.

Completeness in Generalized Ultrametric Spaces

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We analyze the relationship between four notions of completeness for Γ-ultrametric spaces. The notions we consider include Cauchy completeness, strong Cauchy completeness, spherical completeness and injectivity. In the process we show that the category of Γ-ultrametric spaces is equivalent to the category of flabby separated presheaves on Γ^{op}.

1. INTRODUCTION

Γ-ultrametric spaces are spaces which satisfy all the axioms of an ultrametric space except that the distance function takes values in a complete lattice Γ instead of $\mathbb{R}^{\geq 0}$. Γ-ultrametric spaces have been extensively studied as a way to weaken the notion of an ultrametric space[1] while still providing enough structure to be useful (see for example [17], [18], [8]). The many uses of Γ-ultrametric spaces include being a valuable context to study equivalence relations ([19]), trees ([2]), domain theory ([12]), and logic programming ([**?**], [9], [21]).

The goal of this paper is to characterize the relationship between four important notions of completeness for Γ-ultrametric spaces: spherical completeness, Cauchy completeness, strong Cauchy completeness[2] and injectivity.

We begin in Section 2.1 by reviewing the definition of a Γ-ultrametric space and of related notions such as the closed ball functor, γ-Cauchy nets and γ-Cauchy functions. In Section 2.2 we review the definition of spherical completeness and give two characterizations of spherical completeness in terms of γ-Cauchy nets. Finally in Section 2.3 we give the definition of Cauchy completeness, strong Cauchy completeness and injectivity. We also give a characterization of strong Cauchy completeness in terms of the closed ball functor, in terms of spherical completeness, as well as showing that the notion of strong Cauchy completeness coincides with injectivity for Γ-ultrametric spaces.

In Section 3 we will consider the notion of a completion of a Γ-ultrametric space. In Section 3.1 we will show that the category of Γ-ultrametric spaces is equivalent to the category of skeletal symmetric Γ^{op}-enriched categories. We then use this equivalence to show that every Γ-ultrametric space has a Cauchy completion. In Section 3.2 we show not all Γ-ultrametric spaces have a strong Cauchy completions even though each Γ-ultrametric space does have a minimal strong Cauchy complete extension. In the process we will show, in Section 3.2.1, that the category of Γ-ultrametric spaces is equivalent to the category of flabby separated presheaves on Γ^{op}.

1.1. Background

In this paper we will work in a fixed background model of Zermelo-Frankael set theory. In general we will not assume the axiom of choice unless necessary. If a result does use the axiom of choice we will mark it by (∗). All categories in this paper will be locally small. We will use the convention that when C is a category with objects A and B, $C[A,B]$ is the set of morphisms whose domain is A and whose codomain is B. We

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[1] There are several ways in which one can generalize the notion of ultrametric space which we do not consider. For more information on some of these see notions see [13] or [22].

[2] The notion of strong Cauchy completeness is introduced in this paper as a generalization of both Cauchy completeness and spherical completeness.

also abuse notation and say $x \in C$ when x is an object of C. The reader is referred to such standard texts as [14] for general category theoretic notions, to [15] for sheaf theoretic notions, to [23] for enriched category theoretic notions and to [8], [17] and [18] for Γ-ultrametric space notions.

2. Γ-ULTRAMETRIC SPACES

2.1. Γ-Ultrametric Spaces

Definition 2.1. *A partial order* $(\Gamma, \leq_\Gamma)$ *is a* **complete lattice** *if every subset of* Γ *has a supremum and infimum. A complete lattice* $(\Gamma, \leq_\Gamma)$ *is an* **op-frame** *if it satisfies*

- $(\forall x \in \Gamma, A \subseteq \Gamma)\ x \vee_\Gamma (\bigwedge^\Gamma_{\alpha \in A} \alpha) = \bigwedge^\Gamma_{\alpha \in A}(x \vee_\Gamma \alpha)$.

From here on we will refer to a complete lattice $(\Gamma, \leq_\Gamma)$ simply as Γ when no confusion can arise. We will refer to the least element, $\bigwedge^\Gamma\{\gamma : \gamma \in \Gamma\}$, as $\bot_\Gamma$ and the greatest element, $\bigvee^\Gamma\{\gamma : \gamma \in \Gamma\}$, as $\top_\Gamma$. We will also drop the subscripts and superscripts when they are clear from context. If Γ is a complete lattice we let Γ^{op} be the complete lattice with the same underlying set as Γ but with the order reversed. Note that a complete lattice Γ is an op-frame if and only if Γ^{op} is a frame. In particular, if Γ is an op-frame then $\langle \Gamma^{op}, \leq_{op}, \wedge \rangle$ can be viewed as a monodial category (which we will just refer to as Γ^{op}). For the rest of the paper, Γ will always be an op-frame.

Definition 2.2. *A* **Γ-ultrametric space** *is a pair* (M, d_M) *where* M *is a set and* $d_M : M \times M \to \Gamma$ *is a function that satisfies:*

- $(\forall x, y)\ d_M(x, y) = \bot \leftrightarrow x = y$.
- *(Symmetry)* $(\forall x, y \in M)\ d_M(x, y) = d_M(y, x)$.
- *(Strong Triangle Inequality)* $(\forall x, y, z \in M)\ d_M(x, y) \vee d_M(y, z) \geq d_M(x, z)$.

A **non-expanding map** *between Γ-ultrametric spaces* (M, d_M) *and* (N, d_N) *is a function* $f : M \to N$ *such that* $(\forall a, b \in M)\ d_N(f(a), f(b)) \leq d_M(a, b)$. *An* **isometry** *is a map* $f : (M, d_M) \to (N, d_N)$ *where* $(\forall a, b \in M)\ d_N(f(a), f(b)) = d_M(a, b)$.

It is easily checked that Γ-ultrametric spaces and non-expanding maps form a category which we call UMet(Γ). Further, the isometries are the monomorphism in this category.

Definition 2.3. *If* (M, d_M) *is a Γ-ultrametric space then for each* $x \in M$ *and* $\gamma \in \Gamma$ *we define the* **(closed) ball of radius** γ **around** x *to be*

$$B^M(x, \gamma) = \{y : d_M(x, y) \leq \gamma\}.$$

We will omit mention of M *when it is clear from context.*

For the rest of the paper (M, d_M) and (N, d_N) will always be Γ-ultrametric spaces.

Lemma 2.4. *Let*

- $\{x_i : i \in I\} \subseteq M$ *and* $\{\gamma_i : i \in I\} \subseteq \Gamma$.
- $\alpha = \bigwedge\{\gamma_i : i \in I\}$ *and* $\beta = \bigvee\{\gamma_i : i \in I\}$.
- $B = \bigcap_{i \in I} B(x_i, \gamma_i)$.

If $B \neq \emptyset$ *then*

(1) $(\forall i, j \in I) B(x_i, \beta) = B(x_j, \beta)$.

(2) $(\forall x \in B) B = B(x, \alpha)$.

Proof. Let $x \in B$. To see (1) observe that $x \in B(x_i, \gamma_i)$ and $x \in B(x_j, \gamma_j)$ so $B(x_i, \beta) = B(x, \beta) = B(x_j, \beta)$.

To see (2) let $\zeta_B = \bigvee\{d_M(x, y) : x, y \in B\}$. As $(\forall i \in I) x \in B(x_i, \gamma_i)$ we have that $B(x, \gamma_i) = B(x_i, \gamma_i)$. Hence $(\forall i \in I) B \subseteq B(x, \gamma_i)$ and $\zeta_B \leq \gamma_i$. Therefore $\zeta_B \leq \alpha$ and in particular $B \subseteq B(x, \alpha)$. But we also have $(\forall i \in I) B(x, \alpha) \subseteq B(x, \gamma_i)$ and so $B(x, \alpha) \subseteq B$. Therefore (2) holds. □

2.1.1. Closed Ball Functor

An important property of the collection of closed balls (of a fixed radius) of a generalized ultrametric space is that they themselves form a generalized ultrametric space (although with a slightly different set of distances).

Definition 2.5. *Suppose* $(\Gamma, \leq_\Gamma)$ *is an op-frame and* $\gamma \in \Gamma$. *We define* $(\Gamma_\gamma, \leq_{\Gamma_\gamma})$ *where*

- $\Gamma_\gamma = \{\zeta \in \Gamma : \zeta \geq_\Gamma \gamma\}$.
- $(\forall x, y \in \Gamma_\gamma) x \leq_{\Gamma_\gamma} y \leftrightarrow x \leq_\Gamma y$.

It is easy to check that Γ_γ *is an op-frame and for any* $A \subseteq \Gamma_\gamma$, $\bigwedge^\Gamma A = \bigwedge^{\Gamma_\gamma} A$ *and* $\bigvee^\Gamma A = \bigvee^{\Gamma_\gamma} A$.

