a contradiction that there is some $y \in T^{\mathcal{M}_1}$ and some $q \in \mathbb{Q}^{>0}$ with $g_1(y) - \mathbf{cc}(g_0)(y) > \widehat{q}$. Then there must be some $x \in S^{\mathcal{M}_1}$ with $f^{\mathcal{M}_1}(x, y) - g_1(y) < \widehat{q/4}$. But as $f^{\mathcal{M}_1}$, $\mathbf{cc}(g_0)$, and g_1 are continuous and as $\mathcal{M}_0$ is dense in $\mathcal{M}_1$ we can also find $x' \in S^{\mathcal{M}_0}$ and $y' \in T^{\mathcal{M}_0}$ such that $d_R(\mathbf{cc}(g_0)(y'), \mathbf{cc}(g_0)(y)) < \widehat{q/4}$, $d_R(g_1(y'), g_1(y)) < \widehat{q/4}$ and $d_R(f^{\mathcal{M}_1}(x', y'), f^{\mathcal{M}_1}(x, y)) < \widehat{q/4}$. But by the triangle inequality this contradicts the fact $g_1(y) - \mathbf{cc}(g_0)(y) > \widehat{q}$. Therefore $\mathbf{cc}(g_0) = g_1$ and $\mathrm{Th}_{\mathrm{inf}}(S, T, f, g)$ is an absolute definition. □

5.1.1. *Continuous First Order Logic.* In recent years a there has been a great deal of research in continuous first order logic and continuous first order structures (as in [2]). Continuous first order structures, in this context, are structures such that sorts are interpreted by bounded complete metric spaces, functions are interpreted by uniformly continuous maps (with the modulus of continuity as part of the language) and relations are interpreted by uniformly continuous maps to $[0, 1]$ (with the modulus of continuity as part of the language). The distance map then plays the role of equality. In this framework the inf and sup operations correspond to the quantifiers and connectives are arbitrary continuous maps from $[0, 1]^n$ to $[0, 1]$.

Given the framework developed in this paper it is easy to find a theory whose models interpret the continuous first order structures in any given language. Further, as every connective in continuous logic is completely determined by where it takes rational tuples, as inf can be expressed as in Lemma 5.2, and as sup is $1 - \inf$, for every continuous first order sentence φ we could construct a sentence φ^* in an appropriate metric language such that a model satisfies φ^* if and only if the continuous model the structure interprets satisfies φ. In particular, as each such interpretation of continuous first order sentences can be constructed from components which can have definitions that are absolute, every continuous first order formula is absolute (for being a continuous first order structure).

5.2. **Descriptive Set Theory.** There is a close relationship between descriptive set theory and infinitary model theory. As such our theorems applied to discrete metric structures (i.e. structures where all spaces are discrete) have corresponding analogs for (uncountable) descriptive set theory. In particular we will get an analog of the Mostrowski Absoluteness theorem for κ^κ (for an arbitrary κ) from our results. For the rest of this section fix a cardinal κ.

We now define codes which capture various descriptive set theoretic classes.

Definition 5.3. *For a set Z we define the collection of* **basic codes** *on Z to be $Basic(Z) := \{\langle 0, i, j, Z\rangle, \langle 1, i, j, Z\rangle : i \in Z, j \in \kappa\}$. We define the set $\mathcal{CD}_{c,\alpha,\beta}(Z)$ to be the smallest collection such that:*

- $Basic(Z) \subseteq \mathcal{CD}^c_{c,\alpha,\beta}(Z)$
- *If $\gamma < \alpha$ and $f : \gamma \to \mathcal{CD}_{c,\alpha,\beta}(Z)$ then $\langle 2, f\rangle \in \mathcal{CD}^c_{c,\alpha,\beta}(Z)$*
- *If $\gamma < \beta$ and $f : \gamma \to \mathcal{CD}_{c,\alpha,\beta}(Z)$ then $\langle 3, f\rangle \in \mathcal{CD}_{c,\alpha,\beta}(Z)$*

We define the set $\Sigma_1^{c,\alpha,\beta}(Y, Z)$ to be the smallest collection such that:

- $\mathcal{CD}^c_{<\alpha,\beta}(Z \cup Y) \subseteq \Sigma_1^{c,\alpha,\beta}(Y, Z)$ *for all sets Y.*

- *If* $a \in \mathcal{CD}^c_{\alpha,\beta}(Z \cup Y)$ *then* $\langle 4, Y, a\rangle \in \Sigma_1^{c,\alpha,\beta}(Y, Z)$

We define the set $\Sigma_2^{c,\alpha,\beta}(X, Y, Z)$ *to be the smallest collection such that:*

- $\Sigma_1^{c,\alpha,\beta}(Y, X \cup Z) \subseteq \Sigma_2^{c,\alpha,\beta}(X, Y, Z)$
- *If* $\langle 4, Y, a\rangle \in \Sigma_1^{c,\alpha,\beta}(Y, X \cup Z)$ *then* $\langle 5, X, \langle 4, Y, a\rangle\rangle \in \Sigma_2^{c,\alpha,\beta}(X, Y, Z)$

Now we define sets to interpret each code.

