A CHARACTERIZATION OF QUASITRIVIAL n-SEMIGROUPS

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Abstract. We give a complete characterization of those n-semigroups which are quasitrivial. In particular we show that if n is even then every quasitrivial n-semigroup is derived from a (2−)semigroup and if n is odd then every quasitrivial n-semigroup is derived from a 3-semigroup. We also show that contained in every quasitrivial 3-semigroup there is a (unique) maximal sub 3-semigroup derived from a (2−)semigroup and which contains all but two elements.

We also completely characterize those quasitrivial n-semigroups which are ultrahomogeneous, those which are commutative as well as showing for every n that there is a countable quasitrivial n-semigroup which is universal for countable quasitrivial n-semigroups.

1. Introduction

1.1. Preliminaries.

For $\circ_{\mathbb{A}} : A^n \to A$ and for $a_1, \dots, a_{2n-1} \in A$ let

$$\circ^*_A(a_1, \dots, a_{2n-1}) = \circ_{\mathbb{A}}(\circ_{\mathbb{A}}(a_1, \dots, a_n), a_{n+1}, \dots, a_{2n-1}).$$

We say $\circ_{\mathbb{A}}$ is **associative** if for all $a_1, \dots, a_{2n-1} \in A$ and all $1 \le i \le n$

$$\circ^*_{\mathbb{A}}(a_1, \dots, a_{2n+1}) = \circ_{\mathbb{A}}(a_1, \dots, a_{i-1}, \circ_{\mathbb{A}}(a_i, \dots, a_{i+n-1}), a_{i+n}, \dots, a_{2n-1}).$$

The pair $(A, \circ_{\mathbb{A}})$, where $\circ_{\mathbb{A}}$ is associative, is called an **n-ary semigroup** or an **n-semigroup**.

For any operation $\circ_{\mathbb{A}} : A^n \to A$ we say $(A, \circ_{\mathbb{A}})$ is **quasitrivial** if for all $B \subseteq A$, $\{f(b_1, \dots, b_n) : b_1, \dots, b_n \in B\} \subseteq B$, i.e. if every subset is a substructure.

For the rest of this paper $\mathbb{A} := (A, \circ_{\mathbb{A}})$ and its variants will be quasitrivial n-semigroups. We will use a, b, c and their variants for elements of the underlying set of a quasitrivial n-semigroup, and x, y and their variants for variables. We will also adopt the following conventions. First, for a string of elements $a_1 \dots a_i b_1 \dots b_n c_1 \dots c_j \in A$ we will let

$$a_1 \dots a_i (b_1 \dots b_n) c_1 \dots c_j =_{\mathbb{A}} a_1 \dots a_i \ \circ_{\mathbb{A}} (b_1, \dots, b_n)\ c_1 \dots c_j.$$

In other word we use the symbol $=_{\mathbb{A}}$ to signify that we are applying $\circ_{\mathbb{A}}$ to the arguments in parentheses. Second, as we will only be dealing with associative operations, we will omit the parentheses when doing so will help the flow of the argument (as the order of

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application of our functions is irrelevant to the final value). Third, for an element $a \in A$ and a natural number i we will use a^i for the string of i consecutive a's (with a^0 being the empty string).

Definition 1.1. *For $\alpha \in \mathbb{N}$ we define $\mathbb{A}^\alpha := (A, \circ_{\mathbb{A}}^\alpha)$ to be the quasitrivial $\alpha \cdot (n-1)+1$-semigroup where for any sequence $a_1 \dots a_{\alpha\cdot(n-1)+1}$ and $a \in A$, $a_1 \dots a_{\alpha\cdot(n-1)+1} =_{\mathbb{A}^\alpha} a$ if and only if $a_1 \dots a_{\alpha\cdot(n-1)+1} =_{\mathbb{A}} a$. We say the quasitrivial $\alpha\cdot(n-1)+1$-semigroup $(A, \circ_{\mathbb{A}}^\alpha)$ is* **derived** *from $(A, \circ_{\mathbb{A}})$.*

In other words $(A, \circ_{\mathbb{A}}^\alpha)$ is obtained by applying $\circ_{\mathbb{A}}$, α many times, to a string of length $\alpha \cdot (n-1)+1$. The following are immediate.

A **choice function** (of arity n) on a set A is a function $\circ_{\mathbb{A}} : A^n \to A$ where

$$(\forall a_1, \dots, a_n \in A) \bigvee_{1 \le i \le n} \circ_{\mathbb{A}}(a_1, \dots, a_n) = a_i.$$

In other words $\circ_{\mathbb{A}}$ is a choice function if it always returns a value which is among its arguments.

The following lemmas are immediate.

Lemma 1.2. *The following are equivalent:*

- *$(A, \circ_{\mathbb{A}})$ is quasitrivial.*
- *$\circ_{\mathbb{A}}$ is a choice function.*

Lemma 1.3. *For any $a \in A$ and $\alpha \in \mathbb{N}$, $a^{\alpha\cdot(n-1)+1} =_{\mathbb{A}} a$.*

For a first order language L, an L-structure M, and binary relation $R \in L$, or a binary set $R^M \subseteq M \times M$, we will abuse notation and use aRb for $(a, b) \in R^M$. If E is an equivalence relation on a set M we say an E-equivalence class is **trivial** if it contains only one element. We also let $[a]_E = \{b \in M : E(a, b)\}$. We let $\mathbb{Q}$ be the rationals with the implied ordering.

1.2. **Summary.** The characterization of quasitrivial 2-semigroups has been known for almost 60 years and has a simple proof using two results from the 1950s about idempotent semigroups. Recall that an idempotent semigroup $\mathcal{S} = (S, \circ)$ is **left singular** if it satisfies $(\forall x, y)\ x \circ y = x$, **right singular** if it satisfies $(\forall x, y)\ x \circ y = y$ and that it is **rectangular** if it satisfies $(\forall x, y)\ x \circ y \circ x = x$.

A decomposition theorem of McLean ([12]) implies there is a unique congruence relation $\sim$ on any idempotent semigroup $\mathcal{S}$ such that $\mathcal{S}/\sim$ is a semilattice and each $\sim$-equivalence class is a rectangular subsemigroup. Now if $\mathcal{S}$ is a quasitrivial semigroup then the semilattice $\mathcal{S}/\sim$ must be a linear ordering as any semilattice whose operation is a choice function must be a linear order. It is also immediate that, if $\mathcal{S}$ is a quasitrivial semigroup, then $\circ$ is completely determined by $\mathcal{S}/\sim$ and the restriction of $\circ$ to each $\sim$-equivalence class.

However a theorem of Kimura ([8]) says that every rectangular semigroup is (uniquely) the direct product of a left singular semigroup and a right singular semigroup. Further it is easy to check that no quasitrivial semigroup can be direct product non-trivial subsemigroups and so $\circ$ on each $\sim$-equivalence class must be either left singular or right singular. Hence as there is only one left (right) singular semigroup structure on any set, the semigroup $\mathcal{S}$ is completely determined by $\mathcal{S}/\sim$ and whether for each a, $([a]_\sim, \circ)$ is left or right singular.

In this paper we will extend the classification of quasitrivial semigroups to quasitrivial n-semigroups for all n. In particular our main theorem shows that all quasitrivial n-semigroups are almost derived from 2-semigroups.

Theorem 1.4 (Theorem 3.18). *Suppose $\mathbb{A} = (A, \circ_{\mathbb{A}})$ is a quasitrivial n-semigroup.*

- *If n is even then $\mathbb{A}$ is derived from a 2-semigroup.*
- *If n is odd then $\mathbb{A}$ is derived from a 3-semigroup.*
- *If $\mathbb{A}$ is any 3-semigroup not derived from a 2-semigroup then there exists a (unique) pair of elements $b_0, b_1 \in A$ such that*
  - *$(A - \{b_0, b_1\}, \circ_{\mathbb{A}})$ is an n-semigroup derived from a 2-semigroup $\mathbb{A}_2$.*
  - *For any sequence $a_0 \cdots a_n \notin \{b_0, b_1\}^{<\omega}$ with $a_0^* \cdots a_{n^*}^*$ the correspond sequence with b_0 and b_1 removed we have*
  $$a_0^* \cdots a_{n^*}^* =_{\mathbb{A}_2} a \text{ if and only if } a_0 \cdots a_n =_{\mathbb{A}} a$$
  i.e. b_0, b_1 are neutral elements.
  - *If $a_0 \cdots a_n \in \{b_0, b_1\}^{<\omega}$ then $a_0 \cdots a_n =_{\mathbb{A}} b_0$ if and only if b_0 occurs an odd number of times.*

The above characterization of quasitrivial 2-semigroups, while simple, uses in a fundamental way the mentioned decomposition results for idempotent semigroups. When we move to quasitrivial n-semigroups however these decomposition results do not seem to readily generalize, and so we will need a different approach.

Specifically, we will prove Theorem 3.18 by studying the possible values of the quadruple
$$Q_{\mathbb{A}}(a, b) := \langle \circ_{\mathbb{A}}(a^{n-1}b),\ \circ_{\mathbb{A}}(ab^{n-1}),\ \circ_{\mathbb{A}}(ba^{n-1}),\ \circ_{\mathbb{A}}(b^{n-1}a)\rangle$$
for $a, b \in A$. We will show that these values, as a and b range over elements of A, completely determine the quasitrivial n-semigroup.

In Section 2.1 we will show that for any $a, b \in A$ there are only five possible values for this quadruple. We will then individually consider each of these five possibilities. In Section 2.2 and Section 2.3 we will see that for three of these possibilities the collection of pairs which satisfy them each form an equivalence relation. Further we will show that each element can belong to a non-trivial equivalence class for at most one of these equivalence relations. In Section 2.4 we will consider the two remaining possibilities and show that the pairs of elements which satisfy them define a linear order on the equivalence classes from Section 2.2 - 2.3. These equivalence classes and linear order will make up the structure which characterize a quasitrivial n-semigroup and we will define a *representation* to be any structure containing an equivalence relation and linear order with the properties we have defined.

We will show in Section 3 that we can recover a quasitrivial n-semigroup from a representation, i.e. that the operation of going from an quasitrivial n-semigroup to its representation is a functor and that that functor has an inverse. We have to be a little careful about how we do this though as in full generality it only works if the operation is of odd arity. The reason is that one of the possibilities for $Q_{\mathbb{A}}(a, b)$ can only occur if the arity of $\circ_{\mathbb{A}}$ is odd. However, we will see in Section 3.3 that we can also characterize those representations which come from quasitrivial n-semigroups for even n.

