A FIXED POINT THEOREM FOR CONTRACTING MAPS OF SYMMETRIC CONTINUITY SPACES

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Abstract. We prove a generalization of the Banach fixed point theorem for symmetric separated $\mathbb{V}$-continuity spaces. We also give examples to show that in general we cannot weaken our assumptions.

1. Introduction

One of the most important fixed point theorems to arise from the study of metric spaces is the Banach fixed point theorem. This theorem can be stated as follows:

Theorem 1.1 (Banach Fixed Point Theorem)**.** *If (M, d_M) is a Cauchy complete metric space, $m < n \in \mathbb{N}$ and $f : M \to M$ is a function such that*

$$(\forall x, y \in M)\ n \cdot d_M(f(x), f(y)) \leq m \cdot d_M(x, y)$$

then f has a unique fixed point.

It is natural to ask, "On what spaces, other than Cauchy complete metric spaces, and for which classes of maps, can such a fixed point theorem be proved?" In [3] it was shown that for every topological space $(T, \mathcal{O}(T))$ there is a quantale $\mathbb{V}$ and $\mathbb{V}$-continuity space (T, d_T) such that $(T, \mathcal{O}(T))$ is homeomorphic to the topological space of open balls of (T, d_T). This allows us to refine the above question to "For what quantale's $\mathbb{V}$, what $\mathbb{V}$-continuity spaces (M, d_M) and what class of non-expanding maps does a version of the Banach fixed point theorem hold?"

In [5] Priess-Crampe and Ribenboim give a partial answer.[1] They consider the case of generalized ultrametric spaces which are the same as $\mathbb{V}$-continuity spaces when the binary operation on the quantale $\mathbb{V}$ is

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[1]It is worth mentioning that in [2] the authors consider, among other things, the related question of how to generalize the Tarski fixed point theorem to $\mathbb{V}$-continuity spaces

disjunction (for more on generalized ultrametric spaces see [6]). In our terminology they show:

Theorem 1.2 (Priess-Crampe/Ribenboim Fixed Point Theorem)**.** *If* $\langle \Gamma, \leq, \vee \rangle$ *is a quantale,* (M, d_M) *is a separated symmetric spherically complete* $\langle \Gamma, \leq, \vee \rangle$*-continuity space and* $f : M \to M$ *is a function such that*

$$(\forall x, y \in M)\ x \neq y \to d_M(f(x), f(y)) < d_M(x, y)$$

then f *has a unique fixed point.*

The goal of this paper is to extend the work of Priess-Crampe and Ribenboim by proving a generalization of the Banach fixed point theorem for separated symmetric spherically complete $\mathbb{V}$-continuity spaces.

Before we state our main theorem, it is worth observing that in the Banach fixed point theorem, in order to get the existence of a fixed point we do not need the contracting condition for all pairs of points $x, y \in M$ but only those pairs of the form $x, f(x) \in M$. It is only to get the uniqueness of the fixed point that we need the distance is contracting between all pairs of points.

Similarly for the Priess-Crampe/Ribenboim Fixed Point Theorem, as was shown in [7], to get the existance of a fixed point we only need the contracting condition for pairs of points of the form $x, f(x) \in M$.

We are now ready to state, without specific terminology, our main result:

Theorem 1.3 (Main Theorem)**.** *Suppose* $\mathbb{V}$ *is a quantale,* (M, d_M) *is a separated symmetric spherically complete* $\mathbb{V}$*-continuity space and* $f : M \to M$ *is a function such that for some* $m < n \in \mathbb{N}$

- $(\forall x, y \in M)\ d_M(f(x), f(y)) \leq d_M(x, y)$
- $(\forall x \in M)\ x \neq f(x) \to d_M(f(x), f^2(x)) < d_M(x, f(x))$
- $(\forall x \in M)\ n \cdot d_M(f(x), f^2(x)) \leq m \cdot d_M(x, f(x))$

If either

$$m = 1.$$

or

$$(\exists k \in \mathbb{N}, k \geq 2)(\forall x, y \in V)\ x < y \to k \cdot x < k \cdot y.$$

then f *has a fixed point. Further if* f *satisfies*

- $(\forall x, y \in M)\ x \neq y \to d_M(f(x), f(y)) < d_M(x, y)$

then the fixed point is unique.