Definition 2.6. *Suppose* $\gamma \in \Gamma$. *We define* (M_γ, d_{M_γ}) *where*

- $M_\gamma = \{B^M(x, \gamma) : x \in M\}$.
- $d_{M_\gamma}(B^M(x, \gamma), B^M(y, \gamma)) = d_M(x, y) \vee \gamma$.

If $f : (M, d_M) \to (N, d_N)$ *is a map of* Γ*-ultrametric spaces, let* $f_\gamma : (M_\gamma, d_{M_\gamma}) \to (N_\gamma, d_{N_\gamma})$ *be such that* $f(B^M(x, \gamma)) = B^N(f(x), \gamma)$.

In Definition 2.6 the distance function is well defined because whenever $z \in B^M(x, \gamma)$, we have

$$d_{M_\gamma}(B(x, \gamma), B(y, \gamma)) = d_M(x, y) \vee \gamma = d_M(z, y) \vee \gamma = d_{M_\gamma}(B(z, \gamma), B(y, \gamma))$$

with the middle equality following from the strong triangle inequality and the fact that $d_M(x, z) \leq \gamma$. It is then simple to check that (M_γ, d_{M_γ}) is a Γ_γ-ultrametric space. Further, if $y \in B^M(x, \gamma)$ then $f(y) \in B^N(f(x), \gamma)$ (as f is non-expanding) and so f_γ is a well-defined non-expanding map of Γ_γ-ultrametric spaces (that it is non-expanding follows from the fact that f is non-expanding).

The following lemma is then immediate.

Lemma 2.7. $(\cdot)_\gamma$ *is a functor from UMet*(Γ) *to UMet*(Γ_γ).

We call $(\cdot)_\gamma$ a **closed ball functor**.

2.1.2. Cauchy Functions and Nets

We now introduce γ-Cauchy functions and γ-Cauchy nets (which are a special type of γ-Cauchy function). We can think of a γ-Cauchy function as a (possibly filled) "hole" or "idealized point" whose size is that of a ball of radius γ.

Definition 2.8. *For* $\gamma \in \Gamma$, *a* γ**-Cauchy function** *is a function* $t : A \to M$ *where*

- $A \subset \Gamma$ *and* $\bigwedge A = \gamma$.
- $(\forall \gamma', \gamma'' \in A)\ d_M(t(\gamma'), t(\gamma'')) \leq \gamma' \vee \gamma''$.

We say a γ*-Cauchy function,* $t : A \to M$, **converges to** x *(with* $x \in M$*) if*

$$(\forall \zeta \in A)\ d_M(x, t(\zeta)) \leq \zeta.$$

We refer to a $\perp_\Gamma$*-Cauchy function as simply a* **Cauchy function**.

Definition 2.9. *For* $\gamma \in \Gamma$, *a* γ**-Cauchy net**[3] *is a* γ*-Cauchy function* $t : A \to M$ *such that* $(\forall \gamma', \gamma'' \in A)\gamma' \wedge \gamma'' \in A$. *We refer to a* $\perp_\Gamma$*-Cauchy net as simply a* **Cauchy net**.

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[3] We can turn (M, d_M) into a uniform space by considering the entourage generated by $U_\alpha = \{(x, y) : d_M(x, y) \leq \alpha\}$. A map $t : A \to M$ is then a Cauchy net in the uniform space sense if and only if it is a Cauchy net in the sense of Definition 2.9.

Notice

Lemma 2.10. *For $x, y \in M$ the following are equivalent*

(1) For all γ-Cauchy functions t, t converges to x if and only if t converges to y.

(2) For all γ-Cauchy nets t, t converges to x if and only if t converges to y.

(3) $B(x,\gamma) = B(y,\gamma)$.

Proof. (2) $\Rightarrow$ (3) follows from the fact that if $t_x : \{\gamma\} \to M$ with $t_x(\gamma) = x$ then t_x is a γ-Cauchy net and so $d_M(x,y) \leq \gamma$. (3) $\Rightarrow$ (2) follows immediate from the definition of convergence of a γ-Cauchy net.
The proof that (1) $\Leftrightarrow$ (3) is the same as that (2) $\Leftrightarrow$ (3). □

Corollary 2.11. *Suppose t is a Cauchy function. If t converges, then it converges to a unique point.*

2.2. *Spherical Completeness*

The notion of a spherically complete ultrametric space was first introduced by Ingleton in [10] in order to study the Hanh-Banach theorem in non-Archimedean valued fields and has since found applications in many areas, such as study of the p-adics (see [20]) and the study of fixed points (see [16]). The notion of spherical completeness was extended to Γ-ultrametric spaces, for arbitrary partial orders Γ, by Priess-Crampe and Ribenboim ([16]). It is this notion which we will consider here.

Definition 2.12. (M, d_M) *is* **spherically complete** *if, for all infinite cardinals κ, whenever*

- $\{\gamma_i : i < \kappa\} \subseteq \Gamma$ *and* $\{x_i : i < \kappa\} \subseteq M$.
- $B(x_i, \gamma_i) \subseteq B(x_j, \gamma_j)$ *if* $i \geq j$.

then $\bigcap_{i<\kappa} B(x_i, \gamma_i) \neq \emptyset$.

Lemma 2.13. *Suppose (M, d_M) is spherically complete, κ is a cardinal, and*

- $\{\gamma_i : i < \kappa\} \subseteq \Gamma$ *and* $\{x_i : i < \kappa\} \subseteq M$.
- $B(x_i, \gamma_i) \subseteq B(x_j, \gamma_j)$ *if* $i \geq j$.

is a decreasing chain of closed balls. Then $(\forall x \in \bigcap_{i<\kappa} B(x_i, \gamma_i))\ B(x, \bigwedge_{i<\kappa} \gamma_i) = \bigcap_{i<\kappa} B(x_i, \gamma_i)$.

Proof. This is an immediate consequence of Lemma 2.4. □

2.2.1. *Cauchy Nets*

There is a simple characterization of spherical completeness in terms of γ-Cauchy nets.

Proposition 2.14 **(*).** *(M, d_M) is spherically complete if and only if for all $\gamma \in \Gamma$ every γ-Cauchy net converges.*

Proof. Right implies Left:
Assume every γ-Cauchy net in (M, d_M) converges.
Suppose $\{x_i : i < \kappa\} \subseteq M$ and $\{\gamma_i : i < \kappa\} \subseteq \Gamma$ are such that $B(x_i, \gamma_i) \subseteq B(x_j, \gamma_j)$ if $i \geq j$. If $t(\gamma_i) = x_i$, then t is a $\bigwedge_{i<\kappa} \gamma_i$-Cauchy net and hence converges to a point x. For all $i < \kappa$, $d_M(x, x_i) = d_M(x, t(\gamma_i)) \leq \gamma_i$ and hence $x \in B(x_i, \gamma_i)$. So $x \in \bigcap_{i<\kappa} B(x_i, \gamma_i) \neq \emptyset$. Hence, as $\{x_i : i < \kappa\}$ and $\{\gamma_i : i < \kappa\}$ were arbitrary, (M, d_M) is spherically complete.

Left implies Right:
Assume (M, d_M) is spherically complete.
Let $t : A \to M$ be a γ-Cauchy net. We want to show that t converges to a point. We well-order $A = \langle \gamma_i : i < \kappa \rangle$ and define B_α by induction on α as follows:

(Base Case) $B_1 = B(t(\gamma_0), \gamma_0)$.

(Limit Case) $B_\alpha = \bigcap_{j<\alpha} B_j$.

(Successor Case) $B_{\alpha+1} = B(t(\gamma_\alpha), \gamma_\alpha) \cap B_\alpha$.

Notice that if $B_\alpha \neq \emptyset$ then B_α is a ball (by Lemma 2.4). Also notice $B_\alpha \subseteq B_{\alpha'}$ if $\alpha \geq \alpha'$. $\bigcap_{i<\kappa} B_i$ will contain an element which t converges to.

Claim 2.15. *For all $\alpha \leq \kappa$ and for all $\zeta \in A$, $B_\alpha \cap B(t(\zeta), \zeta) \neq \emptyset$.*

Proof. We are going to prove this claim by induction on α. In what follows fix $\zeta \in A$.

Base Case: $\alpha = 1$.
$t(\zeta \wedge \gamma_0) \in B(t(\gamma_0), \gamma_0) \cap B(t(\zeta), \zeta) \neq \emptyset$.