Finally we will end this paper in Section 4 by using our characterization to study quasitrivial n-semigroups with extra properties. We will show there are continuum many quasitrivial n-semigroups (of any $n \geq 2$) which have trivial algebraic closure and continuum many which do not have trivial algebraic closure. For each n we will construct a universal n-semigroup. We will also give a complete classification of the ultrahomogeneous quasitrivial n-semigroups and show that there is no universal ultrahomogeneous quasitrivial n-semigroup. We will also give a complete classification of commutative quasitrivial n-semigroups.

1.3. **Related Work.** While the notion of an n-semigroups is a natural generalization of the notion of semigroup to higher arity operations, it is only one of many such generalizations. Another important generalization is the notion of a Menger algebra, i.e. an algebras $(A, \circ)$ where $\circ$ satisfies the *superassociativity* condition. The case of quasitrivial Menger algebras has been studied by Langer and Skala and in many situations such Menger algebras have been completely characterized. For more on these characterizations see [14], [9], [11] and [10].

Quasitriviality has been studied over the years in many different types of structures and in many cases completely characterizations have been given. For examples in the case of binary operations see [5], [7], or [3]. It is easy to see that for every quasitrivial binary operations $(A, \circ_{\mathbb{A}})$ there corresponds a tournament $(A, \rightarrow)$ where $\rightarrow := \{(x, y) \in A^2 : x \circ y = x\}$. In the case where $\rightarrow$ is an equivalence relation we say $(A, \circ_{\mathbb{A}})$ is an *equivalence algebra*. Equivalence algebras have been studied, for example, in [2] and [6].

Choice functions are of fundamental importance not just in mathematics but also in economics and social choice theory. The view of an associative choice function as repeatedly choosing elements from a substring until only one remains suggests that there is a close connection between associative choice functions and *path independent* choice functions in social choice theory (see [13]).

Another place where quasitrivial n-semigroups arise, and the original motivation for the results of this paper, is in the study of countably infinite n-semigroups which admit a *probabilistic construction*. By a probabilistic construction we mean a probability measure, on the space of all n-ary functions with underlying set $\mathbb{N}$, which is concentrated on the models isomorphic to the desired n-semigroup, but which is also independent under permutations of $\mathbb{N}$. We can think of the condition that our distribution is invariant to permutation of $\mathbb{N}$ as guaranteeing that the manner in which we label our underlying countable set by $\mathbb{N}$ doesn't matter. The quintessential example of such a probabilistic construction is the Erdös-Rényi random construction of the Rado graph. This construction consists of independently flipping a coin for every pair of points and adding an edge if and only if the coin is heads (see [4] for more details). In [1] it is shown that an n-semigroup admits such a probabilistic construction if and only if it has trivial (group theoretic) algebraic closure. It is also easily seen that a necessary condition for an n-semigroup to have trivial algebraic closure (and hence to admit such a probabilistic construction) is for the underlying operation to be a choice function.

2. Two Element Operations

The simplest sequences of length n are those of the form a^n for some $a \in A$. Because $\circ_{\mathbb{A}}$ is a choice function we have $a^n =_{\mathbb{A}} a$ for any $a \in A$ and hence the values of these sequences are completely determined. In this section we consider the second simplest type of sequences of length n, i.e. those which contain only two elements, one of which occurs only once as either the last or first element of the sequence. For any pair (a, b) there are four such sequences: $a^{n-1}b$, ab^{n-1}, ba^{n-1}, and $b^{n-1}a$. In this section we will consider the possible values of applying $\circ_{\mathbb{A}}$ to these four sequences to get a quadruple $Q_{\mathbb{A}}(a, b)$. As we will see the values of $\circ_{\mathbb{A}}$ are completely determined by the values of $Q_{\mathbb{A}}(a, b)$ as a and b range over the elements of A.

Because $\circ_{\mathbb{A}}$ is a choice function there are $2^4 = 16$ possible values for the quadruple $Q_{\mathbb{A}}(a, b)$. It will be useful later on to have list of all of the 16 possibilities, each with a unique label. We give this in Table 1.

2.1. **Non-Valid Values.** In this section we will show that 11 of the 16 possible combinations in Table 1 cannot occur.

Proposition 2.1. *There are no two distinct elements $a, b \in A$ that satisfy any of (X1) - (X11).*

Proof of Proposition 2.1: We show this by first proving the claims.

Claim 2.2. *If $a^{n-1}b =_{\mathbb{A}} a$ then $a^{n-m}b^m =_{\mathbb{A}} a$ for all $0 \leq m < n$.*

Proof of Claim 2.2: Assume $a^{n-k}b^k =_{\mathbb{A}} a$ for some $1 \leq k < n-1$. Then $a =_{\mathbb{A}} a^{n-k}b^k =_{\mathbb{A}} a^{n-k-1}ab^k =_{\mathbb{A}} a^{n-k-1}(a^{n-1}b)b^k$. But because $n-k-1 > 0$ we have $a^{n-k-1}(a^{n-1}b)b^k =_{\mathbb{A}} a^{n-k-2}(a^n)b^{k+1} =_{\mathbb{A}} a^{n-k-2}ab^{k+1} =_{\mathbb{A}} a^{n-(k+1)}b^{k+1}$. The claim then follows by induction.

$\square_{Claim 2.2}$

In particular Claim 2.2 tells us that whenever $a^{n-1}b =_{\mathbb{A}} a$ we must have $ab^{n-1} =_{\mathbb{A}} a$. This rules out the possibility of any of (X1) - (X4) holding for any pair of distinct elements a, b. However by interchanging a and b in Claim 2.2 we also have whenever

TABLE 1. Possible Values of $Q_{\mathbb{A}}(a,b)$

(GT)	$a^{n-1}b =_{\mathbb{A}} a$,	$ab^{n-1} =_{\mathbb{A}} a$,	$ba^{n-1} =_{\mathbb{A}} a$,	$b^{n-1}a =_{\mathbb{A}} a$.
(LT)	$a^{n-1}b =_{\mathbb{A}} b$,	$ab^{n-1} =_{\mathbb{A}} b$,	$ba^{n-1} =_{\mathbb{A}} b$,	$b^{n-1}a =_{\mathbb{A}} b$.
(L)	$a^{n-1}b =_{\mathbb{A}} a$,	$ab^{n-1} =_{\mathbb{A}} a$,	$ba^{n-1} =_{\mathbb{A}} b$,	$b^{n-1}a =_{\mathbb{A}} b$.
(R)	$a^{n-1}b =_{\mathbb{A}} b$,	$ab^{n-1} =_{\mathbb{A}} b$,	$ba^{n-1} =_{\mathbb{A}} a$,	$b^{n-1}a =_{\mathbb{A}} a$.
(O)	$a^{n-1}b =_{\mathbb{A}} b$,	$ab^{n-1} =_{\mathbb{A}} a$,	$ba^{n-1} =_{\mathbb{A}} b$,	$b^{n-1}a =_{\mathbb{A}} a$.
(X1)	$a^{n-1}b =_{\mathbb{A}} a$,	$ab^{n-1} =_{\mathbb{A}} b$,	$ba^{n-1} =_{\mathbb{A}} a$,	$b^{n-1}a =_{\mathbb{A}} a$.
(X2)	$a^{n-1}b =_{\mathbb{A}} a$,	$ab^{n-1} =_{\mathbb{A}} b$,	$ba^{n-1} =_{\mathbb{A}} a$,	$b^{n-1}a =_{\mathbb{A}} b$.
(X3)	$a^{n-1}b =_{\mathbb{A}} a$,	$ab^{n-1} =_{\mathbb{A}} b$,	$ba^{n-1} =_{\mathbb{A}} b$,	$b^{n-1}a =_{\mathbb{A}} a$.
(X4)	$a^{n-1}b =_{\mathbb{A}} a$,	$ab^{n-1} =_{\mathbb{A}} b$,	$ba^{n-1} =_{\mathbb{A}} b$,	$b^{n-1}a =_{\mathbb{A}} b$.
(X5)	$a^{n-1}b =_{\mathbb{A}} a$,	$ab^{n-1} =_{\mathbb{A}} a$,	$ba^{n-1} =_{\mathbb{A}} a$,	$b^{n-1}a =_{\mathbb{A}} b$.
(X6)	$a^{n-1}b =_{\mathbb{A}} b$,	$ab^{n-1} =_{\mathbb{A}} a$,	$ba^{n-1} =_{\mathbb{A}} a$,	$b^{n-1}a =_{\mathbb{A}} b$.
(X7)	$a^{n-1}b =_{\mathbb{A}} b$,	$ab^{n-1} =_{\mathbb{A}} b$,	$ba^{n-1} =_{\mathbb{A}} a$,	$b^{n-1}a =_{\mathbb{A}} b$.
(X8)	$a^{n-1}b =_{\mathbb{A}} b$,	$ab^{n-1} =_{\mathbb{A}} a$,	$ba^{n-1} =_{\mathbb{A}} a$,	$b^{n-1}a =_{\mathbb{A}} a$.
(X9)	$a^{n-1}b =_{\mathbb{A}} b$,	$ab^{n-1} =_{\mathbb{A}} a$,	$ba^{n-1} =_{\mathbb{A}} b$,	$b^{n-1}a =_{\mathbb{A}} b$.
(X10)	$a^{n-1}b =_{\mathbb{A}} a$,	$ab^{n-1} =_{\mathbb{A}} a$,	$ba^{n-1} =_{\mathbb{A}} b$,	$b^{n-1}a =_{\mathbb{A}} a$.
(X11)	$a^{n-1}b =_{\mathbb{A}} b$,	$ab^{n-1} =_{\mathbb{A}} b$,	$ba^{n-1} =_{\mathbb{A}} b$,	$b^{n-1}a =_{\mathbb{A}} a$.

$b^{n-1}a =_{\mathbb{A}} b$ that $ba^{n-1} =_{\mathbb{A}} b$. This also rules out the possibility of any of (X5) - (X7) holding for any pair of distinct elements a, b.

Claim 2.3. *If $a^{n-1}b =_{\mathbb{A}} b$ and $ab^{n-1} =_{\mathbb{A}} a$ then $ba^{n-1} =_{\mathbb{A}} b$ and $b^{n-1}a =_{\mathbb{A}} a$.*

Proof of Claim 2.3: Assume $a^{n-1}b =_{\mathbb{A}} b$ and $ab^{n-1} =_{\mathbb{A}} a$. Then $ba^{n-1} =_{\mathbb{A}} ba^{n-2}(ab^{n-1}) =_{\mathbb{A}} ba^{n-1}bb^{n-2} =_{\mathbb{A}} bbb^{n-2} =_{\mathbb{A}} b$ and $b^{n-1}a =_{\mathbb{A}} (a^{n-1}b)b^{n-2}a =_{\mathbb{A}} a^{n-2}ab^{n-1}a =_{\mathbb{A}} a^{n-2}aa =_{\mathbb{A}} a$. $\square_{Claim 2.3}$

By Claim 2.3 we therefore have that (X8) and (X9) can't hold of any two distinct a, b.