Definition 5.4. *We define* $\widetilde{a}$ *by induction as follows:*

$(a = \langle 0, i, j, Z\rangle)$ $\widetilde{a} = \{f \in \kappa^Z : f(i) = j\}$.
$(a = \langle 1, i, j, Z\rangle)$ $\widetilde{a} = \{f \in \kappa^Z : f(i) \neq j\}$.
$(a = \langle 2, f\rangle)$ $\widetilde{a} = \bigcap_{\zeta \in dom(f)} \widetilde{f(\zeta)}$.
$(a = \langle 3, f\rangle)$ $\widetilde{a} = \bigcup_{\zeta \in dom(f)} \widetilde{f(\zeta)}$.
$(a = \langle 4, Y, a\rangle)$ $\widetilde{a} = \{g \in \kappa^Z : (\exists f : Y \to \kappa) f \cup g \in \widetilde{a}\}$.
$(a = \langle 5, X, a\rangle)$ $\widetilde{a} = \{g : \kappa^Z : (\exists f : X \to \kappa) f \cup g \notin \widetilde{a}\}$.

We say $A \subseteq \kappa^\kappa$ *is* $\mathcal{CD}_{\alpha,\beta}$ *(or* $\Sigma_1^{\alpha,\beta}$ *or* $\Sigma_2^{\alpha,\beta}$*) if there is an element of* a *of* $\mathcal{CD}_{c,\alpha,\beta}(\kappa)$ *(or* $\Sigma_1^{c,\alpha,\beta}(\kappa_0, \kappa)$ *or* $\Sigma_2^{c,\alpha,\beta}(\kappa_0, \kappa_1, \kappa)$*) such that* $A = \widetilde{a}$.

When dealing with uncountably descriptive set theory there are several different analogs of the Borel sets depending on the sizes of the conjunctions and disjunctions you allow. These analogs correspond to the $\mathcal{CD}_{\alpha,\beta}$. Similarly the $\Sigma_1^{\alpha,\beta}$ sets are the uncountable analogs of the $\mathbf{\Sigma}_1^1$ sets and $\Sigma_2^{\alpha,\beta}$ sets are uncountable analogs of the $\mathbf{\Sigma}_2^1$ sets. We can now deduce from our results in Section 4 the following facts.

Proposition 5.5. (1) *For any* $\Sigma_1^{c,\omega_1,\infty}(\kappa_0, \kappa)$ *code* $a \in V_0$ *we have* $\widetilde{a}^{V_0} = \widetilde{a}^{V_1} \cap V_0$.
(2) *For any* $\Sigma_2^{c,\infty,\omega_1}(\kappa_0, \kappa_1, \kappa)$ *code* $a \in V_0$ *we have* $\widetilde{a}^{V_0} \subseteq \widetilde{a}^{V_1}$.

Proof. Let L be a the language with constants $\{\widehat{i} : i \in \kappa\}$ let ψ_κ be the formula $\bigwedge_{0 \leq i < j < \kappa} \widehat{i} \neq \widehat{j} \wedge (\forall x) \bigvee_{i \in \kappa} x = \widehat{i}$ and let $\mathcal{M}_\kappa$ be any model (which is unique up to isomorphism).

In particular if $L_Z = L \cup \{\widehat{z} : z \in Z\}$ then there is a bijection ι between expansions of $\mathcal{M}_\kappa$ to L_Z structures and elements of κ^Z. Further, we have for each $i \in Z$ and $j \in \kappa$ that

- $\iota(\mathcal{N}) \in \widetilde{\langle 0, i, j, Z\rangle}$ if and only if $\mathcal{N} \models \widehat{i} = \widehat{j}$, and
- $\iota(\mathcal{N}) \in \widetilde{\langle 1, i, j, Z\rangle}$ if and only if $\mathcal{N} \models \widehat{i} \neq \widehat{j}$, and

By induction we can therefore assign to each $\mathcal{CD}_{c,\alpha,\beta}$ code a a sentence $\varphi_a \in \mathcal{CD}_{\alpha,\beta}$ such that $\iota(\mathcal{N}) \in \widetilde{a}$ if and only if $\mathcal{N} \models \varphi_a$. Similarly we can find $\Sigma_1^{\alpha,\beta}$ and $\Sigma_2^{\alpha,\beta}$ sentences for each $\Sigma_1^{c,\alpha,\beta}$ and $\Sigma_2^{c,\alpha,\beta}$ code (respectively). Further, the sentence φ_a can be chosen in a way which is independent of the model of set theory (i.e. $\varphi_a^{V_0} = \varphi_a^{V_1}$).