Limit Case: α is a limit and the claim holds for all $j < \alpha$.
Notice that $B(t(\zeta), \zeta) \cap B_\alpha = B(t(\zeta), \zeta) \cap \bigcap_{j<\alpha} B_j = \bigcap_{j<\alpha}[B(t(\zeta), \zeta) \cap B_j]$. But for all $j < \alpha$, $B(t(\zeta), \zeta) \cap B_j \neq \emptyset$ by the inductive hypothesis. Further $\langle B(t(\zeta), \zeta) \cap B_j : j < \alpha \rangle$ is a decreasing sequence of balls. Hence $\bigcap_{j<\alpha} B(t(\zeta), \zeta) \cap B_j \neq \emptyset$ because (M, d_M) is spherically complete.

Successor Case: $\alpha = \alpha' + 1$ and the claim holds for α'.

$$\begin{aligned} B(t(\zeta), \zeta) \cap B_{\alpha'+1} &= B(t(\zeta), \zeta) \cap [B(t(\gamma_{\alpha'}), \gamma_{\alpha'}) \cap B_{\alpha'}] \\ &= (B(t(\zeta), \zeta) \cap B(t(\gamma_{\alpha'}), \gamma_{\alpha'})) \cap B_{\alpha'}. \end{aligned}$$

But $t(\zeta \wedge \gamma_{\alpha'}) \in B(t(\zeta), \zeta) \cap B(t(\gamma'_\alpha), \gamma'_\alpha)$ (as A is a directed set) and so $B(t(\zeta), \zeta) \cap B(t(\gamma'_\alpha), \gamma'_\alpha) = B(t(\zeta \wedge \gamma_{\alpha'}), \zeta \wedge \gamma_{\alpha'})$ by Lemma 2.4. So $B(t(\zeta), \zeta) \cap B_\alpha = B(t(\zeta \wedge \gamma_{\alpha'}), \zeta \wedge \gamma_{\alpha'}) \cap B_{\alpha'}$ and hence is non-empty by the inductive hypothesis. □

In particular Claim 2.15 implies that $\bigcap_{\alpha<\kappa} B_\alpha = B_\kappa \neq \emptyset$. Suppose $x \in \bigcap_{\alpha<\kappa} B_\alpha$. Then for all $\gamma_\alpha \in A$, $x \in B(t(\gamma_\alpha), \gamma_\alpha)$ by construction and hence $d_M(t(\gamma_\alpha), x) \leq \gamma_\alpha$. So t converges to x.
As t was an arbitrary γ-Cauchy net we are done. □

2.2.2. Space of Closed Balls

As we will see in this section, there is also a characterization of spherically complete Γ-ultrametric spaces in terms of Cauchy nets and the closed ball functor.

Proposition 2.16 (*). *(M, d_M) is spherically complete if and only if for all $\gamma \in \Gamma$, every Cauchy net in (M_γ, d_{M_γ}) converges.*

Proof. Left implies Right:
Suppose (M, d_M) is spherically complete, $\gamma \in \Gamma$ and $A \subseteq \Gamma_\gamma$. Let $t : A \to M_\gamma$ be a Cauchy net in (M_γ, d_{M_γ}) with $t(\alpha) = B(t'(\alpha), \gamma)$ for some map $t' : A \to M$. Then $\alpha \vee \beta \geq d_{M_\gamma}(t(\alpha), t(\beta)) = d_M(t'(\alpha), t'(\beta)) \vee \gamma \geq d_M(t'(\alpha), t'(\beta))$. So, as $\gamma = \perp_{\Gamma_\gamma} = \bigwedge A$, t' is a γ-Cauchy net in (M, d_M).
By our assumption (M, d_M) was spherically complete and so, by Proposition 2.14, t' must converge to a point x with $d_M(x, t'(\alpha)) \leq \alpha$ for all $\alpha \in A$. This then implies for all $\alpha \in A$ that $\alpha = \alpha \vee \gamma \geq d_M(x, t'(\alpha)) \vee \gamma = d_{M_\gamma}(B(x, \gamma), t(\alpha))$. Hence t converges to $B(x, \gamma) \in M_\gamma$. As A was arbitrary, all Cauchy nets in M_γ converge.

Right implies Left:
Suppose for all $\gamma \in \Gamma$ every Cauchy net in (M_γ, d_{M_γ}) converges. Let $A \subseteq \Gamma$ and $t : A \to M$ be a γ-Cauchy net in (M, d_M). By the definition of a γ-Cauchy net we have that $\bigwedge A = \gamma = \perp_{\Gamma_\gamma}$.
Let $t^* : A \to M_\gamma$ be given by $t^*(\alpha) = B(t(\alpha), \gamma)$. Then $d_{M_\gamma}(t^*(\alpha), t^*(\beta)) = d_M(t(\alpha), t(\beta)) \vee \gamma \leq \alpha \vee \beta \vee \gamma = \alpha \vee \beta$. Hence t^* is a Cauchy net in (M_γ, d_{M_γ}) and so must converge to a point $B(x, \gamma)$. But then $d_M(t(\alpha), x) \leq d_M(t(\alpha), x) \vee \gamma = d_{M_\gamma}(t^*(\alpha), x) \leq \alpha$ and so t converges to x.
In particular, since $\gamma \in \Gamma$ and t were arbitrary, every γ-Cauchy net in (M, d_M) converges and Proposition 2.14, (M, d_M) is spherically complete.

□

2.3. Cauchy Completeness

Definition 2.17. *We say (M, d_M) is* **Cauchy complete** *if every $(\bot-)$Cauchy function converges. We say (M, d_M) is* **strongly Cauchy complete** *if for each $\gamma \in \Gamma$ every γ-Cauchy function converges.*

2.3.1. Pairwise and Spherical Completeness

We now characterize the relationship between spherical completeness and strong Cauchy completeness.

Definition 2.18. *We say (M, d_M) is* **pairwise complete** *if for all x, y in M and for all γ_x, γ_y in Γ*

$$[B(x, \gamma_x \vee \gamma_y) = B(y, \gamma_x \vee \gamma_y)] \Rightarrow B(x, \gamma_x) \cap B(y, \gamma_y) \neq \emptyset.$$

The following lemma shows us in pairwise complete spaces it suffices to consider Cauchy nets for purposes of completeness.

Lemma 2.19 (*). *If (M, d_M) is pairwise complete and $t : A \to M$ is a γ-Cauchy function, then there is a γ-Cauchy net $\bar{t} : \overline{A} \to M$ with $A \subseteq \overline{A}$ and $t \subseteq \bar{t}$.*

Proof. We define A_α and t_α by induction on α.
Base Case: $\alpha = 0$. $A_0 = A$ and $t_0 = t$.

Limit Case: α is a limit ordinal. $A_\alpha = \bigcup_{i<\alpha} A_i$ and $t_\alpha = \bigcup_{i<\alpha} t_i$.

Successor Case: $\alpha = \alpha' + 1$. We have two cases to deal with.
Case 1: $t_{\alpha'}$ is a γ-Cauchy net.
Let $A_\alpha = A_{\alpha'}$ and $t_\alpha = t_{\alpha'}$.
Case 2: $(\exists a_0, a_1 \in A_\alpha) a_0 \wedge a_1 \notin A_{\alpha'}$.
By virtue of being a γ-Cauchy function $d_M(t_\alpha(a_0), t_\alpha(a_1)) \leq a_0 \vee a_1$ and so $B(t_\alpha(a_0), a_0 \vee a_1) = B(t_\alpha(a_1), a_0 \vee a_1)$. Hence, because our space is pairwise complete, there is an $x \in B(t_\alpha(a_0), a_0) \cap B(t_\alpha(a_1), a_1)$. Let $A_\alpha = A_{\alpha'} \cup \{a_0 \wedge a_1\}$ and $t_\alpha = t_{\alpha'} \cup \{(a_0 \wedge a_1, x)\}$.
Notice that for any $b \in A_{\alpha'}$ we have

$$d_M(x, t_\alpha(b)) \leq d_M(x, t_\alpha(a_0)) \vee d_M(t_\alpha(a_0), t_\alpha(b)) \leq a_0 \vee [a_0 \vee b] = a_0 \vee b.$$

Similarly we have $d_M(x, t_\alpha(b)) \leq a_1 \vee b$. Hence $d_M(x, t_\alpha(b)) \leq [a_0 \vee b] \wedge [a_1 \vee b] = [a_0 \wedge a_1] \vee b$. So t_α is a γ-Cauchy function on A_α.

Eventually this induction must stabilize. If α is the first point at which this happens, t_α is a γ-Cauchy net and we can let $\overline{A} = A_\alpha$ and $\bar{t} = t_\alpha$.