Note that this is a generalization of the Banach fixed point theorem because for metric spaces $n \cdot d_M(f(x), f(y)) \leq m \cdot d_M(x, y)$ implies $d_M(f(x), f(y)) < d_M(x, y)$ if $m < n$ and $(\forall r, s \in \mathbb{R}^{\geq 0} \cup \{\infty\})\ r <$

$s \to 2 \cdot r < 2 \cdot s$. Also note this is a generalization of the Priess-Crampe/Ribenboim fixed point theorem because when $\langle \Gamma, \leq, \vee \rangle$ is a quantale then $(\forall n \in \mathbb{N})(\forall \gamma \in \Gamma)\ n \cdot \gamma = \gamma$.

We end the paper with several examples showing that, in general, the conditions of our main theorem can not be weakened.

Throughout this paper we will be working in a background model of Zermelo-Frankel Set Theory with the Axiom of Choice.

2. Continuity Spaces

The following definitions relating to continuity spaces will be very similar to those in [3].

Definition 2.1. *A (commutative unitial)* **quantale** $\mathbb{V} = \langle V, \leq, + \rangle$ *consists of a complete lattice* $\langle V, \leq \rangle$ *along with an associative and commutative binary operation* $+$ *such that:*

(Q1) *The bottom element of* V, 0, *is a unit of* $+$, *i.e.* $(\forall p \in V)\ p + 0 = p$.

(Q2) *For all* $p \in V$ *and* $Q \subseteq V$, $p + \bigwedge_{q \in Q} q = \bigwedge_{q \in Q}(p + q)$.

We will also abuse notation when $n \in \mathbb{N}$ and $v \in V$ and use $n \cdot v$ for the nth fold sum of v. With the convention that $0 \cdot v = 0$. In what follows $\mathbb{V}$ will always be a quantale.

Definition 2.2. *For* $k \in \mathbb{N}$, *if* $\mathbb{V}$ *satisfies*

$$(\forall x, y \in V)\ x < y \to k \cdot x < k \cdot y.$$

we say $\mathbb{V}$ **respects multiplication by** k

It is worth noting that the term "quantale" is also often used to refer to a one object quantaloid (see [8]). The relation with the above notion is that given any quantale (as Definition 2.1) if you reverse the ordering you get the set of morphisms of a one element quantaloid. Similarly if you have a set of morphisms from a one element quanaloid whose identity is also its greatest element and whose composition is commutative then by reversing the order you get a quantale (as Definition 2.1). We have chosen the above notion of a quanatle because it will make the corresponding notion of a continuity space parallel that of a metric space. However, everything that follows could also be carried out using a one element quantaloid by reversing the ordering. In that case however the notion of a continuity space would parallel that of a category enriched in a (one object) quantaloid (see [9] for more information on such enriched categories).

The following lemma follows immediately from (Q2) and will be important in what follows.

Lemma 2.3. *For all $p, q, r, s \in V$ with $p \leq q$ and $r \leq s$ we have $p + r \leq q + s$.*

We now give the definition of a $\mathbb{V}$-continuity space.

Definition 2.4. *A* **$\mathbb{V}$-continuity space** *is a pair $\mathcal{M} = (M, d_M)$ where M is a set and $d_M : M \times M \to V$ satisfies:*

(Reflexivity) $(\forall x \in M)\ d_M(x, x) = 0$.
(Transitivity) $(\forall x, y, z \in M)\ d_M(x, z) \leq d_M(x, y) + d_M(y, z)$.

We say $\mathcal{M}$ is **symmetric** *if*

- $(\forall x, y \in M)\ d_M(x, y) = d_M(y, x)$.

and we say $\mathcal{M}$ is **separated** *if*

- $(\forall x, y \in M)\ d_M(x, y) \wedge d_M(y, x) = 0 \to x = y$.

In what follows, unless we state otherwise, $\mathcal{M} = (M, d_M)$ and its variants will always be separated symmetric $\mathbb{V}$-continuity spaces. We will omit subscripts when they are clear from context.

In particular $\mathcal{M}$ is a metric space if and only if and $\mathcal{M}$ is a separated symmetric $\langle \mathbb{R}^{\geq 0} \cup \{\infty\}, \leq, +\rangle$-continuity space. We also have that $\mathcal{M}$ is an ultrametric space if and only if $\mathcal{M}$ is a separated symmetric $\langle \mathbb{R}^{\geq 0} \cup \{\infty\}, \leq, \vee\rangle$-continuity space. As such the notion of separated symmetric $\mathbb{V}$-continuity space generalizes the notion of metric and ultrametric spaces.

