# ON $n$-CARDINAL SPECTRA OF ULTRAHOMOGENEOUS THEORIES

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Abstract. We analyze the $n$-cardinal spectra of a sequence of formulas relative to an ultrahomogeneous theory that satisfies strong amalgamation. We also give several examples where we can completely determine the $n$-cardinal spectra.

## 1. Introduction

Given a theory $T$ and a collection of formulas $\Phi = \langle \varphi_1(x), \dots, \varphi_n(x) \rangle$ the **$n$-cardinal spectrum** of $\Phi$ relative to $T$, denoted $\mathrm{Spec}_T(\Phi)$, is the collection of $n$-tuples of cardinals $\langle \gamma_1, \dots, \gamma_n \rangle$ such that there is a model $\mathcal{M}$ of $T$ with $|\{x : \mathcal{M} \models \varphi_i(x)\}| = \gamma_i$ for each $1 \leq i \leq n$. We will omit mention of the free variables when they are clear from context.

People have studied $n$-cardinal spectra, especially for two cardinal spectra, since the early days of modern model theory. Much of what we know is either in the form of transfer theorems, like Vaught's two cardinal theorem ([2] Theorem 12.1.1) or Chang's two cardinal theorem ([2] Theorem 12.1.3) or results which make assumptions on the theory, like Shelah's two cardinal theorem for stable theories ([2] Theorem 12.1.2).

The results in this paper fall into the latter category. We study cardinal spectra of complete theories, in a finite language containing only relations and constants, which are ultrahomogeneous[1] and satisfy the strong amalgamation property[2]. We call such a theory a **strong Fraïssé theory** for short.

The main result of this paper is Theorem 3.2 which says if $T$ is a strong Fraïssé theory and $\langle q_1(x), \cdots, q_n(x) \rangle$ is the collection of complete quantifier

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[1]We say a theory is ultrahomogeneous if it has an ultrahomogeneous model.

[2]We say a theory satisfies the strong amalgamation property if the collection of finite substructures of models of the theory satisfies the strong amalgamation property.

free types in one variable over $T$ then

$$\{\langle \lambda_1, \ldots, \lambda_n \rangle : (\forall i, j \leq n)\ \lambda_i \leq |2^{\lambda_j}|\} \subseteq \mathrm{Spec}_T(\langle q_1(x), \cdots, q_n(x)\rangle).$$

This results are obtained by generalizing the techniques used in [1] to characterize $\mathrm{Spec}_{T_G}(\langle E(c,x), x = x\rangle)$ and $\mathrm{Spec}_{T_G}(\langle \neg E(c,x), x = x\rangle)$ when $T_G$ is the theory of a countable ultrahomogeneous graph with edge relation $E$ and distinguished element $c$.[3]

With Theorem 3.2 in hand we consider situations where we can completely determine all of the $n$-cardinal spectrum. These include the theory of any of the following structures: the generic $m$-hypergraph, the generic partial order, the generic directed graph, as well as the generic $m$-colored graph.

## 2. Background

In this paper we fix a finite language $L$ containing only relations and constants. We also fix a strong Fraïssé theory $T \subseteq \mathcal{L}_{\omega,\omega}(L)$.

We let $K(T)$ be the age of a model of $T$, i.e. the collection of all finite substructures of models of $T$. In particular, every element of $K(T)$ will contain all realizations of constants in the language. We also let $K_n(T)$ be the collection of elements of $K(T)$ which contain exactly $n$ elements that are not realizations of constants. We will abuse notation and do not distinguish between an element $p \in K(T)$ and the (unique) complete quantifier free type which is realized by $p$. We will also let $K_1(T) = \{q_i : 1 \leq |K_1(T)|\}$ and use $K_1(T)$ as a short hand for $\langle q_1, \ldots, q_{|K_1(T)|}\rangle$ when no confusion can arise. When $\varphi(x)$ is a formula of one variable we will abuse notation and let $\varphi[\overline{x}] = \bigwedge_{x \in \overline{x}} \varphi(x)$.