Part (1) of this proposition therefore follows from Theorem 4.1 and Part (2) follows from Lemma 4.4. □

We end this section with the observation that in the case of ω^ω the sets $\mathcal{CD}_{\omega_1,\omega_1}$ are exactly the Borel sets and hence the $\Sigma_n^{\omega_1,\omega_1}$ are exactly the Σ^1_n sets. In particular $\Sigma_2^{\omega_1,\omega_1}$-absoluteness corresponds to Shoenfield absoluteness and in general we do not have $\Sigma_3^{\omega_1,\omega_1}$ absoluteness (as in general Σ^1_3-formulas are not absolute unless we add additional assumptions about V_0 and V_1). In this sense our absoluteness results are optimal.

5.3. **Properties of Metric Spaces.** We now consider the absoluteness of specific complete metric spaces. For the rest of the section $\mathcal{M}_0$ will be a complete metric space in V_0 and $\mathcal{M}_1$ will be the relativization of $\mathcal{M}_0$ to V_1 with respect to being a complete metric space.

Equality of Size: If there is a bijection between a set X and our space.
This property need not be upwards or downwards absolute. For an example of when size is not downwards absolute let Y be a set such that $V_0 \models |X| \neq |Y|$ and $V_1 \models |X| = |Y|$ and let $\mathcal{M}_Y$ be the discrete metric space with underlying set Y.

To see this property isn't upwards absolute consider a V_1 where $\mathfrak{P}(\omega)^{V_0}$ is countable. Let $\mathcal{M}$ be the discrete metric space with underlying set $\mathfrak{P}(\omega)^{V_0}$. Then in V_0 there is an bijection between $\mathbb{R}^{V_0}$ and the underlying set of $\mathcal{M}$. However $\mathbb{R}^{V_0}$ relativizes to $\mathbb{R}^{V_1}$ (as a complete metric space) where $\mathcal{M}$ relativizes to itself. Hence in V_1 there is no bijection between the relativization of $\mathcal{M}$ and $\mathbb{R}^{V_1}$ (as $V_1 \models |\mathcal{M}| = \omega$.

It is worth pointing out that in the above we are not assuming our bijection is a map between (complete) metric spaces. In particular, having a bijection of metric spaces (and not just underlying sets) is upwards absolute.

Image Of Distances Is Countable:
This is upwards absolute but need not be downwards absolute. To see it is upwards absolute suppose $im(d^{\mathcal{M}_0}) = X_0$ where $V_0 \models |X_0| = \omega$. Let $(*)$ be the sentence

$$(\forall x, y) \bigvee_{z \in X_0} \bigwedge_{p \leq z \leq q \in \mathbb{Q}^{\geq 0}} \widehat{p} \leq d(x,y) \wedge d(x,y) \leq \widehat{q}.$$

Then $\mathcal{M}_0 \models (*)$ and $(*)$ is $\Pi_1^{\omega_1,\omega_1}$, and hence upwards absolute by Theorem 4.1.

To see this property need not be downwards absolute let $V_1 \models |\mathbb{R}^{V_0}| = \omega$. Let $\mathcal{M}_0 = (M, d_M)$ where $M = \{0\} \cup [\frac{1}{2}, 1]^{V_0}$, $d_M(0, r) = r$ and $d_M(r_0, r_1) = 1$ if $r, s \in [\frac{1}{2}, 1]$. It is then immediate that $\mathcal{M}_0 \in V_0$ and that $\mathcal{M}_1 = \mathbf{cc}(\mathcal{M}_0) = \mathcal{M}_0$, i.e. $\mathcal{M}_0$ is its own relativization for complete metric spaces. But $V_1 \models |im(d_M)| = \omega$ and $V_0 \models |im(d_M)| = |\mathbb{R}| > \omega$.

Totally Bounded:
This property is absolute. Being totally bounded is equivalent to, for each $\varepsilon \in \mathbb{Q}^{>0}$,

$$(\exists \langle (x_i, q_i) : i \in \omega \rangle) \bigvee_{n \in \omega} (\forall x) \bigvee_{i \leq n} d(x, x_i) < q_i.$$

Hence by Lemma 4.4 being totally bounded is upwards absolute.