Definition 2.5. *If $x \in M$ and $\gamma \in V$ then the* **closed ball** *around x of radius γ is the set $B_{\mathcal{M}}(x, \gamma) := \{y \in M : d(x, y) \vee d_M(y, x) \leq \gamma\}$.*

We will omit the superscripts when the $\mathbb{V}$-continuity space is clear from the context.

Definition 2.6. *$\mathcal{M}$ is* **spherically complete** *if, for all infinite cardinals κ, whenever*

- $\{\gamma_i : i < \kappa\} \subseteq V$ *and* $\{x_i : i < \kappa\} \subseteq M$.
- $B(x_i, \gamma_i) \subseteq B(x_j, \gamma_j)$ *if* $i \geq j$.

then $\bigcap_{i<\kappa} B(x_i, \gamma_i) \neq \emptyset$.

A spherically complete $\mathbb{V}$-continuity space is one where, whenever we shrink the radius of closed balls in a consistent way we always have some point in the intersection. This is different from, but related to the notion of Cauchy completeness which only requires there to be an element when the radius of the balls shrinks to 0. In the general case these notions may be distinct (i.e. neither implies the other). For a discussion of the connections when $\mathbb{V} = \langle \Gamma, \leq, \vee\rangle$ see [1].

Definition 2.7. *Suppose $f : M \to M$. We say f is* **contracting** *if*

$$(\forall x_0, x_1 \in M)\ d_M(x_0, x_1) > 0 \to d_M(f(x_0), f(x_1)) < d_M(x_0, x_1).$$

We say f is **contracting on orbits** *if*

$$(\forall x \in M)\ d_M(x, f(x)) > 0 \to d_M(f(x), f^2(x)) < d_M(x, f(x)).$$

Lemma 2.8. *Suppose M is a $\mathbb{V}$-continuity space and $f : M \to M$ is a contracting map. Then f has at most one fixed point.*

Proof. Suppose to get a contradiction that x, y are fixed points of f with $x \neq y$. Then either $d_M(x, y) > 0$ or $d_M(y, x) > 0$. Without loss of generality we can therefore assume $d_M(x, y) > 0$. Therefore, as f is contracting, we have $d_M(f(x), f(y)) < d_M(x, y)$.

However $d_M(x, y) \leq d_M(x, f(x)) + d_M(f(x), f(y)) + d_M(y, f(y)) = d_M(f(x), f(y))$ as $d_M(x, f(x)) = d(y, f(y)) = 0$. This is a contradiction and hence f has at most one fixed point. □

Lemma 2.9. *Suppose $f : M \to M$ is contracting on orbits and for some $k \in \mathbb{N}$, f^k has a fixed point x. Then for some $0 \leq i \leq k$, $f^i(x)$ is a fixed point of f.*

Proof. Assume to get a contradiction that $f^i(x) \neq f^{i+1}(x)$ for all $i \leq k$. Because f is contracting we have $d(x, f(x)) > d(f(x), f^2(x)) > \cdots > d(f^k(x), f^{k+1}(x))$. But by assumption x is a fixed point of f^k and hence $f^k(x) = x$ and $f^{k+1}(x) = f(x)$. So $d(f^k(x), f^{k+1}(x)) = d(x, f(x))$ giving our contradiction.

We therefore have for some $i \leq k$, $f^i(x) = f^{i+1}(x)$ and $f^i(x)$ is a fixed point of f. □

Definition 2.10. *Suppose $f : M \to M$ and $m, n \in \mathbb{N}$. We say f is (m, n)* **non-expanding** *if*

$$(\forall x_0, x_1 \in M)\ n \cdot d_M(f(x_0), f(x_1)) \leq m \cdot d_M(x_0, x_1).$$

We say f is (m, n) **non-expanding on orbits** *if*

$$(\forall x \in M)\ n \cdot d_M(f(x), f^2(x)) \leq m \cdot d_M(x, f(x))$$

We say a map $f : M \to M$ is **non-expanding** if it is $(1, 1)$ non-expanding, i.e. if $(\forall x, y)\ d_M(f(x), f(y)) \leq d_M(x, y)$.

Note, by Lemma 2.3, if $n \geq n'$ and $m' \geq m$ then any (m, n) non-expanding map is also a (m', n') non-expanding map.

In the case of metric spaces a map $f : M \to M$ is (m, n) non-expanding if and only if $(\forall x, y \in M)\ \frac{m}{n} d_M(x, y) \geq d_N(f(x), f(y))$. So in the case of metric spaces f is C-Lipschitz for some $C < 1$ if and only if there are some $m < n \in \mathbb{N}$ for which f is (m, n) non-expanding.