See [2] for any model theoretic notions or proofs not explicitly given here. Throughout this paper we will be assuming the axiom of choice. In this paper both $\gamma$ and $\lambda$, as well as their variants, will always be cardinals.

**Lemma 2.1.** *The following are axioms for $T$:*

- $(\forall \overline{x}) \bigvee_{p \in K_{|\overline{x}|}(T)} p(\overline{x})$.
- $\bigwedge_{p \subseteq q \in K(T), |q-p|=1} (\forall \overline{x}) p(\overline{x}) \rightarrow (\exists y) q(\overline{x}, y)$.

We will use the following collections of sequences cardinals repeatedly and so we give them a name.

[3] This is not the notation used in [1] as they only consider the case of $\langle \varphi(x), x = x\rangle$.

**Definition 2.2.** *For* $n \in \omega$

- $C(n) = \{\langle \lambda_1, \dots, \lambda_n \rangle : (\forall i, j \leq n)\lambda_i \leq |2^{\lambda_j}|\}$.
- $B(n) = \{(\gamma, \lambda) : \omega \leq \gamma \leq \lambda \leq \beth_n(\gamma)\}$.
- $B^*(n) = \{(\lambda, \gamma) : \omega \leq \gamma \leq \lambda \leq \beth_n(\gamma)\}$.

*where we define* $\beth_0(\gamma) = \gamma$ *and* $\beth_{n+1}(\gamma) = 2^{\beth_n(\gamma)}$.

### 2.1. Inconsistent Formula.

**Lemma 2.3.** *If* $\{\varphi_i : 1 \leq i \leq n\} \subseteq \mathcal{L}_{\omega,\omega}(L)$ *and* $\langle m, \gamma_2, \dots, \gamma_n \rangle \in Spec_T(\langle \varphi_1, \dots, \varphi_n \rangle)$ *with* $m \in \omega$ *then* $m = 0$ *and* $\{(\exists x)\varphi_1(x)\} \cup T$ *is inconsistent.*

*Proof.* Because $T$ satisfies strong amalgamation. □

In particular no formula can be realized by only a non-zero finite number of elements in a model of a strong Fraïssé theory.

### 2.2. Cardinal Spectra of $K_1(T)$.

It is worth recalling that as $T$ admits elimination of quantifiers every complete type is equivalent to a quantifier free formula (relative to $T$) and every formula is equivalent to a finite disjunction of elements of $K_1(T)$ (where we consider inconsistent formulas as equivalent to the empty disjunction). The following lemma is then immediate.

**Lemma 2.4.** *Suppose* $\langle \varphi_1(x), \dots, \varphi_n \rangle$ *is a collection of* $L$ *formulas where* $T \models \varphi_i(x) \leftrightarrow \bigvee_{k \leq m_i} q_{\alpha_{k,i}}(x)$ *for all* $1 \leq i \leq n$. *Then* $\langle \gamma_1, \dots, \gamma_n \rangle \in Spec_T(\Phi)$ *if and only if there exists* $\langle \lambda_1, \dots, \lambda_{|K_1(T)|} \rangle$ *such that*

- $\langle \lambda_1, \dots, \lambda_{|K_1(T)|} \rangle \in Spec_T(K_1(T))$.
- $(\forall i \leq n)\gamma_i = sup\ \{\lambda_{\alpha_{k,i}} : k \leq m_i\}$.

*where here* $sup\ \emptyset = 0$.

Lemma 2.4 shows that the $n$-cardinal spectrum of any sequence of formulas relative to a strong Fraïssé theory $T$ is completely determined by the $|K_1(T)|$-cardinal spectrum of the complete quantifier free types consistent with the theory. With the exception of a few results on 2-cardinal spectra (which we include for concreteness sake) we will now focus our attention on the $|K_1(T)|$-cardinal spectra of the complete quantifier free types.