□

Theorem 2.20. *(*) (M, d_M) is spherically complete and pairwise complete if and only if it is strongly Cauchy complete.*

Proof. Right implies Left:
Assume (M, d_M) is strongly Cauchy complete.
As every γ-Cauchy net is a γ-Cauchy function, every γ-Cauchy net converges. Hence by Proposition 2.14 (M, d_M) is spherically complete.
Similarly suppose $x, y \in M, \gamma_x, \gamma_y \in \Gamma$ and $B(x, \gamma_x \vee \gamma_y) = B(y, \gamma_x \vee \gamma_y)$. If $t(\gamma_x) = x$ and $t(\gamma_y) = y$ then t is a $\gamma_x \wedge \gamma_y$-Cauchy function and so must converge to a point a. But then $d_M(a, x) \leq \gamma_x$ and $d_M(a, y) \leq \gamma_y$ so $a \in B(x, \gamma_x) \cap B(y, \gamma_y)$. Hence, as x, y, γ_x, γ_y were arbitrary (M, d_M) is pairwise complete.

Left implies Right:
Assume (M, d_M) is spherically complete and pairwise complete.
Let $\bar{t}$ and $\overline{A}$ be as in Lemma 2.19. If $\bar{t}$ converges to a point x then so does t. But $\bar{t}$ is a γ-Cauchy net and (M, d_M) is spherically complete. So by Proposition 2.14 $\bar{t}$ converges and hence t converges. So, as t was arbitrary, (M, d_M) is strongly Cauchy complete. □

Even though all strongly Cauchy complete spaces are spherically complete it is not necessarily the case that all spherically complete spaces are strongly Cauchy complete. In particular these two notions coincide if and only if Γ is linearly ordered.

Lemma 2.21. *If Γ is linearly ordered then every Γ-ultrametric space is pairwise complete.*

Proof. Suppose Γ is linearly ordered. Now let $x, y \in M$ and $\gamma_x, \gamma_y \in \Gamma$ with $B(x, \gamma_x \vee \gamma_y) = B(y, \gamma_x \vee \gamma_y)$. We can then assume, without loss of generality, that $\gamma_x \leq \gamma_y$ (as Γ is linearly ordered). But then $B(x, \gamma_y) = B(y, \gamma_y)$ and so $B(x, \gamma_x) \subseteq B(y, \gamma_y)$. Hence $B(x, \gamma_x) \cap B(y, \gamma_y) = B(x, \gamma_x) \neq \emptyset$. So (M, d_M) is pairwise complete. □

Lemma 2.22. *The following are equivalent:*

(a) Γ is linearly ordered.

(b) Every spherically complete Γ-ultrametric space is strongly Cauchy complete.

Proof. (a) implies (b) follows immediately from Theorem 2.20 and Lemma 2.21.

To see $\neg(a)$ implies $\neg(b)$ suppose $\gamma_0, \gamma_1 \in \Gamma$ where $\gamma_0 \not\leq \gamma_1$ and $\gamma_1 \not\leq \gamma_0$. Let $M = \{x, y, z\}$ be the space where $d_M(x, z) = \gamma_0$, $d_M(y, z) = \gamma_1$ and $d_M(x, y) = \gamma_0 \vee \gamma_1$. (M, d_M) is spherically complete as any finite space is spherically complete. But (M, d_M) is not pairwise complete as $B(x, \gamma_1) \cap B(y, \gamma_0) = \emptyset$ and $B(x, \gamma_0 \vee \gamma_1) = B(y, \gamma_0 \vee \gamma_1) = M$. So, by Theorem 2.20, (M, d_M) is not strongly Cauchy complete. □

In particular, if $\gamma_0 \wedge \gamma_1 = \bot$ then (M, d_M) in Lemma 2.22 is spherically complete but not Cauchy complete. We also have:

Corollary 2.23. *The following are equivalent:*

(a) Γ is linearly ordered.

(b) Every Γ-ultrametric space is pairwise complete.

There are also spaces which are Cauchy complete and not spherically complete.

Example 2.24. *Notice that $(\omega + \omega + 1)^{op}$ is an op-frame and that $2^{\omega+\omega}$ can be made into an $(\omega + \omega + 1)^{op}$-ultrametric space by letting $d_{2^{\omega+\omega}}(x, y) = \bigwedge^{(\omega+\omega+1)^{op}}$[4] $\{m \in \omega + \omega + 1 : x|_m$[5] $= y|_m\}$.*

The way $2^{\omega+\omega}$ is made into a $(\omega + \omega + 1)^{op}$-ultrametric space is analogous to the way 2^{ω} is made into an (normal) ultrametric space. By a similar argument, it is clear that $2^{\omega+\omega}$ is strongly Cauchy complete (which is equivalent to being spherically complete as $\omega + \omega + 1$ is linearly ordered).

Now let $E = \{x \in 2^{\omega+\omega} : (\exists n \in \omega)\ x(n) = 0\} \subseteq 2^{\omega+\omega}$. Suppose $\boldsymbol{x} = \langle x_i : i \in A\rangle$ is a Cauchy function in E with $A \subseteq (\omega + \omega + 1)^{op}$ and $\bigwedge^{(\omega+\omega+1)^{op}} A = \bot_{(\omega+\omega+1)^{op}} = \omega + \omega$. Then $\boldsymbol{x}$ converges to a unique point $x' \in 2^{\omega+\omega}$. As $\bigwedge^{(\omega+\omega+1)^{op}} A = \omega + \omega$ there must be an $\alpha \in A$ with $\alpha > \omega$. But $x'|_\omega = x_\alpha|_\omega$ and by construction of E, if $x_\alpha \in E$ then $\{y \in E : y|_\omega = x_\alpha|_\omega\} = \{y \in 2^{\omega+\omega} : y|_\omega = x_\alpha|_\omega\}$. So $x' \in E$ and E is Cauchy complete.

Consider the sequence $\langle x_i : i \in \omega\rangle$ where $x_i(i+1) = 0$ and otherwise $x_i(n) = 1$ for $n \in \omega + \omega - \{i+1\}$. Let $c_i : i \to 2$ be the function which takes the constant value 1. Then $(\forall i \leq j < \omega) d_{2^{\omega+\omega}}(x_i, x_j) = i + 1 > i$ as $x_i|_i = x_j|_i = c_i$. Hence $\langle x_i : i \in \omega\rangle$ is an ω-Cauchy net.

Now suppose $\langle x_i : i \in \omega\rangle$ converges to $x' \in 2^{\omega+\omega}$ (we know such an x' exists as $2^{\omega+\omega}$ is spherically complete). Then $x'(n) = x_n(n) = 1$ for all $n \in \omega$ and $x' \notin E$. So $\langle x_i : i \in \omega\rangle$ does not converge in E and E is not spherically complete.

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[4] This is the supremum with the normal ordering of $\omega + \omega + 1$.

[5] $x|_m$ is the function x restricted to domain m.

2.3.2. Space of Closed Balls

We have the following characterization of strong Cauchy completeness in terms of Cauchy completeness and the closed ball functor. Notice that this proposition is very similar to Proposition 2.16.

Proposition 2.25 (*). *(M, d_M) is strongly Cauchy complete if and only if for all $\gamma \in \Gamma$, every Cauchy function in (M_γ, d_{M_γ}) converges.*

Proof. First notice if (M, d_M) is pairwise complete then for all $\gamma \in \Gamma$, (M_γ, d_{M_γ}) is also pairwise complete.

Left implies Right:
Assume (M, d_M) is strongly Cauchy complete.

By Theorem 2.20 (M, d_M) is both spherically complete and pairwise complete.

Suppose $t : A \to M_\gamma$ is a Cauchy function and let $\bar{t} : \overline{A} \to M_\gamma$ be the Cauchy net in Lemma 2.19. As (M, d_M) is spherically complete we know by Proposition 2.16 that every Cauchy net in (M_γ, d_{M_γ}) converges and hence $\bar{t}$ converges. But by the definition of $\bar{t}$, if $\bar{t}$ converges to x so does t. So every Cauchy function in (M_γ, d_{M_γ}) converges and (M_γ, d_{M_γ}) is Cauchy complete.

Right implies Left:
Assume for all $\gamma \in \Gamma$ every Cauchy function in (M_γ, d_{M_γ}) converges.

Let $A \subseteq \Gamma$ and $t : A \to M$ be a γ-Cauchy function in (M, d_M). By the definition of a γ-Cauchy function we have $\bigwedge A = \gamma = \bot_{\Gamma_\gamma}$.