Definition 2.11. *When* $\mathcal{M} = \langle M, d_M \rangle$ *is a* $\mathbb{V}$*-continuity space and* $k \in \mathbb{N}$ *we define* $\mathcal{M}_k = \langle M, d_M^k \rangle$ *where* $d_M^k(x, y) = k \cdot d_M(x, y)$.

The following lemma is then immediate.

Lemma 2.12. *For each* $k \in \mathbb{N}$ *the following holds:*

- $\mathcal{M}_k$ *is a* $\mathbb{V}$*-continuity space.*
- $\mathcal{M}_k$ *is separated if and only if* $\mathcal{M}$ *separated.*
- $\mathcal{M}_k$ *is spherically complete if* $\mathcal{M}$ *is spherically complete.*
- $\mathcal{M}_k$ *is symmetric if* $\mathcal{M}$ *is symmetric.*
- *If* $f : \mathcal{M} \to \mathcal{M}$ *then* f *is* $(m \times k, n \times k)$ *non-expanding on orbits if and only if* $f : \mathcal{M}_k \to \mathcal{M}_k$ *is* (m, n) *non-expanding on orbits.*

If further $\mathbb{V}$ *respects multiplication by* k *and if* $f : \mathcal{M} \to \mathcal{N}$ *is contracting, then* $f : \mathcal{M}_k \to \mathcal{N}_k$ *is also contracting.*

Lemma 2.13. *If* $f : \mathcal{M} \to \mathcal{M}$ *is* (m, n) *non-expanding on orbits then* $f^k : \mathcal{M} \to \mathcal{M}$ *is* (m^k, n^k) *non-expanding.*

Proof. We know that $(\forall x \in M)\ n \cdot d(f(x), f^2(x)) \leq m \cdot d(x, f(x))$ so

$$n^k \cdot d(f^k(x), f^{k+1}(x)) \leq n^{k-1} \times m \cdot d(f^{k-1}(x), f^k(x)) \leq \cdots$$
$$\leq n^{k-j} \times m^j \cdot d(f^{k-j}(x), f^{k-j+1}(x)) \leq \cdots \leq m^k \cdot d(x, f(x))$$

□

3. Fixed Point Theorems

We begin by proving our fixed point theorem in the special case of non-expanding maps which are also contracting on orbits and $(1, 3)$ non-expanding on orbits. We will then use Lemma 2.3 and Lemma 2.9 to reduce the general case to the case of non-expanding maps which are also contracting or orbits and $(1, 3)$ non-expanding on orbits.

Proposition 3.1. *Suppose* $f : \mathcal{M} \to \mathcal{M}$ *is a map that is*

- *non-expanding*
- *contracting on orbits*
- $(1, 3)$ *non-expanding on orbits.*

Then there is an $x \in M$ *such that* $f(x) = x$ *(called a* **fixed point** *of* f*).*

Proof. Let $x_0 \in M$ be any element and let $\gamma_0 = d(x_0, f(x_0))$. We define x_α and γ_α by induction as follows.

- (Successor Stage) $x_{\alpha+1} = f(x_\alpha)$ and let $\gamma_\alpha = d(x_\alpha, f(x_\alpha))$.
- (Limit Stage) If $\bigcap_{i<\omega\cdot\beta} B(x_i, 3\cdot\gamma_i) \neq \emptyset$ let $x_{\omega\cdot\beta} \in \bigcap_{i<\omega\cdot\beta} B(x_i, 3\cdot\gamma_i)$ be any element. Otherwise let $x_{\omega\cdot\beta}$ be undefined.

Claim 3.2. *If $x_{\alpha+1}$ is defined then $3 \cdot \gamma_{\alpha+1} \leq \gamma_\alpha$.*

Proof. This is because $\gamma_{\alpha+1} = d(x_{\alpha+1}, f(x_{\alpha+1})) = d(f(x_\alpha), f(f(x_\alpha)))$, $\gamma_\alpha = d(x_\alpha, f(x_\alpha))$ and f is $(1,3)$ non-expanding on orbits. □

Claim 3.3. *If $\alpha > \beta$, x_α is defined and $\gamma_\alpha > 0$ then $\gamma_\alpha < \gamma_\beta$.*

Proof. Suppose $\alpha = \beta + \zeta$. We prove this by induction on ζ.