However, by interchanging a with b in Claim 2.3 we have whenever both $b^{n-1}a =_{\mathbb{A}} a$ and $ba^{n-1} =_{\mathbb{A}} b$ hold, both $ab^{n-1} =_{\mathbb{A}} a$ and $a^{n-1}b =_{\mathbb{A}} b$ also hold. Hence (X10) and (X11) can't hold of any two distinct a, b. $\square_{Prop. 2.1}$

In particular Proposition 2.1 limits the possibilities we need to consider to 5. The following corollary, which will greatly simplify what we need to check during the rest of the paper, follows from simply examining the possibilities left in Table 1.

Corollary 2.4. *We have the following implications.*

- *If $a^{n-1}b =_{\mathbb{A}} a$ and $b^{n-1}a =_{\mathbb{A}} a$ then (GT) holds for (a, b).*
- *If $a^{n-1}b =_{\mathbb{A}} b$ and $b^{n-1}a =_{\mathbb{A}} b$ then (LT) holds for (a, b).*
- *If $a^{n-1}b =_{\mathbb{A}} a$ and $b^{n-1}a =_{\mathbb{A}} b$ then (L) holds for (a, b).*
- *If $ab^{n-1} =_{\mathbb{A}} b$ and $ba^{n-1} =_{\mathbb{A}} a$ then (R) holds for (a, b).*
- *If $a^{n-1}b =_{\mathbb{A}} b$ and $ab^{n-1} =_{\mathbb{A}} a$ then (O) holds for (a, b).*

2.2. **(L) And (R) Equivalence Relations.** In this section we will look at those pairs (a, b) that satisfy (L) or (R). We show that the collection of pairs which of satisfy (L) (satisfy (R)) form an equivalence relation $\equiv_L$ ($\equiv_R$). Further we will show that when applying our operation to a sequence all of whose elements are in the same $\equiv_L$ ($\equiv_R$) equivalence class, the result will simply be the left most (right most) element of the sequence.

We will also show that any element can belong to a non-trivial equivalence class in at most one of the equivalence relations $\{\equiv_L, \equiv_R\}$.

Definition 2.5. *Let* $\equiv_L^{\mathbb{A}} := \{(a, b) : (a, b)$ *satisfy (L)*$\}$ *and* $\equiv_R^{\mathbb{A}} := \{(a, b) : (a, b)$ *satisfy (R)*$\}$.

We will omit the superscripts on $\equiv_L^{\mathbb{A}}$ and $\equiv_R^{\mathbb{A}}$ when they are clear from the context.

Proposition 2.6. $\equiv_L$ *is an equivalence relation.*

Proof: Reflexivity and symmetry are immediate. For transitivity assume that $a \equiv_L b$ and $b \equiv_L c$. Then $a^{n-1}c =_{\mathbb{A}} a^{n-2}ac =_{\mathbb{A}} a^{n-2}(ab^{n-1})c =_{\mathbb{A}} a^{n-1}(b^{n-1}c) =_{\mathbb{A}} a^{n-1}b =_{\mathbb{A}} a$. Further $c^{n-1}a =_{\mathbb{A}} c^{n-2}ca =_{\mathbb{A}} c^{n-2}(cb^{n-1})a =_{\mathbb{A}} c^{n-1}b =_{\mathbb{A}} c^{n-1}(b^{n-1}a) =_{\mathbb{A}} c^{n-1}b =_{\mathbb{A}} c$. Hence the proposition follows from Corollary 2.4. □

Lemma 2.7. *Suppose* $a_1, \ldots, a_n$ *are such that* $a_i \equiv_L a_j$ *for any* $i, j \leq n$. *Then* $a_1 a_2 \cdots a_n =_{\mathbb{A}} a_1$.

Proof: We will show that $a_1 a_2 \ldots a_n =_{\mathbb{A}} a_1^n$. Assume it doesn't and let i be the greatest such that $a_1 a_2 \ldots a_n =_{\mathbb{A}} a_1^i a_{i+1} \ldots a_n$. We then have the following $a_1^i a_{i+1} \ldots a_n =_{\mathbb{A}} a_1^{i-1}(a_1^n) a_{i+1} a_{i+2} \ldots a_n =_{\mathbb{A}} a_1^i (a_1^{n-1} a_{i+1}) a_{i+2} \ldots a_n =_{\mathbb{A}} a_1^{i+1} a_{i+2} \ldots a_n$. However, if $i = n-1$ this contradicts the fact that $a_1 a_2 \ldots a_n \neq_{\mathbb{A}} a_1^n$ and if $i < n-1$ this contradicts the fact that i was the greatest such that $a_1 a_2 \ldots a_n =_{\mathbb{A}} a_1^i a_{i+1} \ldots a_n$. Either way our assumption is contradicted and so $a_1 a_2 \ldots a_n =_{\mathbb{A}} a_1^n =_{\mathbb{A}} a_1$. □

Lemma 2.7 shows that for any $a \in A$, $([a]_{\equiv_L}, \circ_{\mathbb{A}}|_{[a]_{\equiv_L}})$ is derived from a left singular 2-semigroup.

The next two results follow by the symmetry of condition (L) and condition (R), i.e. by simply interchanging the order of the elements in the proofs of Proposition 2.6 and Lemma 2.7 respectively.

Proposition 2.8. $\equiv_R$ *is an equivalence relation.*

Lemma 2.9. *Suppose* $a_1, \ldots, a_n$ *are such that* $a_i \equiv_R a_j$ *for any* $i, j \leq n$. *Then* $a_1 a_2 \ldots a_n =_{\mathbb{A}} a_n$.

Lemma 2.9 shows that for any $a \in A$, $([a]_{\equiv_R}, \circ_{\mathbb{A}}|_{[a]_{\equiv_R}})$ is derived from a right singular 2-semigroup.

Finally we show that no element can belong to a non-trivial equivalence class for both $\equiv_L$ and $\equiv_R$.

Proposition 2.10. *Suppose $a \equiv_L b$ and $a \equiv_R c$. Then either $a = b$ or $a = c$.*

Proof: Suppose $c^{n-1}b =_{\mathbb{A}} c$. Then by substituting in $c^{n-1}b$ for c, α many times we get $c =_{\mathbb{A}} c^{\alpha\cdot(n-2)+1}b^{\alpha}$ (for any $\alpha \geq 1$). But then

$$a =_{\mathbb{A}} c^{n-1}a =_{\mathbb{A}} c^{n-2}(c^{(n-1)\cdot(n-2)+1}b^{n-1})a =_{\mathbb{A}} c^{n\cdot(n-2)+1}(b^{n-1}a) =_{\mathbb{A}} c^{n\cdot(n-2)+1}b =_{\mathbb{A}} c.$$

Next suppose $c^{n-1}b =_{\mathbb{A}} b$. By repeated substitution we get $c^{\alpha\cdot(n-2)+1}b^{\alpha} =_{\mathbb{A}} b$ for all $\alpha \geq 1$. Then

$$a =_{\mathbb{A}} ab^{n-1} =_{\mathbb{A}} a(c^{n-1}b)b^{n-2} =_{\mathbb{A}} (ac^{n-1})b^{n-1} =_{\mathbb{A}} cb^{n-1} =_{\mathbb{A}} c^{(n-1)\cdot(n-2)+1}b^{n-1} =_{\mathbb{A}} b.$$

So, as $c^{n-1}b \in \{c, b\}$ we must have either $a = c$ or $a = b$. □

In particular this implies that $\equiv_L \cup \equiv_R$ is an equivalence relation.

2.3. **(O) Equivalence Relation.** In this section we look at those pairs (a, b) which satisfies (O). We will show the collection of pairs (a, b) that satisfy (O) form an equivalence relation and that each non-trivial equivalence class has exactly two elements (in Lemma 2.26 we will see that there is at most one such equivalence class). We will also show that if there is a distinct pair (a, b) which satisfy (O), then the arity of our operation must be odd.

Definition 2.11. *Let $\equiv_O^{\mathbb{A}} := \{(a, b) : (a, b)$ satisfy (O)$\}$.*

We will omit the superscript on $\equiv_O^{\mathbb{A}}$ when it is clear from the context.

Proposition 2.12. *$\equiv_O$ is an equivalence relation.*

Proof: Reflexivity and symmetry are immediate. For transitivity assume $a \equiv_O b$ and $b \equiv_O c$. We then have $a^{n-1}c =_{\mathbb{A}} a^{n-2}(ab^{n-1})c =_{\mathbb{A}} (a^{n-1}b)b^{n-2}c =_{\mathbb{A}} b^{n-1}c =_{\mathbb{A}} c$. Further we have $ca^{n-1} =_{\mathbb{A}} c(b^{n-1}a)a^{n-2} =_{\mathbb{A}} cb^{n-2}(ba^{n-1}) =_{\mathbb{A}} cb^{n-1} =_{\mathbb{A}} c$. Hence by Corollary 2.4 we have $a \equiv_O c$. □

We now give a complete characterization of $\circ_{\mathbb{A}}$ when restricted to two elements which are in the same $\equiv_O$-equivalence class.

Lemma 2.13. *If $a \equiv_O b$ and $1 \leq m \leq n-1$ then*

1. *Whenever m is odd $a^{n-m}b^m =_{\mathbb{A}} b$.*
2. *Whenever m is even $a^{n-m}b^m =_{\mathbb{A}} a$.*

Proof: First observe that if $a = b$ then the lemma is trivial. Next note that it suffices to show if $a \neq b$ then $a^{n-k}b^k \neq_{\mathbb{A}} a^{n-k-1}b^{k+1}$ for all $1 \leq k \leq n-1$ as we know $a^{n-1}b =_{\mathbb{A}} b$.