## 3. Main Construction

The following is the main construction. It is based on the main construction in [1] which is based on the construction of $2^\gamma$ independent subsets of cardinality $\gamma$ in [3] (p. 288).

**Proposition 3.1.** *Suppose $S$ is a model of $T$ and for all $q \in K_1(T)$, $|\{x : S \models q(x)\}| = |S| = \gamma$. Then there is a model $\mathcal{M}_S$ of $T$ with underlying set $M_S$ such that:*

(a) *$M_S = S \cup V_S$ where $V_S = 2^\gamma \times K_1(T)$.*
(b) *For all $\langle \alpha, r\rangle \in V_S$, $\mathcal{M}_S \models r(\langle \alpha, r\rangle)$.*
(c) *For all $s \in S$ and $q \in K_1(T)$, $\mathcal{M}_S \models q(s)$ if and only if $S \models q(s)$.*
(d) *If $S \subseteq S^* \subseteq M_S$ then $S^* \models T$.*

*Proof.* Let $C = \{c^S : c \in L \text{ is a constant}\}$. Let $\mathcal{U}$ be the set of all tuples $U = \langle A_U, X_U, F_U, q_U\rangle$ where $C \subseteq A_U \subseteq S$, $A_U$ is finite, $X_U \subseteq \gamma$ is finite, $q_U \in K_1(T)$, and $F_U$ is a function from $W^*_U = \{W : W \subseteq 2^{X_U} \times K_1(T)\}$ to $K(T)$ such that for all $W \in W^*_U$ the arity of $F_U(W)$ is $|W| + |A_U| + 1$ and $F_U(W)(\overline{x}, \overline{y}, z) \vdash q_U(z)$.

We well-order $S$ so that for each $q \in K_1(T)$ the sets $\{x : S \models q(x)\}$ are co-final in $S$. We then choose for each $U \in \mathcal{U}$ a distinct element $v_U$ such that, $v_U > \sup(A_U)$, $S \models q_U(v_U)$ and $\{v_U : U \in \mathcal{U}, q_U = q\}$ is co-final in $\{x : S \models q(x)\}$ for each $q \in K_1(T)$. The well-ordering of $S$ then induces a well-ordering of $\mathcal{U}$.

Let $v^+_U = \{v \in S : v < v_{U'}$ where $U'$ is the next element of $\mathcal{U}$ after $U\}$ and $v^+_\emptyset = \{v \in S : v < v_U$ for all $U \in \mathcal{U}\}$.

Let $L_\emptyset = L \cup \{c_s : s \in V_S\} \cup \{c_s : s \in v^+_\emptyset\}$, $L_U = L_\emptyset \cup \{c_s : s \in v^+_U\}$ and $L^* = \bigcup_{U \in \mathcal{U}} L_U$ (where each $c_s$ is a new constant). We will abuse notation and let $c_{\overline{a}} = \langle c_{a_i} : a_i \in \overline{a}\rangle$ when $\overline{a} \subseteq M_S$. We now define complete theories $T_U$ in $L_U$ by induction.

Stage 0: Let $T^*_\emptyset = T \cup \{r(c_{\langle \alpha, r\rangle}) : \langle \alpha, r\rangle \in V_S\} \cup \{q(c_s) : s \in v^+_\emptyset, S \models q(s)\}$. $T^*_\emptyset$ is consistent as $T$ satisfies strong amalgamation. Let $T_\emptyset$ be any complete theory in $L_\emptyset$ extending $T^*_\emptyset$.

Stage $v_U$: Assume for all $U' < U$, $T_{U'}$ has been defined.