If ζ is a successor ordinal, i.e. $\zeta = \zeta' + 1$, then $\gamma_\beta \geq \gamma_{\beta+\zeta'}$ by the inductive hypothesis (equality coming when $\zeta' = 0$). We then have

$$\begin{aligned}\gamma_\alpha &= \gamma_{\beta+\zeta'+1} = d(x_{\beta+\zeta'+1}, f(x_{\beta+\zeta'+1})) = d(f(x_{\beta+\zeta'}), f(f(x_{\beta+\zeta'})))\\ &< d(x_{\beta+\zeta'}, f(x_{\beta+\zeta'})) = \gamma_{\beta+\zeta'} \leq \gamma_\beta.\end{aligned}$$

with the first inequality following from the fact that f is contracting.

If ζ is a limit then $x_{\beta+\zeta} \in \bigcap_{i<\beta+\zeta} B(x_i, 3 \cdot \gamma_i)$ and so

$$\begin{aligned}\gamma_\alpha &= d(x_{\beta+\zeta}, f(x_{\beta+\zeta}))\\ &\leq d(x_{\beta+\zeta}, x_{\beta+3}) + d(x_{\beta+3}, f(x_{\beta+3})) + d(f(x_{\beta+3}), f(x_{\beta+\zeta}))\\ &\leq d(x_{\beta+\zeta}, x_{\beta+3}) + d(x_{\beta+3}, f(x_{\beta+3})) + d(x_{\beta+3}, x_{\beta+\zeta})\\ &\leq 3 \cdot \gamma_{\beta+3} + \gamma_{\beta+3} + 3 \cdot \gamma_{\beta+3} = 7 \cdot \gamma_{\beta+3} \leq 9 \cdot \gamma_{\beta+3} \leq 3 \cdot \gamma_{\beta+2} \leq \gamma_{\beta+1} < \gamma_\beta\end{aligned}$$

with the first in equality following from transitivity in the definition of $\mathbb{V}$-continuity space, and the second inequality following from the fact that f is non-expanding. □

Claim 3.4. *For all ordinals $\alpha \geq \beta$, $\emptyset \neq B(x_\alpha, 3 \cdot \gamma_\alpha) \subseteq B(x_\beta, 3 \cdot \gamma_\beta)$.*

Proof. We fix β and let $\alpha = \beta + \zeta$. We will prove this by induction on ζ.

Base Case: When $\zeta = 0$ this is trivial.

Successor Case: $\zeta = \zeta' + 1$
Suppose $z \in B(x_{\beta+\zeta'+1}, 3 \cdot \gamma_{\beta+\zeta'+1})$. Then $d(x_{\beta+\zeta'}, z) \leq d(x_{\beta+\zeta'}, x_{\beta+\zeta'+1}) + d(x_{\beta+\zeta'+1}, z) \leq \gamma_{\beta+\zeta'} + 3 \cdot \gamma_{\beta+\zeta'+1} \leq 2 \cdot \gamma_{\beta+\zeta'} \leq 3 \cdot \gamma_{\beta+\zeta'}$. Hence $z \in B(x_{\beta+\zeta'}, 3 \cdot \gamma_{\beta+\zeta'})$ and so $B(x_{\beta+\zeta'+1}, 3 \cdot \gamma_{\beta+\zeta'+1}) \subseteq B(x_{\beta+\zeta'}, 3 \cdot \gamma_{\beta+\zeta'}) \subseteq B(x_\beta, 3 \cdot \gamma_\beta)$ as z was arbitrary.

Limit Case: ζ is a limit ordinal.
By the inductive hypothesis $\langle B(x_{\beta+i}, 3 \cdot \gamma_{\beta+i}) : i < \zeta \rangle$ is a decreasing sequence of non-empty balls. Hence, as $\mathcal{M}$ is spherically complete, $x_{\beta+\zeta}$ is defined and $x_{\beta+\zeta} \in \bigcap_{i<\zeta} B(x_{\beta+i}, 3 \cdot \gamma_{\beta+i})$. We then have two cases.

Case 1: $(\exists i < \zeta)\ \gamma_{\beta+i} = 0$.
In this case $d(x_{\beta+i}, f(x_{\beta+i})) = 0$ so $x_{\beta+i} = f(x_{\beta+i})$ and $\{x_{\beta+i}\} =$

$B(x_{\beta+i}, 3 \cdot \gamma_{\beta+i}) = B(x_{\beta+\zeta}, 3 \cdot \gamma_{\beta+\zeta})$.