Assume to get a contradiction that $a \neq b$ but $a^{n-k}b^k =_{\mathbb{A}} a^{n-k-1}b^{k+1}$ for some $1 \leq k \leq n-1$. Then for any $1 \leq j < n-1$ we have

$$\begin{aligned} a^{n-j}b^j &=_{\mathbb{A}} a^{n-j-1}(a^n)b^{j-1}(b^n) =_{\mathbb{A}} a^{(n-j-1)+k}(a^{n-k}b^k)b^{(n-k)+(j-1)} \\ &=_{\mathbb{A}} a^{(n-j-1)+k}(a^{n-k-1}b^{k+1})b^{(j-1)+(n-k)} =_{\mathbb{A}} a^{n-j-2}(a^n)b^j(b^n) =_{\mathbb{A}} a^{n-j-1}b^{j+1}. \end{aligned}$$

Hence for all $1 \le i, j \le n-1$ we have $a^{n-j}b^j =_{\mathbb{A}} a^{n-i}b^i$. However we then have $a =_{\mathbb{A}} ab^{n-1} =_{\mathbb{A}} a^{n-1}b =_{\mathbb{A}} b$ contradicting our assumption that $a \neq b$.

In particular we have whenever $a \neq b$ that $a^{n-k}b^k \neq_{\mathbb{A}} a^{n-k-1}b^{k+1}$ for all $1 \le k \le n-1$ and we are done. □

Notice that Lemma 2.13 implies that whenever $|[a]_{\equiv_O}| > 1$ that $([a]_{\equiv_O}, \circ_{\mathbb{A}}|_{[a]_{\equiv_O}})$ is not derived from a 2-semigroup. This is because for any 2-semigroup $\mathbb{A}_2$ and any $m, n \ge 1$, $a^m b^n =_{\mathbb{A}_2} ab$.

Corollary 2.14. *If n is even and $a \equiv_O b$ then $a = b$.*

Proof: We know that $ab^{n-1} =_{\mathbb{A}} a$ because $a \equiv_O b$. But if n is even then $n-1$ is odd and so by Lemma 2.13 (1) we have $ab^{n-1} =_{\mathbb{A}} b$. Hence $a = b$. □

So if n is even there is no pair of distinct elements (a, b) which satisfy (O).

Lemma 2.15. *If $a \equiv_O b$ then $a^{n-m-2k+1}b^{2k-1}a^m =_{\mathbb{A}} b$ for any $0 \le m \le n-2k-1$ (with $n-2k-1 \ge 0$).*

Proof: First notice that if $a = b$ this lemma trivially holds. So without loss of generality we can assume $a \neq b$.

We prove this by induction on k. First let $k = 1$ and suppose to get a contradiction that $a^{n-m-1}ba^m =_{\mathbb{A}} a$. Then $a =_{\mathbb{A}} a^n =_{\mathbb{A}} a^m a a^{n-m-1} =_{\mathbb{A}} a^m(a^{n-m-1}ba^m)a^{n-m-1} =_{\mathbb{A}} (a^{n-1}b)a^{n-1} =_{\mathbb{A}} ba^{n-1} =_{\mathbb{A}} b$ which is a contradiction.

Now assume the lemma holds for k with $n-2(k+1)-1 \ge 0$. We then have

$$\begin{aligned} b &=_{\mathbb{A}} a^{n-m-2k+1}b^{2k-1}a^m =_{\mathbb{A}} a^{n-m-2k+1}b^{n-1}b^{2k-1}a^m =_{\mathbb{A}} a^{n-m-2k-1}(a^2b^{n-2})bb^{2k-1}a^m \\ &=_{\mathbb{A}} a^{n-m-2k-1}b^{2k+1}a^m \qquad \text{by Lemma 2.13.} \end{aligned}$$

Hence the conditions of the Lemma hold for $k+1$ and the result follows by induction. □

The following is a consequence of Lemma 2.13 and Lemma 2.15.

Proposition 2.16. *Suppose $\{a_1, \ldots, a_n\} = \{a^+, a^-\}$, $a^+ \neq a^-$ and $a^+ \equiv_O a^-$. Then $a_1 \ldots a_n =_{\mathbb{A}} a^+$ if and only if there are an odd number of a^+'s in the sequence $a_1 \ldots a_n$.*

We now show that any non-trivial $\equiv_O$-equivalence class contains exactly two elements.

Lemma 2.17. *If $a \equiv_O b$ and $a \equiv_O c$ then one of $c = a$, $c = b$ or $a = b$ must hold.*

Proof: Note $\circ_{\mathbb{A}}(a^{n-2}bc) \in \{a, b, c\}$. The result will then follow by considering the three possible values of $\circ_{\mathbb{A}}(a^{n-2}bc)$.

Case 1: $a^{n-2}bc =_{\mathbb{A}} a$

Then $b =_{\mathbb{A}} a^{n-1}b =_{\mathbb{A}} a^{n-1}bc^{n-1} =_{\mathbb{A}} a(a^{n-2}bc)c^{n-2} =_{\mathbb{A}} aac^{n-2} =_{\mathbb{A}} c$.

Case 2: $a^{n-2}bc =_{\mathbb{A}} b$
Then $a =_{\mathbb{A}} ab^{n-1}c^{n-1} =_{\mathbb{A}} ab^{n-2}(a^{n-1}b)c^{n-1} =_{\mathbb{A}} ab^{n-2}a(a^{n-2}bc)c^{n-2} =_{\mathbb{A}} ab^{n-2}abc^{n-2} =_{\mathbb{A}} a(b^{n-2}ab)c^{n-2} =_{\mathbb{A}} aac^{n-2} =_{\mathbb{A}} c$.

Case 3: $a^{n-2}bc =_{\mathbb{A}} c$
Then $a =_{\mathbb{A}} ab^{n-1}c^{n-1} =_{\mathbb{A}} ab^{n-2}(a^{n-1}b)c^{n-1} =_{\mathbb{A}} ab^{n-2}a(a^{n-2}bc)c^{n-2} =_{\mathbb{A}} ab^{n-2}acc^{n-2} =_{\mathbb{A}} bc^{n-1} =_{\mathbb{A}} b$. □

Suppose $g : \{a^+, a^-\}^3 \to \{a^+, a^-\}$ is idempotent and is such that $g(x_0, x_1, x_2) = y$ if and only if exactly one of x_0, x_1, x_2 equals y. Then Proposition 2.16 and Lemma 2.17 imply that whenever $|[a]_{\equiv_O}| > 1$ that $([a]_{\equiv_O}, \circ_{\mathbb{A}}|_{[a]_{\equiv_O}})$ is derived (a 3-semigroup isomorphic to) $(\{a^+, a^-\}, g)$.

Finally we show that any element can belong to a non-trivial equivalence class for at most on of $\equiv_L$, $\equiv_R$ or $\equiv_O$.

Proposition 2.18. *Suppose $a \equiv_O c$ with $a \neq c$. Then*
(i) *For all b, $a \equiv_L b$ implies $a = b$.*
(ii) *For all b, $a \equiv_R b$ implies $a = b$.*

Proof: Proof of (i): If $a \equiv_L b$ then $c^{n-1}b =_{\mathbb{A}} c^{n-2}(ca^{n-1})b =_{\mathbb{A}} c^{n-1}(a^{n-1}b) =_{\mathbb{A}} c^{n-1}a =_{\mathbb{A}} a$. Hence $a \in \{b, c\}$ and, as $a \neq c$ by assumption we have $a = b$.

Proof of (ii): If $a \equiv_R b$ then $bc^{n-1} =_{\mathbb{A}} b(a^{n-1}c)c^{n-2} =_{\mathbb{A}} (ba^{n-1})c^{n-1} =_{\mathbb{A}} ac^{n-1} =_{\mathbb{A}} a$. Hence $a \in \{b, c\}$ and, as $a \neq c$ by assumption we have $a = b$. □

Definition 2.19. $\equiv^{\mathbb{A}} := \equiv_L^{\mathbb{A}} \cup \equiv_R^{\mathbb{A}} \cup \equiv_O^{\mathbb{A}}$.

We will omit the superscript on $\equiv^{\mathbb{A}}$ when it is clear from context. The flowing lemma follows from Proposition 2.10 and Proposition 2.18.

Lemma 2.20. $\equiv$ *is an equivalence relation.*

2.4. **Linear Order Between Equivalence Classes.** We now consider the final two possibilities from Table 1, i.e. (LT) and (GT).

Definition 2.21. *Let $<^{\mathbb{A}} := \{(a, b) : a \neq b$ and (a, b) satisfy (LT)$\}$ and $>^{\mathbb{A}} := \{(a, b) : a \neq b$ and (a, b) satisfy (GT)$\}$.*

We will omit superscripts on $<^{\mathbb{A}}$ and $>^{\mathbb{A}}$ when they are clear from the context. The following is then immediate from Table 1.

Lemma 2.22. *$a < b$ if and only if $b > a$.*

We now show $<$ is transitive.

Lemma 2.23. *If $a < b$ and $b < c$ then $a < c$.*

Proof: Suppose $a < b$ and $b < c$. Then $a^{n-1}c =_{\mathbb{A}} a^{n-1}(b^{n-1}c) =_{\mathbb{A}} (a^{n-1}b)b^{n-2}c =_{\mathbb{A}} b^{n-1}c =_{\mathbb{A}} c$. Also $c^{n-1}a =_{\mathbb{A}} c^{n-2}(cb^{n-1})a =_{\mathbb{A}} c^{n-1}(b^{n-1}a) =_{\mathbb{A}} c^{n-1}b =_{\mathbb{A}} c$. Finally notice that if $a = c$ then (b, c) satisfies both (LT) and (GT), which is a contradiction. Hence $a < c$. □

We now show that $<$ respects $\equiv$-equivalence classes.

Proposition 2.24. *Suppose $a < b$ and $b < c$*

(1) *If $b_L \equiv_L b$ then $a < b_L$ and $b_L < c$.*
(2) *If $b_R \equiv_R b$ then $a < b_R$ and $b_R < c$.*
(3) *If $b_O \equiv_O b$ then $a < b_O$ and $b_O < c$.*

Proof: First notice by Propositions 2.6, 2.8, and 2.12 along with the fact that *exactly* one of (L), (R), (O), (LT), or (GT) must hold of any pair of distinct elements, that for $b^* \in \{b_L, b_R, b_O\}$ either $a < b^*$ or $b^* < a$ and similarly either $c < b^*$ or $b^* < c$.

Now $b_L^{n-1}a =_{\mathbb{A}} b_L^{n-2}(b_L b^{n-1})a =_{\mathbb{A}} b_L^{n-1}(b^{n-1}a) =_{\mathbb{A}} b_L^{n-1}b =_{\mathbb{A}} b_L$. Hence $b_L \not< a$ and so $a < b_L$. Similarly $b_L^{n-1}c =_{\mathbb{A}} b_L^{n-1}(b^{n-1}c) =_{\mathbb{A}} (b_L^{n-1}b)b^{n-2}c =_{\mathbb{A}} b^{n-1}c =_{\mathbb{A}} c$. Hence $c \not< b_L$ and so $b_L < c$. We have thus shown that (1) holds.