Let $t^* : A \to M_\gamma$ be given by $t^*(\alpha) = B(t(\alpha), \gamma)$. Then $d_{M_\gamma}(t^*(\alpha), t^*(\beta)) = d_M(t(\alpha), t(\beta)) \vee \gamma \leq \alpha \vee \beta \vee \gamma = \alpha \vee \beta$ and t^* is a Cauchy function in (M_γ, d_{M_γ}). By assumption we therefore have that t^* converges to a point $B(x, \gamma) \in M_\gamma$. But $d_M(t(\alpha), x) \leq d_M(t(\alpha), x) \vee \gamma = d_{M_\gamma}(t^*(\alpha), x) \leq \alpha$. So t converges to x. Since $\gamma \in \Gamma$ was arbitrary this implies that (M, d_M) is strongly Cauchy complete. □

2.3.3. Injective Objects

An injective object in a category is one where every partial map into the injective object can be extended to a total map. Injectivity is a type of completeness which has applications in areas ranging from sheaf cohomology to Banach space theory to the theory of modules over a ring (to name just a few).

Injective metric spaces were first introduced in [1] under the name of "hyperconvex spaces". Injective objects in the category of $[0, \infty)$-ultrametric spaces were first studied in [3] under the name of "ultrametrically injective spaces". In [3] it was shown that injectivity was equivalent to spherical completeness for $[0, \infty)$-ultrametric spaces. In this section we extend this result and show that for an arbitrary Γ-ultrametric space injectivity is equivalent to strong Cauchy completeness.

In what follows if C is a category we let $\mathrm{Inj}(C)$ be the full subcategory of C consisting of injective objects in C.

Theorem 2.26 (*). *(M, d_M) is injective in the category of Γ-ultrametric spaces if and only if (M, d_M) is strongly Cauchy complete.*

Proof. Left implies Right
Suppose $f : A \to M$ is a γ-Cauchy function. If $\gamma \in A$ it is immediate that f converges (to $f(\gamma)$). So we assume $\gamma \notin A$. Define $(A', d_{A'})$ where $A' = A \cup \{\gamma\}$ and $d_{A'}(a, b) = a \vee b$ if $a \neq b$ (and $d_{A'}(a, a) = \bot$). We will show $(A', d_{A'})$ is a Γ-ultrametric space.

Symmetry of $(A', d_{A'})$ is immediate. Next notice that if any of $x, y, z \in A'$ are equal then $d_{A'}(x, z) \leq d_{A'}(x, y) \vee d_{A'}(y, z)$. So to show $(A', d_{A'})$ satisfies the strong triangle inequality it suffices to consider unique $x, y, z \in A'$. But then $(\forall x, y, z \in A') d_{A'}(x, z) = x \vee z \leq (x \vee z) \vee y = (x \vee y) \vee (z \vee y) = d_{A'}(x, y) \vee d_{A'}(y, z)$. Hence $(A', d_{A'})$ satisfies the strong triangle inequality and $(A', d_{A'})$ is a Γ-ultrametric space.

Next notice that $d_M(f(a), f(b)) \leq a \vee b = d_{A'}(a, b)$ for all $a, b \in A$ and hence f is a non-expanding map from $(A, d_{A'})$ to (M, d_M). Now because (M, d_M) is injective there is a $g : A' \to M$ such that $(\forall a \in A) g(a) = f(a)$. Hence $d_M(g(\gamma), f(a)) \leq d_{A'}(\gamma, a) = \gamma \vee a = a$ for all $a \in A$ and so $g(\gamma)$ realizes f.

In particular, as f was an arbitrary γ-Cauchy function this means that (M, d_M) is strongly Cauchy complete.

Right implies Left

Assume (M, d_M) is strongly Cauchy complete. Let (D, d_D) be a Γ-ultrametric space with $C \subseteq D$ and let $f : C \to M$ be a non-expanding map. We need to show that there is a $g : D \to M$ which is non-expanding and which agrees with f on C.

By induction it suffices to consider the case where $D = C \cup \{*\}$. Let $\gamma = \bigwedge\{d_D(c, *) : c \in C\}$. There is then a set $A \subseteq \Gamma$ and a map $t : A \to C$ such that:

- $A = \{d_D(c, *) : c \in C\}$.
- $(\forall \zeta \in A) d_D(t(\zeta), *) = \zeta$.

In particular t is a γ-Cauchy function on D and so $f \circ t$ is a γ-Cauchy function on (M, d_M). Hence, as (M, d_M) is strongly Cauchy complete, $f \circ t$ must be realized by an element a. Let $g = f \cup \{(*, a)\}$.

All that is left is to show that g is a non-expanding map. Choose $c \in C$ with $\zeta = d_D(c, *)$. Then $d_M(g(c), g(*)) = d_M(f(c), a) \leq d_M(f(c), f(t(\zeta))) \vee d_M(f(t(\zeta)), a) \leq d_D(c, t(\zeta)) \vee \zeta \leq [d_D(c, *) \vee d_D(*, t(\zeta))] \vee \zeta = \zeta = d_D(c, *)$. So g is a non-expanding map and, as f was arbitrary, (M, d_M) is injective. □

3. COMPLETIONS

In this section we consider the notion of a minimal complete extension of a Γ-ultrametric space for the various notions of completeness we have studied. In what follows let **** be one of these notions of completeness.

Definition 3.1. *The* ******-completion** *of a Γ-ultrametric space $\mathcal{M}$ is an isometry $c_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}^c$ such that*

- *$\mathcal{M}^c$ is a ****-complete Γ-ultrametric space.*
- *Whenever $\alpha : \mathcal{M} \to \mathcal{N}$ with $\mathcal{N}$ a ****-complete Γ-ultrametric space there is a unique map $\alpha' : \mathcal{M}^c \to \mathcal{N}$ such that $\alpha = \alpha' \circ c_{\mathcal{M}}$.*

The following is a well known fact about such completions.

Definition 3.2. *Let *UMet(Γ) be the full subcategory of UMet(Γ) consisting of the ****-complete Γ-ultrametric spaces and let IN : *UMet(Γ) $\to$ UMet(Γ) be the inclusion functor. Then the following are equivalent*

- *IN has a left adjoint where the unit is a monic natural transformation.*
- *Every Γ-ultrametric space has a ****-completion.*

As we will see in Section 3.1 every Γ-ultrametric space will have a Cauchy completion. However, in general, a Γ-ultrametric space may not have a strong Cauchy completion and so we will need a weaker notion of "minimal complete extensions" which we will discuss in Section 3.2.

3.1. Cauchy Completions

For any op-frame Γ, Γ^{op} is a frame and hence $\Gamma^{op} = (\Gamma^{op}, \wedge, \top)$ can be viewed as a monodial category. In particular we can consider the category CAT(Γ^{op}) of skeletal symmetric Γ^{op}-enriched categories and Γ^{op}-functors (see [23] for more on enriched categories). We call an object of CAT(Γ^{op}) a Γ^{op}-**category**. The following is immediate from the definitions.

Proposition 3.3. *There is an isomorphism of categories $E_\Gamma : CAT(\Gamma^{op}) \rightleftarrows UMet(\Gamma) : U_\Gamma$.*

Proof. Given a Γ-ultrametric space (M, d_M) we let $U_\Gamma(M, d_M)$ be the Γ^{op}-category $\mathbb{A}$ where $\mathbb{A}_o = M$ and $\mathbb{A}[a, b] = d_M(a, b)$. We also let $U_\Gamma(f) = f$ for any non-expanding map f.

If $\mathbb{A}$ is a Γ^{op}-category we let $E_\Gamma(\mathbb{A})$ be the Γ-ultrametric space (M, d_M) where $M = \mathbb{A}_o$ and $d_M(a, b) = \mathbb{A}[a, b]$. We also let $E_\Gamma(f) = f$ for any Γ^{op}-function.

It is immediate that $E_\Gamma \circ U_\Gamma = id_{\mathrm{UMet}(\Gamma)}$ and $U_\Gamma \circ E_\Gamma = id_{\mathrm{CAT}(\Gamma^{op})}$. □

Proposition 3.3 is a generalization of the fact shown in [13] that the category of (generalized) metric spaces is equivalent to the category of $\mathbb{R}^{\geq 0}$-enriched categories. In [13] Lawvere gives a characterization of Cauchy completeness in terms of distributors, i.e. that an enriched category X is Cauchy complete if and only if every adjoint pair of distributors from Y to X arises from an enriched functor from Y to X (see [23] for a more general characterization of Cauchy completeness in enriched categories). We will now give a characterization of Cauchy completeness for skeletal symmetric Γ^{op}-enriched categories.

Definition 3.4. *We define a* **virtual element** *of* (M, d_M) *to be a function* $v : M \to \Gamma$ *satisfying:*

(a) $(\forall x, y \in M)\ d_M(x, y) \vee v(x) \geq v(y)$.

(b) $(\forall x, y \in M)\ v(x) \vee v(y) \geq d_M(x, y)$.

(c) $\bigwedge_{x \in M} v(x) = \bot$.