Case 2: $(\forall i < \zeta)\ \gamma_{\beta+i} > 0$
Suppose $y \in B(x_{\beta+\zeta}, 3 \cdot \gamma_{\beta+\zeta})$ and $i < \zeta$. Then

$$\begin{aligned} d(x_{\beta+i}, y) &\leq d(x_{\beta+i}, x_{\beta+i+1}) + d(x_{\beta+i+1}, x_{\beta+\zeta}) + d(x_{\beta+\zeta}, y) \\ &\leq \gamma_{\beta+i} + 3 \cdot \gamma_{\beta+i+1} + 3 \cdot \gamma_{\beta+\zeta} \leq 3 \cdot \gamma_{\beta+i} \end{aligned}$$

Hence $y \in B(x_{\beta+i}, 3 \cdot \gamma_{\beta+i})$ and so $B(x_{\beta+\zeta}, 3 \cdot \gamma_{\beta+\zeta}) \subseteq B(x_{\beta+i}, \cdot \gamma_{\beta+i})$ as y was arbitrary. In particular this is true when $i = 0$. □

Claim 3.4 implies that x_α and γ_α are defined for all ordinals α. Therefore, by Claim 3.3, there is an α such that $\gamma_\alpha = 0$. Let α be the least such. Then $0 = \gamma_\alpha = d(x_\alpha, x_{\alpha+1}) = d(x_\alpha, f(x_\alpha))$. Hence $x_\alpha = f(x_\alpha)$, as $\mathcal{M}$ is symmetric and separated, and so f has a fixed point. □

Now that we have the fixed point theorem for contracting $(1, 3)$ non-expanding maps we want to use it to prove our main theorem.

Theorem 3.5. *Suppose* $f : \mathcal{M} \to \mathcal{M}$ *is*

- *non-expanding*
- *contracting on orbits*
- (m, n) *non-expanding on orbits where* $m < n \in \mathbb{N}$.

If either

(a) $m = 1$.
(b) $\mathbb{V}$ *respects multiplication by* k *for some* $k \in \mathbb{N}$, $k \geq 2$.

then there is an $x \in M$ *such that* $f(x) = x$ *(called a* **fixed point** *of* f*). If* f *is also contracting then* x *is the unique fixed point of* f.

Proof. First observe that the uniqueness of a fixed point when f is contracting, if a fixed point exists, follows from Lemma 2.8.

Case (a): By Lemma 2.3 and Proposition 3.1 it suffices to consider the case when f is $(1, 2)$ non-expanding on orbits. In this case, by Lemma 2.13, we have f^2 is contracting and $(1, 4)$ non-expanding on orbits and hence also $(1, 3)$ non-expanding on orbits. Therefore by Proposition 3.1 f^2 has a fixed point and hence by Lemma 2.9 f has a fixed point.

Case (b): As $m < n$ there is some i such that $3 \cdot m^{k^i} < n^{k^i}$. By Lemma 2.3 and Lemma 2.13 $f^{k^i} : \mathcal{M}_{m^{k^i}} \to \mathcal{M}_{m^{k^i}}$ is $(1, 3)$ non-expanding on orbits. Further, by an iterated use of Lemma 2.12 and the fact that $\mathbb{V}$ respects multiplication by k, we have that f^{k^i} is contracting on orbits. But then by Proposition 3.1 there is an $x \in M$ such that $f^{k^i}(x) = x$ and hence by Lemma 2.9 f has a fixed point. □

4. Counterexamples

In this section we give examples showing that, in general, the assumptions of Theorem 3.5 cannot be weakened.

We now show we can't replace the assumption of spherical completeness with Cauchy completeness. The notion of Cauchy completeness makes sense for any quantale $\mathbb{V}$, however in the example below our quantale will be $\langle \mathbb{R}, \leq, \vee \rangle$ and so the notion of Cauchy completeness coincides with that for metric spaces (see [4] for more on the general notion of Cauchy completeness and [1] for more on the connection between Cauchy completeness and spherical completeness).

Example 4.1. *Notice that $\langle \mathbb{R}^{\geq 0} \cup \{\infty\}, \leq, \vee \rangle$ respects multiplication by k for all $k \in \mathbb{N}$. We will define a $\langle \mathbb{R}^{\geq 0} \cup \{\infty\}, \leq, \vee \rangle$-continuity space $(2^{\omega+\omega}, d_{\omega+\omega})$. For $x, y \in 2^{\omega+\omega}$ let $m(x, y) = sup\{\alpha : (\forall \beta < \alpha)\ x(\beta) = y(\beta)\} \in \omega + \omega + 1$. We now define $d_{\omega+\omega} : 2^{\omega+\omega} \times 2^{\omega+\omega} \to \Gamma$ as follows:*