(2) follows from (1) by symmetry, i.e. reversing the order of the sequences in the proof of (1).

Finally, notice $b_O^{n-1}c =_{\mathbb{A}} b_O^{n-1}(b^{n-1}c) =_{\mathbb{A}} (b_O^{n-1}b)b^{n-2}c =_{\mathbb{A}} b^{n-1}c =_{\mathbb{A}} c$. Therefore $c \not< b_O$ and so $b_O < c$. Similarly $a^{n-1}b_O =_{\mathbb{A}} a^{n-1}(b^{n-1}b_O) =_{\mathbb{A}} (a^{n-1}b)b^{n-2}b_O =_{\mathbb{A}} b^{n-1}b_O =_{\mathbb{A}} b_O$. So we have $b_O \not< a$ and hence $a < b_O$. □

Note that for any $a \in A$, $a \not< a$, as if we let $a < a$ then $<$ wouldn't respect non-trivial equivalence classes, because $a \not< b$ if $a \neq b$ and $a \equiv b$.

Corollary 2.25. *$<$ is a linear order on $A/\equiv$.*

Proof: We have shown in Proposition 2.24 that $<$ respects $\equiv$. Lemma 2.23 shows that $<$ is transitive and hence a partial ordering on $A/\equiv$. Further, by Proposition 2.1 we know that for any $a \not\equiv b$ either $a < b$ or $a > b$. Linearity then follows from Lemma 2.22. □

We are almost done defining the structure we need to characterize quasitrivial n-semigroups. We just need one more lemma.

Lemma 2.26. *Suppose $a \equiv_O b$ and $a \neq b$. Then $(\forall c) c \equiv_O a \vee a < c$.*

Proof: First notice that if $c < a$ then by repeated use of the equality $a^{n-1}c =_{\mathbb{A}} a$ we have $a^{\alpha\cdot(n-2)+1}c^{\alpha} =_{\mathbb{A}} a$. Similarly if $c < b$ we have by repeated use of the equality $cb^{n-1} =_{\mathbb{A}} b$ that $c^{\alpha}b^{\alpha\cdot(n-2)+1} =_{\mathbb{A}} b$.

Now suppose to get a contradiction that $c \not\equiv_O a$ and $a \not< c$. By Proposition 2.18 we also have $c \not\equiv a$. Hence by Corollary 2.25 we have $c < a$. But by Proposition 2.24 this also implies $c < b$.

We then have $ac^{n-2}b =_{\mathbb{A}} a^{(n-2)\cdot(n-1)+1}c^{n-2}b =_{\mathbb{A}} a^{n-2}(a^{(n-2)\cdot(n-2)+1}c^{n-2})b =_{\mathbb{A}} a^{n-2}ab =_{\mathbb{A}} b$. However we also have $ac^{n-2}b =_{\mathbb{A}} ac^{n-2}b^{(n-2)\cdot(n-1)+1} =_{\mathbb{A}} a(c^{n-2}b^{(n-2)\cdot(n-2)+1})b^{n-2} =_{\mathbb{A}} abb^{n-2} =_{\mathbb{A}} a$. But this contradicts our assumption that $a \neq b$ and so we must have $a < c$. □

In other words Lemma 2.26 tells us that there is at most one $\equiv_O$ equivalence class, and if it exists it is the minimal element of the ordering of $A/\equiv$ by $<$.

3. Characterization

We now give a precise definition of the structures which we use to characterize quasitrivial n-semigroups. We also define appropriate categories of quasitrivial n-semigroups as well as categories of representations and show these categories are isomorphic.

3.1. Categories.

We now define our category of quasitrivial n-semigroups as well our category of representations.

3.1.1. *Category of Choice n-Semigroups.*

Definition 3.1. *Let $L_{QSG_n} = \{\circ\}$ where $\circ$ is an n-ary function symbol and let Th_{QSG_n} be the theory which says $\circ$ is an associative choice function.*

Definition 3.2. *Let QSG_n be the category where:*

- *The objects of QSG_n are the L_{QSG_n}-structures which satisfy Th_{QSG_n}.*
- *The morphisms are the injective homomorphisms of L_{QSG_n}-structures.*

Note that we only consider injective homomorphisms as we want the maps to preserve the formulas $x \neq y$ as well as all relations.

The following five L_{QSG_n}-formulas will be useful:

$\hat{L}(x) := (\exists y) y \neq x \wedge f(y^{n-1}x) = y \wedge f(x^{n-1}y) = x.$
$\hat{R}(x) := (\exists y) y \neq x \wedge f(xy^{n-1}) = y \wedge f(yx^{n-1}) = x.$
$\hat{O}(x) := (\exists y) y \neq x \wedge f(y^{n-1}x) = x \wedge f(yx^{n-1}) = y.$

$$x \widehat{\equiv} y := [f(x^{n-1}y) = x \wedge f(y^{n-1}x) = y] \vee [f(xy^{n-1}) = y \wedge f(yx^{n-1}) = x] \vee [f(x^{n-1}y) = y \wedge f(xy^{n-1}) = x].$$

$$x \widehat{<} y := x \neq y \wedge f(x^{n-1}y) = y \wedge f(y^{n-1}x) = y.$$

It is then immediate that for any $a, b \in A$, $a \equiv^{\mathbb{A}} b$ if and only if $\mathbb{A} \models a \widehat{\equiv} b$ and $a <^{\mathbb{A}} b$ if and only if $\mathbb{A} \models a \widehat{<} b$. Further we have for any $a \in \mathbb{A}$ that $\mathbb{A} \models \hat{L}(a)$ if and only if a is in a non-trivial $\equiv_L^{\mathbb{A}}$ equivalence class, $\mathbb{A} \models \hat{R}(a)$ if and only if a is in a non-trivial $\equiv_R^{\mathbb{A}}$ equivalence class and $\mathbb{A} \models \hat{O}(a)$ if and only if a is in a non-trivial $\equiv_O^{\mathbb{A}}$ equivalence class.

Lemma 3.3. *Any morphism of QSG_n preserves the formulas $\hat{L}(x), \hat{R}(x), \hat{O}(x), x\hat{\equiv}y, x\hat{<}y$.*

Proof: As all morphisms of QSG_n are homomorphisms they preserve all positive quantifier free formulas (i.e. quantifier free formulas which contain no negations). In particular this means $\hat{\equiv}$ and $\hat{<}$ are preserved by all morphisms of QSG_n.

Each of $\hat{L}(x), \hat{R}(x)$, and $\hat{O}(x)$ are of the form $(\exists y)x \neq y \wedge \psi(x,y)$ where $\psi(x,y)$ is positive and quantifier free. But as each morphism is injective, it also preserves $x \neq y$ and hence also preserves $(\exists y)x \neq y \wedge \psi(x,y)$. □

In the definition of QSG_n, we chose only to consider injective homomorphisms instead of all homomorphisms, precisely so Lemma 3.3 would hold.

3.1.2. *Category of Representations.*

Definition 3.4. *Let $L_{RO} = \{L, R, O\} \cup \{\equiv, <\}$ where L, R, O are unary relations and $\equiv, <$ are binary relations. Let Th_{RO} be the theory which says the following:*

- *Each element satisfies at most one of $L(x), R(x)$ or $O(x)$.*
- *$\equiv$-Axioms:*
  - *$\equiv$ is an equivalence relation.*
  - $(\forall x)[(\exists y)x \neq y \wedge x \equiv y] \leftrightarrow [L(x) \vee R(x) \vee O(x)]$.
  - $(\forall x, y)x \neq y \wedge x \equiv y \rightarrow [L(x) \leftrightarrow L(y)]$.
  - $(\forall x, y)x \neq y \wedge x \equiv y \rightarrow [R(x) \leftrightarrow R(y)]$.
  - $(\forall x, y)x \neq y \wedge x \equiv y \rightarrow [O(x) \leftrightarrow O(y)]$.
  - $(\forall x, y, z)[x \equiv y \wedge y \equiv z \wedge O(x)] \rightarrow [x = y \vee y = z \vee x = z]$.
- *$<$-Axioms*
  - *For all x, y exactly one of the following holds: $x \equiv y$, $x < y$, $x > y$.*
  - $(\forall x, y, z)x \equiv y \rightarrow [x < z \leftrightarrow y < z] \wedge [z < x \leftrightarrow z < y]$
  - $(\forall x, y)O(x) \rightarrow [x \equiv y \vee x < y]$

We call a model of Th_{RO} a **representation** *(of a quasitrivial n-semigroup).*

Definition 3.5. *Let RO be the category where*

- *The objects of RO are those L_{RO}-structures that satisfy Th_{RO}.*
- *The morphisms are the injective L_{RO}-homomorphisms.*

Let RE be the full subcategory of RO consisting of those representations which satisfy $(\forall x)\neg O(x)$.

While a morphism in RO is not required to preserve all negations of relations, it will preserve negations of $\equiv$ and $<$.

Lemma 3.6. *Suppose $\mathcal{X}, \mathcal{Y}$ are objects of RO (with underlying sets X, Y respectively) and $g : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of RO. Then the following hold:*

(1) $(\forall x, y \in X)x \equiv^{\mathcal{X}} y \leftrightarrow g(x) \equiv^{\mathcal{Y}} g(y)$.
(2) $(\forall x, y \in X)x <^{\mathcal{X}} y \leftrightarrow g(x) <^{\mathcal{Y}} g(y)$.

Proof: The implication of left to right in (1) and (2) is because g is a homomorphism and hence preserves all relations. The implications from right to left follow because for all $x, y \in X$, exactly one of $x <^{\mathcal{X}} y$, $x \equiv^{\mathcal{X}} y$, or $y <^{\mathcal{X}} x$ holds and exactly one of $g(x) <^{\mathcal{Y}} g(y)$, $g(x) \equiv^{\mathcal{Y}} g(y)$, or $g(y) <^{\mathcal{Y}} g(x)$ holds. □

3.2. **Functors.** In this section we introduce functors between QSG_n and RO. We will also prove isomorphisms between the categories QSG_{2n+1} and RO and between QSG_{2n} and RE (for any n).

Definition 3.7. *If $\mathbb{A}$ is an L_{QSG_n}-structure which satisfies Th_{QSG_n} then let $F_n(\mathbb{A})$ be the L_{RO}-structure such that*

- *The underlying set of $F(\mathbb{A})$ and $\mathbb{A}$ are the same.*
- $<^{F_n(\mathbb{A})} (x,y) := \widehat{\lessgtr}^{\mathbb{A}}(x,y)$, $\equiv^{F_n(\mathbb{A})} (x,y) := \widehat{\equiv}^{\mathbb{A}}(x,y)$, $L(x)^{F_n(\mathbb{A})} := \hat{L}^{\mathbb{A}}(x)$, $R^{F_n(\mathbb{A})}(x) := \hat{R}^{\mathbb{A}}(x)$, *and* $O^{F_n(\mathbb{A})}(x) := \hat{O}(x)^{\mathbb{A}}$.