If $\overline{f} = \langle\langle f_1, r_1\rangle, \ldots, \langle f_n, r_n\rangle\rangle \subset V_S$ we let $\overline{f}|_{X_U} = \langle\langle f_1|_{X_U}, r_1\rangle, \ldots, \langle f_n|_{X_U}, r_n\rangle\rangle$. When $\overline{f} \subset V_S$ we say $U$ is $\overline{f}$-consistent if

(i) For each $\langle f_i, r_i\rangle, \langle f_j, r_j\rangle \in \overline{f}$, $f_i|_{X_U} \neq f_j|_{X_U}$.
(ii) The collection $\{F_U(\overline{g}|_{X_U}) : \overline{g} \subseteq \overline{f}\}$ are compatible types.
(iii) If $t^U_{\overline{f}}(\overline{x}, \overline{y}, z) = F_U(\overline{f}|_{X_U})(\overline{x}, \overline{y}, z)$ then $\bigcup_{U'<U} T_{U'} \cup \{t^U_{\overline{f}}(c_{A_U}, c_{\overline{f}}, c_{v_U})\}$ is consistent.

Let $T^*_U = \bigcup_{U'<U} T_{U'} \cup \{t^U_{\overline{f}}(c_{A_U}, c_{\overline{f}}, c_{v_U}) : F_U$ is $\overline{f}$-consistent$\} \cup \{q(c_s) : s \in v^+_U, S \models q(s)\}$.

We want to show that $T^*_U$ is consistent. First observe if $\overline{f}, \overline{h} \subset V_S$ and $\overline{g} = \overline{f} \cap \overline{h}$ then $t^U_{\overline{f}}(c_{A_U}, c_{\overline{f}}, c_{v_U}) \models F_U(\overline{g}|_U)(c_{A_U}, c_{\overline{g}}, c_{v_U})$ and $t^U_{\overline{h}}(c_{A_U}, c_{\overline{h}}, c_{v_U}) \models F_U(\overline{g}|_U)(c_{A_U}, c_{\overline{g}}, c_{v_U})$ by (ii) above. In other words $t^U_{\overline{f}}$ and $t^U_{\overline{h}}$ agree on the quantifier free type of their common elements.

Now let $T' \subseteq T^*_U$ be a finite subset. Then $T' \subseteq T \cup T''$ where $T'' = \{p_i(c_{\overline{g}_i}) : i \leq m\}$ is finite, each $p_i(x)$ is a complete quantifier free type, and $T''$ is pairwise consistent. But as $T$ satisfies strong amalgamation we can amalgamate any pairwise consistent collection $T''$ and hence $T'$ is consistent. By compactness $T^*_U$ is then consistent and we can find a complete theory $T_U$ in $L_U$ extending $T^*_U$.

We end the construction by letting $T^* = \bigcup_{U\in\mathcal{U}} T_U$, which is a complete $L^*$ theory, and letting $\mathcal{M}_S \models R(s_1, \ldots, s_n)$ if and only if $R(c_{s_1}, \ldots, c_{s_n}) \in T^*$ and as $S \subseteq \mathcal{M}_S$, we let $d^{\mathcal{M}_S} = d^S$ for any constant $d$ in $L$.

In the construction we know that if $x \in S$ then the 1-type of $x$ as an element of $S$ and as an element of $\mathcal{M}_S$ agree. However, this is not necessarily the case for larger types. As such, for the rest of the argument, we will assume all structure is the structure induced by $\mathcal{M}_S$.

It is immediate that $\mathcal{M}_S$ satisfies condition (a), (b) and (c) of Proposition 3.1. To see $\mathcal{M}_S$ satisfies conditions (d) let $S \subseteq S^* \subseteq \mathcal{M}_S$. Suppose $\overline{a} \in S, \overline{f} \in S^* - S$ with $S^* \models r(\overline{a}, \overline{f})$ (where $r \in K(T)$). Let $t(\overline{x}, \overline{y}, z) \in K(T)$ be a quantifier free type extending $r(\overline{x}, \overline{y})$ and suppose $t(\overline{x}, \overline{y}, z) \vdash q(z)$ for $q \in K_1(T)$.