- *If $m(x, y) = \omega + \omega$ then $d_{\omega+\omega}(x, y) = 0$.*
- *If $m(x, y) = \omega + n$ then $d_{\omega+\omega}(x, y) = \frac{1}{n+1}$.*
- *If $m(x, y) = n < \omega$ then $d_{\omega+\omega}(x, y) = 1 + \frac{1}{n+1}$.*

It is easily checked that $(2^{\omega+\omega}, d_{\omega+\omega})$, as a $\langle \mathbb{R}^{\geq 0} \cup \{\infty\}, \leq, \vee \rangle$-continuity space, is separated, symmetric and both Cauchy complete and spherically complete. Note that this construction is a direct analog of the standard manner in which 2^{ω} is turned into a $\langle \mathbb{R}^{\geq 0} \cup \{\infty\}, \leq, \vee \rangle$-continuity space.

Now let $E = \{x \in 2^{\omega+\omega} : (\exists n \in \omega)\ x(n) = 0\} \subseteq 2^{\omega+\omega}$. For any Cauchy sequence $\langle x_i : i \in \mathbb{N} \rangle \subseteq E$ there is an n such that $(\forall n', n'' > n)$ $d_{\omega+\omega}(x_{n'}, x_{n''}) < \frac{1}{2}$. We therefore have $(\forall n' > n)(\forall m \in \omega) x_n(m) = x_{n'}(m)$. Hence, if $\langle x_i : i \in \mathbb{N} \rangle$ converges to x in $2^{\omega+\omega}$, $x \in E$ and so E is Cauchy complete.

Let $f : 2^{\omega+\omega} \to 2^{\omega+\omega}$ be the function where $f(x)(0) = f(x)(\omega) = 1$ and $f(x)(\alpha + 1) = x(\alpha)$. It is then immediate that $d_{\omega+\omega}(f(x), f(y)) < d_{\omega+\omega}(x, y)$ and so f is contracting. Further, as $(\forall x \in \mathbb{R})\ x \vee x = x$ we have f is (m, n) non-expanding for all $m, n \in \mathbb{N}$. So, by Theorem 3.5, f has a unique fixed point. If $c_1 : \omega + \omega \to 2$ is the constant function 1 then it is easily checked that $f(c_1) = c_1$ and hence that c_1 is the unique fixed point.

However, we also have if $x \in E$ then $f(x) \in E$ and so $f|_E : E \to E$ is a contracting (m, n) non-expanding map for all $m, n \in \mathbb{N}$. However $f|_E$ does not have a fixed point as $c_1 \notin E$.

We now show that if we only require our map $f : \mathcal{M} \to \mathcal{M}$ to be contracting then f may not have a fixed point.

Example 4.2. *Notice that* $\langle \mathbb{R}^{\geq 0} \cup \{\infty\}, \leq, +\rangle$ *respects multiplication by* k *for any* $k \in \mathbb{N}$*. Let* $s_n = \sum_{i=1}^{n} \frac{1}{i}$ *and let* $M_H = \{s_n : n \in \mathbb{N}\}$*. Also let* $\mathcal{M}_H = (M_H, d_{\mathbb{R}})$ *where* $d_{\mathbb{R}}(x, y) = |x - y|$*. As* $M_H \subseteq \mathbb{R}$*,* $\mathcal{M}_H$ *is separated and symmetric. Further, as every closed ball is either the whole space or contains at most a finite number of elements of* M_H*,* $\mathcal{M}_H$ *is spherically complete.*

Let $f_H : M_H \to M_H$ *be the function where* $f_H(s_n) = s_{n+1}$*. If* $m < n$ *then* $d_{\mathbb{R}}(f(s_m), f(s_n)) = \sum_{i=m+2}^{n+1} \frac{1}{i} < \sum_{i=m+1}^{n} \frac{1}{i} = d_{\mathbb{R}}(s_m, s_n)$ *and so* f_H *is contracting. However it is immediate from the definition of* f_H *that* f_H *has no fixed points.*

Next we now show that if we only require our map to be (m, n) non-expanding we may not have a fixed point.