If $f : \mathbb{A} \to \mathbb{B}$ is a morphism of QSG_n then let $F_n(f) = f$. Note $F_n(f)$ is well defined as $\mathbb{A}$ and $F_n(\mathbb{A})$ have the same underlying and $\mathbb{B}$ and $F_n(\mathbb{B})$ have the same underlying set. We call $F_n(\mathbb{A})$ the **representation** *of $\mathbb{A}$.*

Lemma 3.8. *F_n is a functor from QSG_n to RO.*

Proof: By the results of Section 2, whenever $\mathbb{A}$ is an object of QSG_n, $F_n(\mathbb{A})$ is an L_{RO}-structure that satisfies Th_{RO}, and hence an object of RO. To show that F_n takes morphisms to morphisms we need to show that whenever $g : \mathbb{A} \to \mathbb{B}$ is a homomorphism of L_{QSG_n} structures, it is also a homomorphism of L_{RO} structures, i.e. it preserves all relations. However this follows immediately from Lemma 3.3. Therefore F_n is a functor. □

The main purpose of Section 2 was so that we could define the functor F_n.
Now we will show that there is a functor going the other way when n is odd.

Definition 3.9. *When $\mathcal{X}$ is an L_{RO}-structure which satisfies Th_{RO} let $G_n(\mathcal{X})$ be the L_{QSG_n}-structure such that*

- *The underlying set of $G_n(\mathcal{X})$ and $\mathcal{X}$ are the same .*
- *We define $\circ_{G_n(\mathcal{X})}(a_1, \ldots, a_n)$ in the following way:*
  - (0) *First remove all elements which are not $<$-maximal to get a subsequence $a_{i_1} \cdots a_{i_m}$ with $i_1 < i_2 < \cdots < i_m$.*
  - (1) *If $L(a_{i_1})$ then $\circ_{G_n(\mathcal{X})}(a_1, \ldots, a_n) = a_{i_1}$.*
  - (2) *If $R(a_{i_1})$ then $\circ_{G_n(\mathcal{X})}(a_1, \ldots, a_n) = a_{i_m}$.*
  - (3) *If $O(a_{i_1})$ and n is odd then*
    - (i) *If a_1 occurs an odd number of times in $a_1, \ldots, a_n$ then $\circ_{G_n(\mathcal{X})}(a_1, \ldots, a_n) = a_1$.*
    - (ii) *Otherwise $\circ_{G_n(\mathcal{X})}(a_1, \ldots, a_n) = a_i$ for the smallest i such that $a_i \neq a_1$.*

(4) *Otherwise* $\circ_{G_n(\mathcal{X})}(a_1, \ldots, a_n) = a_{i_1}$.

If $g : \mathcal{X} \to \mathcal{Y}$ *is a morphism in RO then let* $G_n(g) = g$.

Lemma 3.10. G_n *is a functor from RO to* QSG_n *when* n *is odd and is a functor from RE to* QSG_n *when* n *is even.*

Proof: First we need to show that $G_n(\mathcal{X})$ is always an object of QSG_n, i.e. a quasitrivial n-semigroup. Suppose $G_n(\mathcal{X}) = (A, \circ_{\mathbb{A}})$. It is immediate from the construction that $\circ_{\mathbb{A}}$ is always a choice function and so we just need to show it is associative. Suppose $a_1, \ldots, a_{2n-1} \in A$. Then by two uses of (0) in the construction and the fact that $<$ is a linear order on equivalence classes, that $\circ_{\mathbb{A}}(a_1, \ldots, a_j, \circ_{\mathbb{A}}(a_{j+1}, \ldots, a_{j+n}), a_{j+n+1}, \ldots, a_{2n-1})$ is an element of $<$-largest equivalence class among those containing an element of $\{a_1, \ldots, a_{2n-1}\}$.

We now have four cases to consider depending on which of (1)-(4) holds of elements of that equivalence class.

Case (1): $L(x)$ holds

No matter what j is, $a_1 \ldots a_j(a_{j+1} \ldots a_{j+n})a_{j+n+1} \ldots a_{2n-1} =_{\mathbb{A}} a_i$ where i is the smallest less than or equal to $2n-1$ with a_i in the $<$-largest equivalence class.

Case (2): $R(x)$ holds

No matter what j is, $a_1 \ldots a_j(a_{j+1} \ldots a_{j+n})a_{j+n+1} \ldots a_{2n-1} =_{\mathbb{A}} a_i$ where i is the largest less than or equal to $2n-1$ with a_i in the $<$-largest equivalence class.

Case (3): $O(x)$ holds

In this case all elements of $a_1, \ldots, a_{2n-1}$ are in the same equivalence class because by Lemma 2.26 there is a unique equivalence class with elements that satisfy $O(x)$ and that equivalence class is the $<$-minimal one. Further the set $\{a_1, \ldots, a_{2n-1}\} = \{a^+, a^-\}$, i.e. it consists of exactly two elements. Because n is odd, we can assume without loss of generality that a^+ occurs an odd number of times and a^- occurs an even number of times. We then have two cases.

Case (i): a^+ occurs an even number of times among $a_{j+1} \ldots a_{j+n}$.

In this case a^+ occurs an odd number of times among $a_1 \ldots a_j a_{j+n+1} \ldots a_n$. But we also have $a_{j+1} \ldots a_{j+n} =_{\mathbb{A}} a^-$. Hence a^+ occurs an odd number of times among the tuple $a_1 \ldots a_j a^- a_{j+n+1} \ldots a_n$ and so

$$a_1 \ldots a_j(a_{j+1} \ldots a_{j+n})a_{j+n+1} \ldots a_{2n-1} =_{\mathbb{A}} a_1 \ldots a_j a^+ a_{j+n+1} \ldots a_n =_{\mathbb{A}} a^+.$$

Case (ii): a^+ occurs an odd number of times among $a_{j+1} \ldots a_{j+n}$.

In this case a^+ occurs an even number of times among $a_1 \ldots a_j a_{j+n+1} \ldots a_n$. But we also hav $a_{j+1} \ldots a_{j+n} =_{\mathbb{A}} a^+$. Hence a^+ occurs an odd number of times among

$a_1 \dots a_j a^+ a_{j+n+1} \dots a_n$ and so

$$a_1 \dots a_j(a_{j+1} \dots a_{j+n})a_{j+n+1} \dots a_{2n-1} =_{\mathbb{A}} a_1 \dots a_j a^+ a_{j+n+1} \dots a_n =_{\mathbb{A}} a^+.$$

In either case the result does not depend on j and so we are done.

Case (4): Otherwise

In this case there is only one element of the largest $\equiv$-equivalence class. Hence we have that $a_1 \dots a_j(a_{j+1} \dots a_{j+n})a_{j+n+1} \dots a_{2n-1}$ must be that element (no matter what the value of j).

Notice that if $\mathcal{X}$ is an object of RE then Case (3) does not occur. However Case (3) is the only case which requires n to be odd, and so if n is even we have $G_n : RE \to QSG_n$.

Finally notice that there is a quantifier free formula $\text{graph}_f(x_1 \dots x_n y)$ such that for any $\mathcal{X}$ an object of RO, and any $a_1, \dots, a_n, b \in \mathcal{X}$, $\circ_{G_n(\mathcal{X})}(a_1 \dots a_n) = b$ if and only if $\mathcal{X} \models \text{graph}_f(a_1 \dots a_n b)$. Therefore any function which is a morphism in RO is also a morphism in QSG_n and G_n is a functor.

□

Lemma 3.11. *For every Σ_1 formula φ in L_{QSG_n} there is a Σ_1 formula φ_n^+ in L_{RO} such that for all $\mathbb{A}$ an object of QSG_n, $\{a_1 \dots a_k \in A : \mathbb{A} \models \varphi(a_1 \dots a_k)\} = \{a_1 \dots a_k \in A : F_n(\mathbb{A}) \models \varphi_n^+(a_1 \dots a_k)\}$.*

For every Σ_1 formula ψ in L_{RO} there is a Σ_1 formula ψ_n^- in L_{QSG_n} such that for all $\mathbb{A}$ an object of QSG_n, $\{a_1 \dots a_k \in A : \mathbb{A} \models \psi_n^-(a_1 \dots a_k)\} = \{a_1 \dots a_k \in A : F_n(\mathbb{A}) \models \psi(a_1 \dots a_k)\}$.

Proof: This is because the formula $\text{graph}_f(x_1 \dots x_n y)$ is definable by a quantifier free formula in L_{RO} and each of $\widehat{<}, \widehat{\equiv}, \hat{L}, \hat{R}, \hat{O}$ is definable by a Σ_1 formula in L_{QSG_n}. □

3.3. **Isomorphism of Categories.** The following lemma is immediate from Definition 3.7 and Definition 3.9.

Lemma 3.12.

- *For any $\mathcal{X}$ an object of RO and any n, $F_{2n+1} \circ G_{2n+1}(\mathcal{X}) = \mathcal{X}$.*
- *For any $\mathcal{X}$ an object of RE and any n, we have $F_{2n} \circ G_{2n}(\mathcal{X}) = \mathcal{X}$.*
- *For any $\mathbb{A}$ an object of QSG_n and any n we have $G_n \circ F_n(\mathbb{A}) = \mathbb{A}$.*

In particular this implies

Proposition 3.13. *For any $n \in \mathbb{N}$,*

- *$F_{2n+1} : QSG_{2n+1} \to RO$ is an isomorphism of categories with inverse G_{2n+1}.*

- *$F_{2n} : QSG_{2n} \to RE$ is an isomorphism of categories with inverse G_{2n}.*

Proof: This follows from Lemma 3.12 and the fact that both F_n and G_n are injective on morphisms.

□

We can now deduce that every quasitrivial n-semigroup is derived from either a quasitrivial 2-semigroup or a quasitrivial 3-semigroup.