Choose a finite $X \subseteq \gamma$ such that for all $\langle f_i, r_i\rangle, \langle f_j, r_j\rangle \in \overline{f}$, $f_i|_X \neq f_j|_X$. Let $F$ be any function on $\{W : W \subseteq 2^X \times K_1(T)\}$ where for each $Y \subseteq \overline{f}$, $t(\overline{x}, \overline{f}, z) \models F(Y|_X)(\overline{x}, Y, z)$. In particular $F(\overline{f}|_X) = t(\overline{x}, \overline{f}, z)$ and if $U = \langle \overline{a}, X, F, q\rangle$ then $U$ is $\overline{f}$-consistent so $S^* \models t(\overline{a}, \overline{f}, v_U)$. This means $S^* \models$

$(\exists v)t(\overline{a}, \overline{f}, v)$. But as $t$ was arbitrary we have by Lemma 2.1 that $S^* \models T$ and condition (d) is satisfied. Hence $\mathcal{M}_S$ is a model which witnesses the proposition. □

We can now use the above proposition to prove the main theorem of the paper.

**Theorem 3.2.** *$C(|K_1(T)|) \subseteq Spec_T(K_1(T))$.*

*Proof.* First observe by compactness and the downward Löwenheim-Skolem theorem, for each $\gamma \geq \omega$ there is a model $S_\gamma \models T$ where $|\{x : S_\gamma \models q(x)\}| = \gamma$ for all $q \in K_1(T)$. Then by Proposition 3.1 there is model $\mathcal{M}_\gamma \models T$ such that $|\{x : \mathcal{M}_\gamma \models q(x)\}| = |2^\gamma|$ for all $q \in K_1(T)$ and whenever $S_\gamma \subseteq \mathcal{M} \subseteq \mathcal{M}_\gamma$ then $\mathcal{M} \models T$. The theorem then follows immediately. □

In particular in the case of two formulas we have.

**Corollary 3.3.** *If $\varphi_0(x), \varphi_1(x)$ are formulas such that $T \not\models (\forall x)\varphi_1(x) \rightarrow \varphi_0(x)$ then $B(1) \subseteq Spec_T(\varphi_0, \varphi_1)$.*

*Proof.* Because $T$ is complete we have $(\exists x)\varphi_1(x) \wedge \neg\varphi_0(x)$. Hence there is a $r \in K_1(T)$ such that $r(x) \vdash \varphi_1(x)$ and $r(x) \vdash \neg\varphi_0(x)$. □

3.1. **Counterexamples.** Note that Theorem 3.2 may fail for languages with function symbols or for theories which do not satisfy strong amalgamation, as can be seen in the following two examples.

**Example 3.4.** *Consider the language $L_{func} = \{U, f_0, f_1\}$ where $U$ is a unary predicate and $f_0, f_1$ are 1-place functions. Let $T_{func}$ be the theory which says:*

- *$U$ and $\neg U$ are infinite.*
- *The range of $f_0$ is $U$ and $f_0$ is the identity on $U$.*[4]
- *The range of $f_1$ is $\neg U$ and $f_1$ is the identity on $\neg U$.*
- *$f_0 \circ f_1$ is the identity on $U$ and $f_1 \circ f_0$ is the identity on $\neg U$.*

*In any model of $T_{func}$ $f_1$ gives a bijection from $U$ to $\neg U$ with inverse $f_0$. Therefore $Spec_{T_{func}}(\langle U, \neg U\rangle) = \{\langle\gamma, \gamma\rangle : \omega \leq \gamma\}$. Further $U(x)$ and $\neg U(x)$ are equivalent to complete quantifier free types over $T$. It is also immediate that the unique countable model of $T_{func}$ is ultrahomogeneous and locally finite.*

[4]Here we associate a unary formula with the set of its realizations.