We say v *is* **realized** *if there is an element* $m \in M$ *such that*

$$(\forall x \in M)\ v(x) = d_M(m, x).$$

We have the following relationships between Cauchy completeness (in the enriched category sense) and virtual elements.

Lemma 3.5. *The following are equivalent:*

- *Every virtual element of* (M, d_M) *is realized.*
- $U_\Gamma(M, d_M)$ *is Cauchy complete (as a* Γ^{op}*-enriched category).*

Proof. Let $*_\Gamma$ be a one point Γ-ultrametric space (any two of which are isomorphic). By results in [4] the Cauchy completion of $U_\Gamma(M, d_M)$ is symmetric as $U_\Gamma(M, d_M)$ is symmetric. Therefore every adjoint pair of distributors $(\varphi_l, \varphi_r) : U_\Gamma(*_\Gamma) \rightleftarrows U_\Gamma(M, d_M)$ is of the form (φ, φ). But it is immediate from the definition of virtual element that $(\varphi, \varphi) : U_\Gamma(*_\Gamma) \rightleftarrows U_\Gamma(M, d_M)$ is an adjoint pair of distributors if and only if φ is a virtual element of (M, d_M). It is also immediate that φ is realized by a point x if and only if (φ, φ) arises from the Γ^{op}-functor $\hat{x} : U_\Gamma(*_\Gamma) \to U_\Gamma(M, d_M)$ which takes a constant value of x.

We therefore have that every virtual element of (M, d_M) is realized if and only if every adjoint pair of distributors $(\varphi, \varphi) : U_\Gamma(*_\Gamma) \rightleftarrows U_\Gamma(M, d_M)$ arises from a Γ^{op}-functor. But this holds if and only if every pair of distributors $(\varphi_l, \varphi_r) : X \rightleftarrows U_\Gamma(M, d_M)$ arises from a Γ^{op}-functor from X to $U_\Gamma(M, d_M)$ i.e. if and only if $U_\Gamma(M, d_M)$ is Cauchy complete. □

There is also a close relationship between virtual elements and Cauchy functions.

Definition 3.6 (*). *If* $t : A \to M$ *is a Cauchy function define* v_t *by* $v_t(m) = \bigwedge_{a \in A} d_M(t(a), m) \vee a$. *If* v *is a virtual element and* $A_v = range(v)$ *then define* $t_v : A_v \to M$ *such that* $v(t_v(a)) = a$ *for all* $a \in A_v$.

Lemma 3.7 (*). *Suppose* $t : A \to M$ *is a Cauchy function and* v *is a virtual element*

(i) v_t *is a virtual element.*

(ii) v_t *is realized by* x *if and only if* t *converges to* x.

(iii) t_v *is a Cauchy function.*

(iv) t_v *converges to* x *if and only if* v *is realized by* x.

(v) $v_{t_v} = v$.

Proof. (i):
Condition (a) from Definition 3.4 follows because for any $x, y \in M$ we have $d_M(x, y) \vee v_t(x) = d_M(x, y) \vee [\bigwedge_{a \in A} d_M(t(a), x) \vee a] = \bigwedge_{a \in A} d_M(t(a), x) \vee a \vee d_M(x, y) \geq \bigwedge_{a \in A} d_M(t(a), y) \vee a = v_t(y)$.

Condition (b) of Definition 3.4 follows because for any $x, y \in M$,

$$\begin{aligned} v_t(x) \vee v_t(y) &= \left[\bigwedge_{a\in A} d_M(t(a),x) \vee a\right] \vee \left[\bigwedge_{a\in A} d_M(t(a),y) \vee a\right] \\ &= \bigwedge_{a,a'\in A} [d_M(t(a),x) \vee d_M(t(a'),y)] \vee [a \vee a'] \\ &= \bigwedge_{a,a'\in A} [d_M(t(a),x) \vee d_M(t(a'),y)] \vee [d_M(t(a),t(a')) \vee a \vee a'] \\ &\geq \bigwedge_{a,a'\in A} d_M(x,y) \vee a \vee a' = d_M(x,y) \vee \bigwedge_{a,a'\in A} a \vee a' \\ &= d_M(x,y) \vee \bigwedge_{a\in A} a \vee \bigwedge_{a'\in A} a' = d_M(x,y). \end{aligned}$$

where the third equality follows from the fact that t is a Cauchy function. Condition (c) of Definition 3.4 follows from the fact that

$$\begin{aligned} \bigwedge_{x\in M} v_t(x) &= \bigwedge_{a\in A, x\in M} d_M(t(a),x) \vee a \\ &\leq \bigwedge_{a,a'\in A} d_M(t(a),t(a')) \vee a \vee a' = \bigwedge_{a,a'\in A} a \vee a' = \bot. \end{aligned}$$

(ii):
If t converges to x and $m \in M$ then

$$\begin{aligned} v_t(m) &= \bigwedge_{a\in A} d_M(t(a),m) \vee a \leq \bigwedge_{a\in A} d_M(t(a),x) \vee d_M(m,x) \vee a \\ &= d_M(m,x) \vee \bigwedge_{a\in A} d_M(t(a),x) \vee a \leq d_M(m,x) \vee \bigwedge_{a\in A} a = d_M(m,x). \end{aligned}$$

In particular $v_t(x) = \bot$. But then $v_t(m) = v_t(x) \vee v_t(m) \geq d_M(x,m)$. So $v_t(m) = d_M(x,m)$.
Finally, note that if $a' \in A$ then $v_t(t(a')) = \bigwedge_{a\in A} d_M(t(a),t(a')) \vee a \leq \bigwedge_{a\in A} a' \vee a = a' \vee \bigwedge_{a\in A} a = a'$. So if x realizes v_t then for any $a \in A$, $d_M(t(a),x) \leq v_t(t(a)) \vee v_t(x) \leq a$. Hence t converges to x.

(iii):
First note by Definition 3.4 (c) $\bigwedge A_v = \bot$. Now

$$(\forall a, a' \in A_v)\ d_M(t_v(a), t_v(a')) \leq v(t_v(a)) \vee v(t_v(a')) = a \vee a'.$$

Hence t_v is a Cauchy function.

(v):
For $m \in M$ we have that $v_{t_v}(m) = \bigwedge_{a\in A_{v_{t_v}}} d_M(t_v(a),m) \vee a \leq \bigwedge_{a\in A_v} v(t_v(a)) \vee v(m) \vee a = \bigwedge_{a\in A_v} v(m) \vee a = v(m) \vee \bigwedge_{a\in A_v} a = v(m)$. But we also have by Definition 3.4 (a) that for $a \in A_v$, $v(m) \leq d_M(t_v(a),m) \vee v(t_v(a)) = d_M(t_v(a),m) \vee a$. So $v(m) \leq \bigwedge_{a\in A_v} d_M(t_v(a),m) \vee a = v_{t_v}(a)$. Hence $v = v_{t_v}$.

(iv):
This follows (ii) and (v). □

Corollary 3.8. *For any Γ-ultametric space (M, d_M) the following are equivalent:*

- *Every Cauchy function converges.*
- *Every virtual element is realized.*

Let cCAT(Γ^{op}) be the full subcategory of CAT(Γ^{op}) consisting of the Cauchy complete Γ^{op}-enriched categories and let cUMet(Γ) be the full subcategory of UMet(Γ) consisting of Cauchy complete Γ-ultrametric spaces. We then have

Proposition 3.9 (*). *The isomorphisms $E_\Gamma : CAT(\Gamma^{op}) \rightleftarrows UMet(\Gamma) : U_\Gamma$ (from Proposition 3.3) restricts to isomorphisms of categories $E_\Gamma : cCAT(\Gamma^{op}) \rightleftarrows cUMet(\Gamma) : U_\Gamma$.*

Proof. This is immediate from Corollary 3.8 and Lemma 3.5. □

In particular this implies that every Γ-ultrametric space has a Cauchy completion.

Proposition 3.10. *If $IN_\Gamma : cUMet(\Gamma) \to UMet(\Gamma)$ is the identity functor then IN_Γ has a left adjoint.*

Proof. This follows from Proposition 3.3, Proposition 3.9, and the fact, shown in [4] that the Cauchy completion functor for Γ^{op}-enriched categories takes skeletal symmetric Γ^{op}-categories to skeletal symmetric Γ^{op}-categories. □

3.2. Strong Cauchy Completions

As we saw in Section 3.1, every Γ-ultrametric space has a Cauchy completion. However it is not the case that every Γ-ultrametric space has a strong Cauchy completion.

Lemma 3.11. *There is a $(\omega + \omega + 1)$-ultrametric space which does not have a strong Cauchy completion.*

Proof. Recall the definition of $E \subseteq 2^{\omega+\omega}$ from Example 2.24. Suppose to get a contradiction that E had a strong Cauchy completion, i.e. a strongly Cauchy complete $(\omega + \omega + 1)$-ultrametric space $s(E)$ along with an inclusion $i_E : E \to s(E)$ satisfying Definition 3.1.