Corollary 3.14. *Suppose $i_{3,2n+1} : QSG_3 \to QSG_{2n+1}$ is the functor which takes $(A, \circ_{\mathbb{A}})$ to $(A, \circ_{\mathbb{A}}^{n-1})$ and is the identity on morphisms. Then $i_{3,2n+1}$ is an isomorphism of categories.*

Proof: This follows from the fact that $F_{2n+1} \circ i_{3,2n+1} = F_3$. □

Similarly we have

Corollary 3.15. *Suppose $i_{2,2n} : QSG_2 \to QSG_{2n}$ is the functor which takes $(A, \circ_{\mathbb{A}})$ to $(A, \circ_{\mathbb{A}}^{2n-1})$ and is the identity on morphisms. Then $i_{2,2n}$ is an isomorphism of categories.*

Proof: This follows from the fact that $F_{2n} \circ i_{2,2n} = F_2$. □

In particular we have that every quasitrivial $2n$-semigroup is derived from a quasitrivial 2-semigroup and every quasitrivial $2n+1$-semigroup is derived from an quasitrivial 3-semigroup. The following is a characterization of those quasitrivial n-semigroups which are derived from a quasitrivial 2-semigroups.

Corollary 3.16. *A quasitrivial n-semigroup $\mathbb{A}$ is derived from a quasitrivial 2-semigroup if and only if $\mathbb{A} \models (\forall x)\neg\hat{O}(x)$.*

Recall that $Q_{\mathbb{A}}(a, b)$ is the quadruple of values obtained by applying $\circ_{\mathbb{A}}$ to each of $a^{n-1}b$, ab^{n-1}, b, a^{n-1} and b^{n-1}, a.

Corollary 3.17. *If $(A, \circ_{\mathbb{A}})$ and (A, g_A) are two quasitrivial n-semigroups such that $(\forall a, b)\ Q_{(A,\circ_A)}(a, b) = Q_{(A,g_A)}(a, b)$, then $(A, \circ_{\mathbb{A}}) = (A, g_A)$.*

Proof: Because then $F_n(A, \circ_{\mathbb{A}}) = F_n(A, g_A)$. □

Combining all of these results we now get the main theorem of the paper.

Theorem 3.18. *Suppose $\mathbb{A} = (A, \circ_{\mathbb{A}})$ is a quasitrivial n-semigroup.*

- *If n is even then $\mathbb{A}$ is derived from a 2-semigroup.*
- *If n is odd then $\mathbb{A}$ is derived from a 3-semigroup.*
- *If $\mathbb{A}$ is any 3-semigroup not derived from a 2-semigroup then there exists a (unique) pair of elements $b_0, b_1 \in A$ such that*
  - *$(A - \{b_0, b_1\}, \circ_{\mathbb{A}})$ is an n-semigroup derived from a 2-semigroup $\mathbb{A}_2$.*

– *For any sequence* $a_0 \cdots a_n \notin \{b_0, b_1\}^{<\omega}$ *with* $a_0^* \cdots a_{n^*}^*$ *the correspond sequence with* b_0 *and* b_1 *removed we have*

$$a_0^* \cdots a_{n^*}^* =_{\mathbb{A}_2} a \text{ if and only if } a_0 \cdots a_n =_{\mathbb{A}} a$$

i.e. b_0, b_1 *are* neutral elements*.*

– *If* $a_0 \cdots a_n \in \{b_0, b_1\}^{<\omega}$ *then* $a_0 \cdots a_n =_{\mathbb{A}} b_0$ *if and only if* b_0 *occurs an odd number of times.*

We end this section with an observation about why we used injective homomorphisms and not embeddings. Notice that if $g : \mathcal{X} \to \mathcal{Y}$ is an embedding of models of Th_{RO} then $g : G_n(\mathcal{X}) \to G_n(\mathcal{Y})$ is also an embedding of models of Th_{QSG_n}. However, the converse need not hold. The reason is that $g : G_n(\mathcal{X}) \to G_n(\mathcal{Y})$ need not preserve the negations of formulas $\hat{L}(x), \hat{R}(x)$, and $\hat{O}(x)$.

To see this note that there is a unique (up to isomorphism) one element quasitrivial n-semigroup $\langle \{*\}, \circ_* \rangle$ and that for every other quasitrivial n-semigroup A with $a \in A$ the map which takes $*$ to a is an embedding. However, such a map is only a homomorphism in $F_n(\mathbb{A})$ if a is in a trivial $\equiv$-equivalence class.

We do however have the following immediate, and important, lemma concerning homomorphisms of models of Th_{QSG_n} and morphisms in the category QSG_n.

Lemma 3.19. *For any* $\mathbb{A}, \mathbb{B}$ *models of* Th_{QSG_n}*, if* $\alpha : \mathbb{A} \to \mathbb{B}$ *is an isomorphism of* L_{QSG_n}*-structures, then* α *is also an isomorphism in RO.*

Proof: This follows immediately from Proposition 3.13 and the fact that automorphisms preserve all first order formulas. □

4. Classification of Properties

Now that we have a characterization of quasitrivial n-semigroups we will use the characterization to describe those quasitrivial n-semigroups with various nice properties.

4.1. **Universal Choice n-Semigroups.** Even though there are continuum many countable quasitrivial n-semigroups, in this section we will show that (for any n) there is a countable universal quasitrivial n-semigroup, i.e. one in which all other countable quasitrivial n-semigroups embed. This follows immediately from the following lemma.

Lemma 4.1. *There are* L_{RO}*-structures* $Univ_2$*,* $Univ_3$ *such that*

- $Univ_2, Univ_3 \models Th_{RO}$ *and* $Univ_2 \models (\forall x)\neg O(x)$*.*
- *If* $X_3 \models Th_{RO}$ *is countable then there is an embedding of* X *in* $Univ_3$*.*
- *If* $X_2 \models Th_{RO} \wedge (\forall x)\neg O(x)$ *and is countable then there is an embedding of* X *in* $Univ_2$*.*

Proof: We first construct Univ$_2$. Let $c : \mathbb{Q} \to \{0, 1, 2\}$ be any function such that

- $c^{-1}(i)$ is dense in $\mathbb{Q}$ for each $i \in \{0, 1, 2\}$.

- For all $i, j \in \{0, 1, 2\}$, $c^{-1}(i)$ is dense in $c^{-1}(\{i, j\})$.

Let $U = [c^{-1}(\{1, 2\}) \times \omega] \cup [c^{-1}(0) \times \{0\}]$. We say $\text{Univ}_2 \models (a_q, a_w) \equiv (b_q, b_w)$ if and only if $a_q = b_q$. We say $\text{Univ}_2 \models (a_q, a_w) < (b_q, b_w)$ if and only if $a_q < b_q$. Finally we say $\text{Univ}_2 \models L(a_q, a_w)$ if and only if $c(a_q) = 1$ and $\text{Univ}_2 \models R(a_q, a_w)$ if and only if $c(a_q) = 2$.

Suppose $\mathcal{X}$ is a countable model of Th_{RO} and $(\forall x)\neg O(x)$ where $X = \{x_0, x_1, x_2, \dots\}$. We can construct an embedding α of $\mathcal{X}$ in Univ_2 as follows. First, let $\alpha(x_0)$ be any element of Univ_2 which satisfies the same set of formulas from $\{L(x), R(x)\}$ as x_0 does in $\mathcal{X}$. Now suppose we have defined α on $x_1 \dots x_n$ and we wish to define $\alpha(x_{n+1})$. First, if x_{n+1} is in the same $\equiv$-equivalence class as any element of $x_0 \dots x_n$ let $\alpha(x_{n+1})$ be any element of that equivalence which is not already in the image of α (we can do this as every non-trivial $\equiv$-equivalence class in Univ_2 is infinite). If however this is not the case then suppose $\mathcal{X} \models x_i < x_{n+1} < x_j$. Because of the properties of c we can then find some a such that $\text{Univ}_2 \models L(a)$ if and only if $\mathcal{X} \models L(x_{n+1})$ and similarly for $R(a)$ and $\text{Univ}_2 \models \alpha(x_i) < a < \alpha(x_j)$. Then let $\alpha(x_{n+1}) = a$. We treat the cases where $\mathcal{X} \models \bigwedge_{i \le n} x_i < x_{n+1}$ and $\mathcal{X} \models \bigwedge_{i \le n} x_{n+1} < x_i$ similarly. In particular this shows Univ_2 satisfies the conditions of Lemma 4.1.

Finally, let $\text{Univ}_3 = \text{Univ}_2 \cup \{d_0, d_1\}$ where $\text{Univ}_3 \models d_0 \equiv d_1 \wedge O(d_0) \wedge O(d_1)$ and $\text{Univ}_3 \models (\forall a \in \text{Univ}_2)\ (d_0 < a \wedge d_1 < a) \vee d = a_0 \vee d = a_1$. It is then immediate that Univ_3 satisfies the conditions of Lemma 4.1. □

Proposition 4.2. *For every n there is a universal quasitrivial n-semigroup.*

Proof: If n is even then by Lemma 4.1 and Proposition 3.13 we have that $G_n(\text{Univ}_2)$ is a universal quasitrivial n-semigroup. Similarly if n is odd then $G_n(\text{Univ}_3)$ is a universal quasitrivial n-semigroup. □

4.2. **Ultrahomogeneous.**

Ultrahomoeneous structures are those which have the maximum amount of symmetry possible. Specifically a structure is ultrahomogeneous if every isomorphism between finitely generated substructures extends to an automorphism. We now give a complete classification of countable ultrahomogeneous quasitrivial n-semigroups.

Theorem 4.3. *A countable quasitrivial n-semigroup $\mathbb{A}$ is ultrahomogeneous if and only if it satisfies all of the following:*

1. *Either there is a single $\equiv$-equivalence class or the ordering $<$ (on $A/\equiv$) is isomorphic to $\mathbb{Q}$.*
2. *Every $\equiv$-equivalence class is the same size.*
3. *$\mathbb{A}$ satisfies the following sentences:*
   - $(\forall x, y)\hat{L}(x) \leftrightarrow \hat{L}(y)$
   - $(\forall x, y)\hat{R}(x) \leftrightarrow \hat{R}(y)$

– $(\forall x, y)\hat{O}(x) \leftrightarrow \hat{O}(y)$

Proof: Suppose $\mathbb{A}$ satisfies (1), (2) and (3) and $\overline{a}, \overline{b}$ are subtuples of A with $\alpha : \overline{a} \cong \overline{b}$ an isomorphism. Because all equivalence classes have the same size (by (2)) and all pairs of elements satisfy the same formulas from $\{\hat{L}(x), \hat{R}(x), \hat{O}(x)\}$ (by (3)) we can extend α to an isomorphism α' between those equivalence classes containing some element of $\overline{a}$ and those containing some element of $\overline{b}$. But by (1) we know that either α' is an isomorphism of our structure or it can be extended to an automorphism α^* as $\mathbb{Q}$ is ultrahomogeneous and $A/\equiv$ is isomorphic to $\mathbb{Q}$. Hence any quasitrivial n-semigroup which satisfies (1), (2) and (3) must be ultrahomogeneous.