**Example 3.5.** *Consider the language $L_{bij} = \{U, E\}$ where $U$ is a unary predicate and $E$ is a binary relation. Let $T_{bij}$ be the theory which says:*

- *$U$ and $\neg U$ are infinite.*
- *$E(x, y)$ implies $U(x)$ and $\neg U(y)$*
- *For all $x, x', y, y'$ we have:*
  - *$E(x, y) \wedge E(x, y') \to y = y'$.*
  - *$E(x, y) \wedge E(x', y) \to x = x'$.*

*In any model of $T_{bij}$, $E$ is the graph of a bijection between $U$ and $\neg U$. Hence $Spec_{T_{bij}}(\langle U, \neg U\rangle) = \{(\gamma, \gamma) : \omega \leq \gamma\}$. Further $U(x)$ and $\neg U(x)$ are equivalent to complete quantifier free types over $T$. It is also immediate that the unique countable model of $T_{bij}$ is ultrahomogeneous. However $T_{bij}$ does not satisfy strong amalgamation.*

## 4. Complete Characterizations

In this section we mention a couple of situations where we can completely determine the spectra of any two formula.

### 4.1. Unary Languages.

The $n$-cardinal spectra of unary languages is particularly easy to describe.

**Proposition 4.1.** *If $L$ has only unary relations and constants then $Spec_T(K_1(T)) = \{\langle \lambda_1, \ldots, \lambda_{|K_1(T)|}\rangle : (\forall i)\ \omega \leq \lambda_i\}$.*

*Proof.* Because $L$ has only unary relations and constants every complete quantifier free type $q(x_1, \ldots, x_n) \in K_n(T)$ is equivalent to $\bigwedge_{i \leq n} q_i(x_i)$ for complete quantifier free types $q_i \in K_1(T)$. Hence any $L$-structure $\mathcal{M}$ which satisfies $(\forall x) \bigvee_{q \in K_1(T)} q(x)$ and $\bigwedge_{q \in K_1(T)} (\exists^\infty x) q(x)$ is a model of $T$.[5] $\square$

### 4.2. Restrictions.

**Definition 4.2.** *We say a formula $\varphi(x)$* **restricts** *a formula $\psi(x)$ if there is an $m$ such that*

$$T \vdash (\forall \overline{y}_0, \overline{y}_1)\big[(\overline{y}_0 \neq \overline{y}_1) \wedge \psi[\overline{y}_0, \overline{y}_1]\big] \to \left[(\exists \overline{z})\varphi[\overline{z}] \wedge \bigvee_{p \neq q \in K(T)} p(\overline{y}_0, \overline{z}) \wedge q(\overline{y}_1, \overline{z})\right].$$

[5] $(\exists^\infty x)$ is shorthand for "there exist infinitely many $x$".

*where $\overline{y}_0, \overline{y}_1$ are tuples of arity $m$ and we allow $\overline{z}$ to range over all finite tuples.*[6]

**Lemma 4.3.** *If $\varphi(x)$ restricts $\psi(x)$ then for all models $\mathcal{M} \models T$*

$$|\{a : \mathcal{M} \models \psi(a)\}| \leq \left|2^{\{a:\mathcal{M}\models\varphi(a)\}}\right|.$$

*Proof.* For each $\overline{y} \in \mathcal{M}$ of arity $m$ such that $\mathcal{M} \models \psi[\overline{y}]$ let $f_{\overline{y}} : \{x : \mathcal{M} \models \varphi(x)\}^{<\omega} \to K(T)$ be such that $f_{\overline{y}}(\overline{x})$ is the quantifier free type of $(\overline{y}, \overline{x})$. Then the condition that $\varphi(x)$ restricts $\psi(x)$ is exactly the statement that whenever $\overline{y}_0$ and $\overline{y}_1$ are distinct tuples all of whose elements satisfy $\psi(x)$, $f_{\overline{y}_0} \neq f_{\overline{y}_1}$. But there are at most $|2^{\{a:\mathcal{M}\models\varphi(a)\}}|$ such functions, hence there can be at most $|2^{\{a:\mathcal{M}\models\varphi(a)\}}|$ many $m$-tuples all of whose elements satisfy $\psi(x)$. □

In particular, if every pair of formulas each restricts the other Lemma 4.3 allows us to completely characterize the $n$-cardinal spectra of a theory.