As $2^{\omega+\omega}$ is strongly Cauchy complete and $E \subseteq 2^{\omega+\omega}$ there is a subset $E^* \subseteq 2^{\omega+\omega}$ with $E^* \cong s(E)$. Now notice that if $C_1 \in 2^{\omega+\omega}$ is the constant function 1, then $E \cup \{C_1\}$ is strongly Cauchy complete. Hence as $E \subsetneq E^* \subseteq E \cup \{C_1\}$ we must have that $E^* \cong E \cup \{C_1\}$.

But there are many different injections from E^* into $2^{\omega+\omega}$ which are constant on E and hence the map $s : E \to 2^{\omega+\omega}$ doesn't factor through the inclusion of E in E^* in a unique way. □

Even though there is no strong Cauchy completion functor there is still always a minimal strongly Cauchy complete extension.

3.2.1. Flabby Separated Presheaves

In order to show that each Γ-ultrametric space has a minimal strongly Cauchy complete extension we first to show that the category of Γ-ultrametric spaces is equivalent to the category of flabby separated presheaves on Γ^{op}. To do this we will simultaneously deal with both $(\Gamma, \leq)$ and $(\Gamma^{op}, \leq_{op})$. So, to avoid confusion, for the rest of this section, $\leq$, $\vee$, $\bigwedge$, $\perp$, etc. will always refer to the ordering in Γ.

The following notation will be useful:

- $\text{Flab}(\Gamma^{op})$: The category of flabby separated presheaves on Γ^{op}.
- $\text{Sep}(\Gamma^{op})$: The category of separated presheaves on Γ^{op}.
- $i_\Gamma : \text{Flab}(\Gamma^{op}) \to \text{Sep}(\Gamma^{op})$ is the inclusion functor.

Note $\text{Flab}(\Gamma^{op})$ is a reflexive subcategory of $\text{Sep}(\Gamma^{op})$.

Theorem 3.12. *There is an equivalence of categories between $Flab(\Gamma^{op})$ and $UMet(\Gamma)$.*

Proof. We define maps $G_\Gamma : \text{Flab}(\Gamma^{op}) \to \text{UMet}(\Gamma)$ and $F_\Gamma : \text{UMet}(\Gamma) \to \text{Flab}(\Gamma^{op})$ which form an equivalence of categories.

We begin by defining the functor F_Γ.

Definition 3.13. *If $(M, d_M) \in UMet(\Gamma)$ then we define $F_\Gamma(M, d_M)(\gamma) = (M_\gamma, d_{M_\gamma})$ and $B(a, \gamma)|_{\gamma^*} = B(a, \gamma^*)$ for all $\gamma^* \geq \gamma$.*

If $(M, d_M), (N, d_N) \in UMet(\Gamma)$ and $f : (M, d_M) \to (N, d_N)$ is a non-expanding map then for all $B^M(a, \gamma) \in F_\Gamma(M, d_M)(\gamma)$ let $F_\Gamma(f)_\gamma(B^M(a, \gamma)) = B^N(f(a), \gamma)$.

Claim 3.14. *If $(M, d_M) \in UMet(\Gamma)$ then $F_\Gamma(M, d_M) \in Flab(\Gamma^{op})$.*

Proof. First, in order to show $F_\Gamma(M, d_M)$ is a presheaf, we need to check that restriction is well defined. Suppose $B(a,\gamma) = B(b,\gamma) \in F_\Gamma(M,d_M)(\gamma)$, $\gamma^* \geq \gamma$ and $x \in B(a,\gamma^*)$ (i.e. $d_M(x,a) \leq \gamma^*$). $d_M(a,b) \leq \gamma \leq \gamma^*$, so $d_M(x,b) \leq d_M(x,a) \vee d_M(a,b) \leq \gamma^* \vee \gamma = \gamma^*$ and therefore $x \in B(b,\gamma^*)$. Hence $B(a,\gamma^*) = B(b,\gamma^*)$ and restriction is well defined.

To see that $F_\Gamma(M,d_M)$ is flabby notice that if $B(a,\gamma) \in F_\Gamma(M,d_M)(\gamma)$ then $B(a,\perp) = \{a\} \in F_\Gamma(M,d_M)(\perp)$ and $\{a\}|_\gamma = B(a,\gamma)$.

Finally, to see $F_\Gamma(M,d_M)$ is separated, suppose $\gamma = \bigwedge_{i\in I} \lambda_i$ and $\boldsymbol{B} = \{B(x_i,\lambda_i) \in F_\Gamma(M,d_M)(\lambda_i) : i \in I\}$ is such that $B(x_i,\lambda_i)|_{\lambda_i\vee\lambda_j} = B(x_i,\lambda_i \vee \lambda_j) = B(x_j,\lambda_i \vee \lambda_j) = B(x_j,\lambda_j)|_{\lambda_i\vee\lambda_j}$. Further suppose $B(x,\gamma), B(y,\gamma) \in F_\Gamma(M,d_M)(\gamma)$ are both compatible with $\boldsymbol{B}$ (to show they must be equal). Then $B(x,\gamma)|_{\lambda_i} = B(x,\lambda_i) = B(x_i,\lambda_i)$. Hence $x \in B(x_i,\lambda_i)$ for all i and $x \in \bigcap_{i\in I} B(x_i,\lambda_i)$. But by Lemma 2.4 $B(x,\gamma) = \bigcap_{i\in I} B(x_i,\lambda_i)$. By a similar argument we also have $B(y,\gamma) = \bigcap_{i\in I} B(x_i,\lambda_i)$ and so $B(x,\gamma) = B(y,\gamma)$. Hence $F_\Gamma(M,d_M)$ is separated. □

Claim 3.15. *If $f : (M,d_M) \to (N,d_N)$ is a non-expanding map then $F_\Gamma(f) : F_\Gamma(M,d_M) \to F_\Gamma(N,d_N)$ is a map of flabby separated presheaves.*

Proof. First we need to show that $F_\Gamma(f)_\gamma(B^M(a,\gamma))$ doesn't depend on our choice of a. Suppose $B^M(a,\gamma) = B^M(b,\gamma)$ and $x \in B^N(f(a),\gamma)$. Then $d_N(f(a),f(b)) \leq d_M(a,b) \leq \gamma$ and $d_N(x,f(a)) \leq \gamma$. So $d_N(x,f(b)) \leq d_N(x,f(a)) \vee d_N(f(a),f(b)) \leq \gamma$ and $x \in B^N(f(b),\gamma)$. Hence we have $F_\Gamma(f)_\gamma(B^M(a,\gamma)) = F_\Gamma(f)_\gamma(B^N(b,\gamma))$.

Next we need to show $F_\Gamma(f)$ is a natural transformation, i.e. if $\lambda \geq \gamma$ then $F_\Gamma(f)_\lambda(B^M(a,\gamma)|_\lambda) = (F_\Gamma(f)_\gamma(B^M(a,\gamma)))|_\lambda$. But we have

$$\begin{aligned} F_\Gamma(f)_\lambda(B^M(a,\gamma)|_\lambda) &= F_\Gamma(f)_\lambda(B^M(a,\lambda)) = B^N(f(a),\lambda) \\ &= B^N(f(a),\gamma)|_\lambda = (F_\Gamma(f)_\gamma(B^M(a,\gamma)))|_\lambda. \end{aligned}$$

So $F_\Gamma(f)$ is a natural transformation from $F_\Gamma(M,d_M)$ to $F_\Gamma(N,d_N)$. □

Now we define our functor G_Γ.

Definition 3.16. *If $A \in Flab(\Gamma^{op})$ let $G_\Gamma(A)(A(\perp), d_{A_\perp})$ and where*

$$d_\perp(a,b) = \bigwedge\{\gamma : a|_\gamma = b|_\gamma\}.$$

If $A, C \in Flab(\Gamma^{op})$ and $f : A \Rightarrow C$ is a map of presheaves then $G_\Gamma(f) = f_\perp : A(\perp) \to C(\perp)$.

Claim 3.17. *If A is a flabby separated presheaf on Γ then $G_\Gamma(A)$ is a Γ-ultrametric space.*

Proof. To see $G_\Gamma(A)$ is a Γ-ultrametric space let $a,b,c \in A(\perp)$ with $d_\perp(a,b) \leq \gamma$ and $d_\perp(b,c) \leq \gamma$. Then $a|_\gamma = b|_\gamma = c|_\gamma$ and hence $d_\perp(a,c) \leq \gamma$. In particular this means $d_\perp(a,c) \leq d_\perp(a,b) \vee d_\perp(b,c)$ and $G_\Gamma(A)$ satisfies the strong triangle inequality. Symmetry is immediate.