For downwards absoluteness suppose that $\mathcal{M}_1$ is totally bounded and $\{B^{\mathcal{M}_1}(x_i, \varepsilon/2) : i \leq n\}$ is a cover of $\mathcal{M}_1$. Then because $\mathcal{M}_0$ is dense in $\mathcal{M}_1$ we can find a collection of elements $y_i \in \mathcal{M}_0 \cap B^{\mathcal{M}_1}(x_i, \varepsilon/2)$. But then by the triangle inequality we have $B(x_i, \varepsilon/2) \subseteq B(y_i, \varepsilon)$ and hence $\{B(y_i, \varepsilon) : i \leq n\}$ is a cover of $\mathcal{M}_0$. Further, as $\{y_i : i \leq n\}$ is finite, it is also in V_0.

Compact Spaces:
This is absolute. This follows from the fact that being compact is equivalent to being totally bounded and complete.

Ultrametric Space:
This is absolute. This is because being an ultrametric space is equivalent to satisfying $(\forall x, y, z) d(x, z) \leq d(x, y) \vee d(x, z) \leq d(y, z)$ which is absolute by Theorem 4.1.

6. Local Properties

In this section we will show that for any property P, if P is upwards or downwards absolute for complete metric spaces, then the property of being "locally P" is also upwards or downwards for absolute complete metric spaces. First though we need to show that two collections of open balls having the same union is absolute.

Proposition 6.1. *Suppose $\mathcal{M}_0$ is a complete metric space*

- $\langle a_i : i \in K_a \rangle \cup \langle b_i : i \in K_b \rangle$ *are elements of* $\mathcal{M}_0$.
- $\langle q_i : i \in K_a \rangle \cup \langle r_i : i \in K_b \rangle \subseteq \mathbb{Q}^{>0}$.

Let $(*)$ *be the following statement:*

(*) $\bigcup_{i \in K_a} B(a_i, q_i) \subseteq \bigcup_{i \in K_b} B(b_i, r_i)$

(∗) *is absolute for being a complete metric space.*

Proof. First notice that (*) is equivalent to the statement

$$(\forall x)\left[\left[\bigvee_{i\in K_a} d_S(x,a_i) < q_i\right] \rightarrow \left[\bigvee_{i\in K_b} d_S(x,b_i) < r_i\right]\right]$$

and hence is also equivalent to:

$$(\forall x)\bigwedge_{i\in K_a}\left[d_S(x,a_i) \geq q_i \vee \bigvee_{i\in K_b} d_S(x,b_i) < r_i\right]$$

and finally to

$$\bigwedge_{i\in K_a}(\forall x)\left[\mathfrak{op}_-(d_S(x,a_i),q_i) = 0 \vee \bigvee_{i\in K_b}\bigvee_{e\in Q^{>0}} \mathfrak{op}_-(\mathfrak{op}_+(d_S(x,b_i),e),r_i) = 0\right]$$

But this is equivalent to a conjunction of formulas whose negations meet the criteria of Proposition 4.5 and hence are absolute. □

Definition 6.2. *Suppose P is some property of a metric space. We say that $\mathcal{M}$ satisfies* **local** *P if there is a collection of open balls $\langle B(x_i, r_i) : i \in I\rangle$ and closed sets $\langle C_i : i \in I\rangle$ such that*

- *For each $i \in I$, $B(x_i, q_i) \subseteq C_i$.*
- *For each $i \in I$, the induced metric structure on C_i has property P.*
- *$\mathcal{M} = \bigcup_{i\in I} B(x_i, q_i)$.*

In what follows let $\mathcal{M}_0 \in V_0$ be a complete metric space with relativization $\mathcal{M}_1$ to V_1 for being a complete metric space.

Lemma 6.3. *If $\mathcal{P}$ is some property which is upwards absolute then satisfying "locally $\mathcal{P}$", is upwards absolute.*

Proof. In what follows let $\langle B_i : i \in I\rangle$ and $\langle C_i : i \in I\rangle$ be witnesses to local P holding in V_0. First notice that for any closed set $C \subseteq \mathcal{M}_0$ we have that the completion of C (as a metric space) in V_1 is isomorphic to the closure of C in $\mathcal{M}_1$ (by Corollary 3.27). Hence for each C_i, P holds in V_1 of the closure of C_i as $\mathcal{P}$ is upwards absolute for complete metric spaces.

Now suppose $\langle y_j : j \in \omega\rangle \subseteq \mathcal{M}_0$ is a Cauchy sequence in V_1 which converges to an element $y \in B^{\mathcal{M}_1}(x_i, r_i)$. Then we must have $d(x_i, y) = r' < r_i$ and in particular this means that all but finitely many elements of $\langle y_i : i \in \omega\rangle$ are in $B^{\mathcal{M}_0}(x_i, q_i) \subseteq C_i$.