**Corollary 4.4.** *Suppose $q_0(x)$ restricts $q_1(x)$ for all $q_0(x), q_1(x) \in K_1(T)$. Then $Spec_T(K_1(T)) = C(|K_1(T)|)$.*

*Proof.* This follows immediately from Theorem 3.2 and Lemma 4.3. □

**Corollary 4.5.** *Suppose $\varphi_0(x)$ restricts $\varphi_1(x)$ and $\varphi_1(x)$ restricts $\varphi_0(x)$. Then there are three possibilities for $Spec_T(\langle\varphi_0, \varphi_1\rangle)$.*

- $\underline{T \vdash (\forall x)\varphi_0(x) \to \varphi_1(x)}$: *$Spec_T(\langle\varphi_0, \varphi_1\rangle) = B(1)$.*
- $\underline{T \vdash (forall x)\varphi_1(x) \to \varphi_0(x)}$: *$Spec_T(\langle\varphi_0, \varphi_1\rangle) = B^*(1)$.*
- *Otherwise*: *$Spec_T(\langle\varphi_0, \varphi_1\rangle) = B(1) \cup B^*(1)$.*

The notion of restriction allows us to completely characterize the $n$-cardinal spectra of several important theories.

**Example 4.6.** *Let $\mathcal{M}^*$ be any one of the following structures with a finite number of elements named:*

- *The generic countable $n$-hypergraph.*
- *The generic countable directed graph.*
- *The generic countable $n$-colored graph.*
- *The generic countable partial order.*

[6] While this can't be expressed in a first order way, it can be expressed in $\mathcal{L}_{\omega_1,\omega}(L)$ with a single disjunction over the arity of $\overline{z}$.

*Let $T^*$ be the theory of $\mathcal{M}^*$. It is then immediate that $T^*$ is a strong Fraïssé theory and any quantifier free formulas $\varphi(x)$ restricts $(x = x)$. Hence for any two $q_0(x), q_1(x) \in K_1(T)$ restrict each other. Therefore by Corollary 4.5 Spec$_{T^*}(K_1(T)) = C(|K_1(T)|)$.*

4.3. **Examples With Larger Spectra.** Up until this point we have only seen cardinal spectra where all cardinals in the sequence were bound between $\gamma$ and $|2^\gamma|$ for some $\gamma$. In this section we give an example of a strong Fraïssé theory where this isn't the case. Our strong Fraïssé theories, $Bp_n$ for $n \geq 1$, will be constructed from the theory of the generic bipartite graph. These theories will be such that there is a unary relation $R_n$ where $\text{Spec}_{Bp_n}(\langle R_n, x = x\rangle) = B(n)$.

Let $L(P,Q) = \{P,Q,E\}$ where $P,Q$ are unary relations and $E$ is a binary relation. Let $BG(P,Q)$ be the (incomplete) theory which says that $\{x : P(x) \vee Q(x)\}$ is a generic bipartite graph with edge relation $E$ and partitions $\{x : P(x)\}$ and $\{x : Q(x)\}$.

**Example 4.7.** *Let $L^{Bp}_n = \{R_0, \dots, R_n, E\}$ where each $R_i$ is a unary relation and $E$ is a binary relation. Let $Bp_n$ be the theory which contains:*

- $(\forall x) \bigvee_{0 \leq i \leq n} R_i(x)$.
- $(\forall x) \bigwedge_{0 \leq i < j \leq n} \neg R_i(x) \wedge R_j(x)$.
- *For all $0 \leq i < n$, $BG(R_i, R_{i+1})$.*
- $(\forall x, y) E(x,y) \to \bigvee_{0 \leq i < n} [R_i(x) \wedge R_{i+1}(y)] \vee [R_i(y) \wedge R_{i+1}(x)]$.