Next notice for any $a,b \in A(\perp)$, $a|_{d_\perp(a,b)} = b|_{d_\perp(a,b)}$ because $\{(a|_\gamma,\gamma) : \gamma \leq d_\perp(a,b)\}$ is a compatible collection of elements covering both $a|_{d_\perp(a,b)}$ and $b|_{d_\perp(a,b)}$. In particular this means if $d_\perp(a,b) = \perp$ then $a = a|_\perp = b|_\perp = b$. So $G_\Gamma(A)$ is a Γ-ultrametric space. □

Claim 3.18. *If $A, C \in Flab(\Gamma^{op})$ and $f : A \Rightarrow C$ is a map of presheaves then $G_\Gamma(f) : G_\Gamma(A) \to G_\Gamma(C)$ is a non-expanding map.*

Proof. Fix $a,b \in A(\top)$ and let $\zeta = d_\perp(a,b)$. Then $a|_\zeta = b|_\zeta$ and so $f_\perp(a)|_\zeta = f_\zeta(a|_\zeta) = f_\zeta(b|_\zeta) = f_\perp(b)|_\zeta$ and hence $\zeta = d_\perp(a,b) \geq d_{C_\perp}(f(a),f(b))$. □

We now show that F_Γ and G_Γ form an equivalence of categories.

Claim 3.19. *There is a natural isomorphism $\eta : F_\Gamma \circ G_\Gamma \Rightarrow id_{Flab(\Gamma^{op})}$ such that $(\forall x \in A(\perp))\eta_A(\{x\}) = x$ when $A \in Flab(\Gamma^{op})$.*

Proof. For all $a, b \in A(\bot)$, $a|_\gamma = b|_\gamma$ if and only if $d_\bot(a,b) \leq \gamma$ if and only if $B^{G_\Gamma(A)}(a,\gamma) = B^{G_\Gamma(A)}(b,\gamma)$. So the maps $\eta_A(B^{G_\Gamma(A)}(a,\gamma)) = a|_\gamma$ is a well defined and injective natural transformation. Further, because A is flabby, η_A is also surjective and hence an isomorphism for all γ.

To show that η_A is a natural isomorphism we need to show for any map $f \in \text{Flab}(\Gamma^{op})[A, C]$ we have $\eta_C \circ F_\Gamma(G_\Gamma(f)) = f \circ \eta_A$. Now $(\forall x \in A(\bot))\eta_A(\{x\}) = x$, and $(\forall x \in C(\bot))\eta_C(\{x\}) = x$. Further we have $(\forall x \in A(\bot))\ F_\Gamma(G_\Gamma(f))_\bot(\{x\}) = \{f_\bot(x)\}$. So $(\eta_C \circ F_\Gamma(G_\Gamma(f)))_\bot = (f \circ \eta_A)_\bot$. But, because A is a flabby presheaf this implies that $\eta_C \circ F_\Gamma(G_\Gamma(f)) = f \circ \eta_A$.

Hence $\eta : F_\Gamma \circ G_\Gamma \Rightarrow id_{\text{Flab}(\Gamma^{op})}$ is a natural isomorphism. □

Claim 3.20. *There is a natural isomorphism $\varepsilon : id_{UMet(\Gamma)} \Rightarrow G_\Gamma \circ F_\Gamma$.*

Proof. If $(M', d_{M'}) = G_\Gamma(F_\Gamma(M, d_M))$ then $M' = \{\{a\} : a \in M\}$ and for all $a, b \in M$, $d_{M'}(\{a\},\{b\}) = \bigwedge\{\gamma : B^M(a,\gamma) = B^M(b,\gamma)\} = d_M(a,b)$. Hence the map $\varepsilon_{(M,d_M)}(a) = \{a\}$ is an isomorphism of Γ-ultrametric spaces. It also follows immediately that $\varepsilon : id_{\text{UMet}(\Gamma)} \Rightarrow F_\Gamma \circ G_\Gamma$ is a natural isomorphism. □

Hence F_Γ and G_Γ are equivalences of categories. □

3.2.2. Injective Hull

An injective hull of an object is a minimal injective object containing it. Injective hulls occur in many areas of mathematics. Examples include the Dedekind-MacNeille completion of a partial order as well as the algebraic closure of a field.

Injective hulls were first discovered in the category of metric spaces by Isabell in 1964 ([11]). The concept was then rediscovered in 1984 by Dress ([7]) and again by Chrobak and Larmore in 1994 ([5]). In [3] injective hulls in the category of $[0,\infty)$-ultrametric spaces were studied and it was shown that every $[0,\infty)$-ultrametric space has an injective hull.[6]

In this section we show that every Γ-ultrametric space has an injective hull.

Definition 3.21. *Suppose C is a category and $A \in C$. An* **injective hull**[7] *for A is a monic $e : A \to I$ where*

(i) I is injective.

(ii) If $k : A \rightarrowtail I'$ is a monomorphism with I' injective, there is a monomorphism $k' : I \rightarrowtail I'$ such that $k' \circ e = k$.

By Theorem 2.26 we know that a Γ-ultrametric space is injective if and only if it is strongly Cauchy complete. So an injective hull of a Γ-ultrametric space is a minimal extension of the Γ-ultrametric space which is strongly Cauchy complete.

Proposition 3.22 (*). *For each Γ-ultrametric space (M, d_M) there is an injective hull $e_M : (M, d_M) \to I_M$.*

Proof. By by Theorem 3.12 it suffices to show that every object of $\text{Flab}(\Gamma^{op})$ has an injective hull.

Claim 3.23 (*). *For every object $\mathcal{F}$ of Flab(Γ^{op}) the following are equivalent.*

(1) $\mathcal{F}$ is injective in Flab(Γ^{op}).

(2) $i_\Gamma(\mathcal{F})$ is injective in Sep(Γ^{op}).

Proof. It is immediate that (2) implies (1). To see that (1) implies (2) we need some notation. If A is a separated presheaf on Γ let A^* be the flabby separated presheaf such that:

- $A^*(V) = \bigcup_{U \leq V} A(U) \times \{U\} \times \{V\}$.
- If $U \leq V$ and $\langle x, W, V\rangle \in A^*(V)$ then $\langle x, W, V\rangle|_U = \langle x|_{U\wedge W}, U \wedge W, U\rangle \in A^*(U)$

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[6] These were called “Ultrametrically injective envelopes”.
[7] An injective hull is sometimes referred to as an injective envelope.

If $m : A \to B$ let $m^* : A^* \to B^*$ be the map where $m^*(\langle x, W, V\rangle) = \langle m(x), W, V\rangle$ if $\langle x, W, V\rangle \in \bigcup_{V\in\Gamma} A^*(V)$. Notice that if m is a monic so is m^*. Let $e_A : A \to A^*$ be the map such that $e_A(a) = e_A(\langle a, V, V\rangle)$ for all $a \in A(V)$. It 12is immediate that e_A is a monic and that $e_B \circ m = m^* \circ e_A$.

Now suppose $\mathcal{F}$ is injective in Flab(Γ^{op}), we have a map $f : A \to i_\Gamma(\mathcal{F})$ in Sep(Γ^{op}) and a monomorphism $m : A \to B$ in Sep(Γ^{op}). For $U \in \Gamma$ and $a \in A(U)$ let $a_* \in \mathcal{F}(\top)$ be such that $a_*|_U = f(a)$. We can always find such a a_* because $\mathcal{F}$ is flabby. Let $f^* : A^*(\top) \to \mathcal{F}(\top)$ be such that if $a \in A(U)$ then $f^*(\langle a, U, \top\rangle) = a^*$. As A^* is flabby any map from A^* is determined by where it takes $A^*(\top)$ and so f^* can be extended to a unique map $f^* : A^* \to \mathcal{F}$. It is also easy to see that $f^* \circ e_A = f$. Because $\mathcal{F}$ is injective, there is a map $h : B^* \to \mathcal{F}$ such that $h \circ g^* = f^*$. But then we also have $h \circ g = h \circ g^* \circ e_A = f^* \circ e_A = f$. We therefore have $\mathcal{F}$ is injective in Sep(Γ^{op}) and we are done with the proof of the claim. □

We also have

Claim 3.24. *The injective objects of Sep*(Γ^{op}) *are exactly the flabby sheaves and every object of Sep*(Γ^{op}) *has an injective hull.*

Proof. See [6]. □

So, because Flab(Γ^{op}) is a full subcategory of Sep(Γ^{op}) and the injective objects in Flab(Γ^{op}) are exactly those which are injective in Sep(Γ^{op}) we have that every object in Flab(Γ^{op}) has an injective hull. □

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