We now show that if $\mathbb{A}$ is ultrahomogeneous it must satisfy (1), (2) and (3). First recall by Lemma 1.2 that for any $a, b \in A$, both $\{a\}$ and $\{b\}$ are finitely generated substructures of $\mathbb{A}$. The unique map from $\{a\}$ to $\{b\}$ is an isomorphism for any $a, b \in A$. Hence if $\mathbb{A}$ is ultrahomogeneous, for any $a, b \in A$ there must be an automorphism $g_{a,b} : \mathbb{A} \to \mathbb{A}$ with $g_{a,b}(a) = b$.

Because automorphisms must preserve all formulas, a and b must satisfy the same formulas among $\{\hat{L}(x), \hat{R}(x), \hat{O}(x)\}$. Hence $\mathbb{A}$ must satisfy (3). Further there is a formula $\zeta_m(x)$ which says that x has exactly m elements in its $\equiv$-equivalence class. But any two $a, b \in A$ must satisfy the same collection of formulas from $\{\zeta_m : m \in \omega\}$. Hence either all $\equiv$-equivalence classes have the same finite size or they are all infinite. But, as $\mathbb{A}$ is countable, all infinite $\equiv$-equivalence classes have the same size and hence (2) holds.

Finally suppose $<$ has order type $(T, \prec)$ on $A/\equiv$. If $i_T : \overline{a}_T \cong \overline{b}_T$ is an isomorphism of finite substructures of $(T, \prec)$ then we can find $\overline{a}, \overline{b}$ consisting of exactly one element from each equivalence class corresponding to each element in $\overline{a}_T, \overline{b}_T$. Because $\mathbb{A}$ satisfies (3) there is an isomorphism $i : \overline{a} \cong \overline{b}$. But then as $\mathbb{A}$ is ultrahomogeneous i extends to an automorphism i_* of $\mathbb{A}$. However as $\equiv$ is definable, i_* must preserve $\equiv$-equivalence classes and therefore give an automorphism of $(\mathbb{A}/\equiv, <) \cong (T, \prec)$ which extends i. Hence T is ultrahomogeneous and so must be isomorphic to either the one point set or $\mathbb{Q}$. ☐

Corollary 4.4. *For any n there is a unique ultrahomogeneous quasitrivial $2n+1$-semigroup, $\mathbb{A}^O_{2n+1}$ which is not derived from a quasitrivial 2-semigroup. Further $|A^O_{2n+1}| = 2$ and $\mathbb{A}^O_{2n+1} \models (\forall x)\hat{O}(x)$.*

Corollary 4.5. *There is no ultrahomogeneous universal quasitrivial n-semigroup (for any n).*

Proof: This is immediate from condition (3) in Theorem 4.3. ☐

It is worth mentioning that while Theorem 4.3 gives a classification of all ultrahomogeneous quasitrivial n-semigroups there are ultrahomogeneous L_{RO}-structures which do not come from ultrahomogeneous quasitrivial n-semigroups. The reason is that the condition of being an isomorphism of a finite substructure in the case of L_{RO}-structures is more restrictive than in the case of quasitrivial functions, hence less automorphism are required to exists.

Example 4.6. *Consider the L_{RO} structure with two infinite $\equiv$-equivalence classes, one which satisfies $L(x)$ and one which satisfies $R(x)$ (with all elements which satisfy $L(x)$ being $<$-less than those which satisfies $R(x)$). This is a representation and is an ultrahomogeneous L_{RO}-structure even though the quasitrivial n-semigroup it represents is not ultrahomogeneous (as a L_{QSG_n}-structure).*

4.3. **Commutative.** We now give a complete classification of commutative quasitrivial n-semigroups.

Definition 4.7. *An n-ary function $\circ_{\mathbb{A}} : A^n \to A$ is* **commutative** *if for all permutations σ of n:*

$$(\forall a_1, \ldots, a_n \in A) \ \circ_{\mathbb{A}} (a_1, \ldots, a_n) = \circ_{\mathbb{A}}(a_{\sigma(a)}, \ldots, a_{\sigma(n)}).$$

Theorem 4.8. *A quasitrivial n-semigroup is commutative if and only if no element satisfies $\hat{L}(x)$ or $\hat{R}(x)$.*

Proof: The implication from right to left follows immediately from Table 1. The implication from left to right follows from the fact that the right side implies any non-trivial $\equiv$-class is the minimal one with elements satisfying $O(x)$. But $\circ_{\mathbb{A}}$ is necessarily commutative on those elements satisfying $O(x)$. $\square$

Corollary 4.9. *All countable commutative ultrahomogeneous quasitrivial n-semigroups are derived from one of the following three:*

- *The unique one element quasitrivial n-semigroup.*
- *The two element quasitrivial 3-semigroup $\mathbb{A}_3^O$.*
- *The quasitrivial 2-semigroup for which $\equiv$ is trivial and $<$ has order type $\mathbb{Q}$.*

Corollary 4.10. *A quasitrivial n-semigroup is commutative and derived from a quasitrivial 2-semigroup if and only if it is the* max *operation on a linear order.*

4.4. **Trivial Algebraic Closure.** In this section we give a continuum size collection of quasitrivial n-semigroups all of which have trivial "group theoretic" algebraic closure and a continuum sized collection which do not. We will also completely classify those ultrahomogeneous quasitrivial n-semigroup which have trivial (group theoretic) algebraic closure.

Definition 4.11. *The* **algebraic closure** *of a tuple $\overline{a}$, denoted $acl(\overline{a})$, is the set of b such that the collection*

$$\{g(b) : g \text{ is an automorphism which fixes } \overline{a} \text{ pointwise}\}$$

is finite. We say $\mathbb{A}$ has **trivial algebraic closure** *if $\overline{a} = acl(\overline{a})$ for all finite tuples $\overline{a}$ of $\mathbb{A}$.*

Lemma 4.12. *If a quasitrivial n-semigroup $\mathbb{A}$ has a finite non-trivial $\equiv$-class then it does not have trivial algebraic closure.*

Proof: This is because if a is in such an $\equiv$-equivalence class E then $E \subseteq acl(a)$. $\square$

Lemma 4.13. *If every $\equiv$-equivalence class is infinite then $\mathbb{A}$ has trivial algebraic closure.*

Proof: First note that if $\mathbb{A} \models a\hat{\equiv}b$ and $g_{a,b}$ is the automorphism of the set A where $g_{a,b}(a) = b, g_{a,b}(b) = a$ and $g_{a,b}(c) = c$ for $c \in A - \{a, b\}$, then $g_{a,b}$ is in fact an automorphism of the L_{QSG_n}-structure $\mathbb{A}$. Now suppose $\overline{a}$ is a finite subset of A and $a \in A - \overline{a}$. Because each $\equiv$-equivalence class is infinite and $\overline{a}$ is finite we can find a collection $\{a_i : i \in \omega\} \subseteq A - \overline{a}$ such that $\mathbb{A} \models a \equiv a_i$ for all $i \in \omega$. Then g_{a,a_i} is an automorphism of $\mathbb{A}$ which fixes $\overline{a}$ and so $\{a_i : i \in \omega\} \subseteq \{g(a) : g$ is an automorphism which fixes $\overline{a}$ pointwise$\}$. But this implies $a \notin acl(\overline{a})$ and so, as a was arbitrary, we have $acl(\overline{a}) = \overline{a}$ and $\mathbb{A}$ has trivial algebraic closure. $\square$

Corollary 4.14. *For every linear ordering $(T, \prec)$ there are quasitrivial n-semigroup $\mathbb{A}_0^T, \mathbb{A}_1^T$ where*

- $(A_0^T/\equiv, <) \cong (A_1^T/\equiv, <) \cong (T, \prec)$
- $\mathbb{A}_0^T$ *has trivial algebraic closure but* $\mathbb{A}_1^T$ *has non-trivial algebraic closure.*

Proof: For $i \in \{0, 1\}$ let $\mathbb{A}_i^T$ be such that $(A_i^T/\equiv, <) \cong (T, \prec)$ and such that $(\forall a)\hat{L}(a)$. Further let $\mathbb{A}_0^T$ be such that all equivalence classes have infinitely many elements and let $\mathbb{A}_1^T$ be such that all equivalence classes have 2 elements. $\square$

In particular this gives a continuum sized collection of quasitrivial n-semigroups which have trivial algebraic closure and a continuum sized collection which don't.

Proposition 4.15. *A countable ultrahomogeneous quasitrivial n-semigroup $\mathbb{A}$ has trivial algebraic closure if and only if it satisfies one of the following:*

1. *There is only one $\equiv$-equivalence class and it is infinite or contains a single element.*
2. *We have*
   - *Each $\equiv$-equivalence class is infinite (and in particular $\mathbb{A} \models (\forall x)\neg\hat{O}(x)$).*
   - *The ordering $<$ (on $A/\equiv$) is isomorphic to $\mathbb{Q}$.*
   - $\mathbb{A} \models (\forall x, y)\hat{L}(x) \leftrightarrow \hat{L}(y)$ *(and hence also* $\mathbb{A} \models (\forall x, y)\hat{R}(x) \leftrightarrow \hat{R}(y)$*).*
3. *Every equivalence class is trivial and the ordering $<$ is isomorphic to $\mathbb{Q}$.*

Proof: It is clear from Lemma 4.12 and Theorem 4.3 that any countable ultrahomogeneous quasitrivial n-semigroup with trivial algebraic closure must be among (1), (2) or (3). Further it is immediate from Lemma 4.13 that (1) and (2) are both ultrahomogeneous and have trivial algebraic closure. Finally, because $\mathbb{Q}$ also has trivial algebraic closure the unique quasitrivial n-semigroup satisfying (3) must also. □

Lemma 4.16. *A countable commutative quasitrivial n-semigroup $\mathbb{A}$ has trivial algebraic closure if and only if A has a single element or $\widehat{<}$ on A is isomorphic to $\mathbb{Q}$.*

Proof: By Theorem 4.8 no commutative quasitrivial n-semigroup can have an infinite non-trivial $\equiv$-equivalence class. But by Lemma 4.12 no quasitrivial n-semigroup with trivial algebraic closure can have an $\equiv$-equivalence class which is finite but non-trivial. Hence every $\equiv$-equivalence class must be trivial and $\widehat{<}$ must be a linear ordering. But the only countable linear orderings with trivial algebraic closure are the one point linear order and $\mathbb{Q}$. □

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