*It is then immediate that $Bp_n$ is a strong Fraïssé theory (where a model consists of $n$ (linked) copies of a generic bipartite graph). Also over $Bp_n$ every complete quantifier free type of one variable is equivalent to a formula $R_i(x)$ for some $0 \leq i \leq n$. As such we will simply refer to the elements of $K_1(Bp_n)$ by the relation they are equivalent to.*

*Let $D = \{\langle \lambda_0, \dots, \lambda_n\rangle : (\forall 1 \leq i \leq n) \lambda_{i-1} \leq |2^{\lambda_i}|$ and $\lambda_i \leq |2^{\lambda_{i-1}}|\}$.*

**Lemma 4.8.** *Spec$_{Bp_n}(\langle R_0, \dots R_n\rangle) = D$.*

*Proof.* First notice that for each $i$, $R_{i+1}(x)$ restricts $R_i(x)$ and $R_i(x)$ restricts $R_{i+1}(x)$, hence by repeated use of Lemma 4.3, $\text{Spec}_{Bp_n}(\langle R_0, \dots R_n\rangle) \subseteq D$.

Let $\langle \lambda_0, \dots, \lambda_n\rangle \in D$ and $X_0, \dots, X_n$ be disjoint sets where $|X_i| = \lambda_i$ for $0 \leq i \leq n$. By Corollary 4.5 we can construct a model $M_{\gamma,i}$ of $BG(R_i, R_{i+1}) \cup \{(\forall x) R_i(x) \vee R_{i+1}(x)\}$ where $\{x : M_{\gamma,i} \models R_i(x)\} = X_i$ and $\{x : M_{\gamma,i} \models$

$R_{i+1}(x)\} = X_{i+1}$. But as relation in $L(R_i, R_{i+1}) \cap L(R_{i+1}, R_{i+2}) = \{R_{i+1}, E\}$, the union of the $\mathcal{M}_{\gamma,i}$ gives us an $L_n^{Bp}$-structure $\mathcal{M}^*$ whose underlying set is $\bigcup_{i\leq n} X_i$. However it is immediate from the construction that for each $i$, $\mathcal{M}^*|_{L(R_i,R_{i+1})}$ satisfies $BG(R_i, R_{i+1})$ and so $\mathcal{M}^*$ satisfies $Bp_n$. $\square$

**Corollary 4.9.** *$Spec_{Bp_n}(R_n, x = x) = B(n)$.*

## 5. Conjectures

We will end with two related conjectures. First recall by "Vaught's theorem on cardinals far apart" ([2] Theorem 12.1.4) that if $\bigcup_{n\in\omega} B(n) \subseteq \mathrm{Spec}_T(\varphi_0, \varphi_1)$ then $\{(\gamma, \lambda) : \omega \leq \gamma \leq \lambda\} \subseteq \mathrm{Spec}_T(\varphi_0, \varphi_1)$.

**Conjecture 5.1.** *Suppose $T \vdash (\forall x)\varphi_0(x) \to \varphi_1(x)$. Then one of the following holds:*

1. *$Spec_T(\varphi_0, \varphi_1) = \{(\gamma, \lambda) : \omega \leq \gamma \leq \lambda\}$.*
2. *$Spec_T(\varphi_0, \varphi_1) = B(n)$ for some $n$.*

**Conjecture 5.2.** *Suppose $T \not\vdash (\forall x)\varphi_0(x) \to \varphi_1(x)$ and $T \not\vdash (\forall x)\varphi_1(x) \to \varphi_0(x)$. Then one of the following holds:*

1. *$Spec_T(\varphi_0, \varphi_1) = \{(\gamma, \lambda) : \omega \leq \gamma, \omega \leq \lambda\}$.*
2. *$Spec_T(\varphi_0, \varphi_1) = B(n) \cup B^*(n)$ for some $n$.*

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