Example 4.3. *Let* $d_{\mathbb{N}}(x, y) = 0$ *if* $x = y$ *and* $d_{\mathbb{N}}(x, y) = 1$ *if* $x \neq y$*. Then* $(\mathbb{N}, d_{\mathbb{N}})$ *as a* $\langle \mathbb{R}^{\geq 0} \cup \{\infty\}, \leq, \vee\rangle$*-continuity space which is spherically complete and such that any map* $f : \mathbb{N} \to \mathbb{N}$ *is* (m, n) *non-expanding for every* $m, n \in \mathbb{N}$*. In particular the map* $f(x) = x + 1$ *is* (m, n) *non-expanding and has no fixed point.*

Next we show that if $m \geq 2$ then we cannot, in general, remove the assumption that our quantale $\mathbb{V}$ respects multiplication by k for some $k \in \mathbb{N}$.

Example 4.4. *Define the quantale* $\mathbb{V}' = \langle V', \leq, +'\rangle$ *where:*

- $V' = \mathbb{R}^{\geq 0} \cup \{\infty\}$ *and* $\leq$ *is the normal ordering on* $\mathbb{R}^{\geq 0} \cup \{\infty\}$*.*
- $(\forall x, y \in V')$ $x = 0$ *or* $y = 0$ *or* $x +' y = \infty$*.*

If $\mathcal{M}$ *is any symmetric separated* $\mathbb{V}$*-continuity space and* $f : \mathcal{M} \to \mathcal{M}$ *is any map then* f *is* (m, n) *non-expanding for all* $2 \leq m$ *and* $2 \leq n$*. In particular* $f : \mathcal{M}_H \to \mathcal{M}_H$ *from Example 4.2 is contracting and* (m, n) *non-expanding, but doesn't have a fixed point.*

4.1. **Non-Symmetric $\mathbb{V}$-Continuity Spaces.** In this subsection we drop the assumption that (M, d_M) need be symmetric. Without the assumption of symmetry Theorem 3.5 might not hold.

Example 4.5. *Let* $d^*_{\mathbb{N}}(x, y) = 0$ *if* $x \leq y$ *and* $d^*_{\mathbb{N}}(x, y) = 1$ *if* $x > y$*. Then* $(\mathbb{N}, d^*_{\mathbb{N}})$ *is a* $\langle \mathbb{R}^{\geq 0} \cup \{\infty\}, \leq, \vee\rangle$*-continuity space which is spherically complete and such that any non-decreasing map* $f : \mathbb{N} \to \mathbb{N}$ *is* (m, n) *non-expanding for every* $m, n \in \mathbb{N}$ *and contracting on orbits. In particular the map* $f(x) = x+1$ *is such a map which has no fixed point.*

The reason why Theorem 3.5 fails for non-symmetric continuity spaces is that the notion of contracting on orbits and the notion of (m, n) non-expanding on orbits only deals with the distance between x and $f(x)$

and says nothing about the distance between $f(x)$ and x. However, it turns out that this is the only obstacle to Theorem 3.5 holding.

Definition 4.6. *For all $x, y \in M$ let $d_M^\circ(x,y) := d_M(y,x)$ and let $d_M^\vee(x,y) := d_M(x,y) \vee d_M(y,x)$.*

The following lemma is immediate.

Lemma 4.7. *We have*

- *(M, d_M) is a separated $\mathbb{V}$-continuity space if and only if (M, d_M°) is a separated $\mathbb{V}$-continuity space if and only if $(M, d_M^\vee)$ is a separated $\mathbb{V}$-continuity space.*
- *(M, d_M) is spherically complete if and only if (M, d_M°) is spherically complete if and only if $(M, d_M^\vee)$ is spherically complete.*
- *$(M, d_M^\vee)$ is symmetric. Further $(M, d_M) = (M, d_M^\vee)$ if and only if (M, d_M) is symmetric.*

We then have the following non-symmetric version of Theorem 3.5

Theorem 4.8. *Suppose $f : M \to M$ is*

- *non-expanding,*
- *contracting on orbits for both (M, d_M) and (M, d_M°),*
- *(m, n) non-expanding on orbits for both (M, d_M) and (M, d_M°) where $m < n \in \mathbb{N}$.*

If either

(a) *$m = 1$.*
(b) *$\mathbb{V}$ respects multiplication by k for some $k \in \mathbb{N}$, $k \geq 2$.*

then there is an $x \in M$ such that $f(x) = x$ (called a **fixed point** *of f). If f is also contracting then x is the unique fixed point of f.*

Proof. Our assumptions ensure that f, considered as a map from $(M, d_M^\vee)$ to $(M, d_M^\vee)$, is non-expanding, contracting on orbits, and (m, n) non-expanding. Further by Lemma 4.7 we know that $(M, d_M^\vee)$ is a spherically complete symmetric separated $\mathbb{V}$-continuity space. Hence the result follows from Theorem 3.5.

□

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