So y must be in the closure of C_i in V_1. Hence in V_1 we have $B^{\mathcal{M}_1}(x_i, q_i) \subseteq \overline{C_i}$. But by assumption, in V_1, $\overline{C_i}$ satisfies P.

Next notice $\bigcup_{i\in I} B^{\mathcal{M}_0}(x_i, q_i) = \bigcup_{q\in\mathbb{Q}^{>0}} B^{\mathcal{M}_0}(x, q)$ for any $x \in \mathcal{M}_0$. Hence by Proposition 6.1 we also have $\bigcup_{i\in I} B^{\mathcal{M}_1}(x_i, q_i) = \bigcup_{q\in\mathbb{Q}^{>1}} B^{\mathcal{M}_1}(x, q)$. Therefore we have $\bigcup_{i\in I} B^{\mathcal{M}_1}(x_i, q_i) = \mathcal{M}_1$ and $\mathcal{M}_1$ satisfies locally $\mathcal{P}$. □

Lemma 6.4. *If $\mathcal{P}$ is some property which is downwards absolute and preserved by all closed subspaces then satisfying "local $\mathcal{P}$" is downwards absolute.*

Proof. Suppose $\mathcal{M}_1$ satisfies locally $\mathcal{P}$.

Let $X = \{(x, q) : x \in \mathcal{M}_0, q \in \mathbb{Q}^{>0}$ and the closure of $B(x, q)$ satisfies $\mathcal{P}\}$. As $\mathcal{P}$is closed under subsets we see that for $i \in \{0, 1\}$, $\mathcal{M}_i$ satisfies local $\mathcal{P}$if and only if $V_i \models \mathcal{M}_i = \left[\bigcup_{(x,q)\in X} B(x, q)\right]^{V_i}$ holds in $\mathbb{V}_i$.

However notice that for any $(x, q) \in X^{V_1}$ we have $B^{\mathcal{M}_0}(x, q) \subseteq B^{\mathcal{M}_1}(x, q)$ and hence $V_1 \models \overline{B^{\mathcal{M}_0}(x, q)} \subseteq \overline{B^{\mathcal{M}_1}(x, q)}$. But as $\mathcal{P}$ is preserved by taking closed subsets, and as $\overline{B^{\mathcal{M}_1}(x, q)}$ satisfies $\mathcal{P}$ by our assumption on (x, q) we also have in V_1, $\overline{B^{\mathcal{M}_0}(x, q)}$ satisfies $\mathcal{P}$.

Now let $C_{x,q}$ be a complete metric space in V_0 which is isomorphic to $(\overline{B^{\mathcal{M}_0}(x, q)})^{V_0}$. Then by Corollary 3.27 the relativization of $C_{x,q}$ to V_1 is isomorphic to the closure of $(\overline{B^{\mathcal{M}_0}(x, q)})^{V_0}$ in V_1. But the closure of $(\overline{B^{\mathcal{M}_0}(x, q)})^{V_0}$ in V_1 is the same as the closure of $B^{\mathcal{M}_0}(x, q)$ in V_1, i.e. $\overline{B^{\mathcal{M}_0}(x, q)}^{V_1}$. But by the previous paragraph we know that $\overline{B^{\mathcal{M}_0}(x, q)}^{V_1}$ satisfies $\mathcal{P}$ in V_1 and hence by the downward absoluteness of $\mathcal{P}$ we have that $C_{x,q}$ satisfies $\mathcal{P}$ as well. In particular this implies that $(x, q) \in X^{V_0}$ and so $X^{V_1} \subseteq X^{V_0}$.

But because we have assumed that $\mathcal{M}_1$ satisfies locally $\mathcal{P}$ we have for any $x_0 \in \mathcal{M}_0$, $V_1 \models \bigcup_{(x,q)\in X^{V_1}} B(x, q) = \bigcup_{p\in\mathbb{Q}^{>0}} B(x_0, p)$. Hence by Proposition 6.1 we also have $V_0 \models \bigcup_{(x,q)\in X^{V_1}} B(x, q) = \bigcup_{p\in\mathbb{Q}^{>0}} B(x_0, p) = \mathcal{M}_0$. But as $X^{V_1} \subseteq X^{V_0}$ we then have $V_0 \models \bigcup_{(x,q)\in X^{V_1}} B(x, q) = \mathcal{M}_0$ and so $\mathcal{M}_0$ satisfies local $\mathcal{P}$. □

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