The Number of Countable Models in Realizability Toposes

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Abstract

The goal of this paper is to extend Morley's results in [9] to realizability toposes. We consider two natural notions of "countable model" in this context. We show for both of these notions of countable and for any first order theory T in a countable language, that there is either a perfect set of non-isomorphic models of T or there are at most $\aleph_1$ many non-isomorphic models of T in the realizability topos over any countable PCA.

Key words: Countable models, realizability, topos hyperdoctrine, Vaught's conjecture

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1. Introduction

1.1. Summary

One of the oldest open questions in model theory is "how many countable models can a countable first order theory have?" This problem was first proposed by Vaught in his seminal paper [12] where he asked "Can it be proved, without the use of the continuum hypothesis, that there exists a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?". Since that time the statement that every first order theory has either countably many or continuum many countable models, a statement which would imply a negative answer to Vaught's question, has become known as "Vaught's conjecture".

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In his paper [9] Morley took one of the most significant steps towards resolving Vaught's conjecture. Morley not only proved that any first order theory which did not have a perfect set of countable models had at most $\aleph_1$ many countable models but he also showed that this result holds for arbitrary sentences of $\mathcal{L}_{\omega_1,\omega}$. Morley's result was significant not only because it provided a concrete bound on the ways in which Vaught's conjecture could fail, but also because it extended Vaught's conjecture into the realm of infinitary logic and as such opened up a large collection of new methods for studying the problem.

In [1] the author showed that, under a mild determinacy hypothesis, Morley's result[1] could be extended to categories of sheaves on a (countably generated) site. Just as Morley's original result led to a generalization of Vaught's conjecture for sentences of $\mathcal{L}_{\omega_1,\omega}$ the results in [1] lead to an extension of Vaught's conjecture to the realm of Grothendieck Toposes. Further, the results of [1] suggest the question: "Are there other toposes (with a natural notion of countable) for which Morley's theorem holds?"

The goal of this paper is to further extend Morley's theorem to those realizability toposes which come from a countable partial combinatorial algebra (PCA). We will do this by reducing the question of how many countable models of a theory exists in a realizability topos to a question about the number of models in the category of sets of a (related) sentence of $\mathcal{L}_{\omega_1,\omega}(\sigma)$.

Our proof consists of three parts. As is well known, every realizability topos can be constructed as the category of partial equivalence relations on a first order hyperdoctrine. The first step in the proof is to show that we can represent any model in a realizability topos inside of this hyperdoctrine using what we call "interpretations". In the second part of the proof we show how to define the collection of interpretations which satisfy a given theory by models in the category of sets which satisfy a given formula of $\mathcal{L}_{\infty,\omega}$. In this way we get, for every model of a theory in our realizability topos, a model in the category of

[1]As was observed in [1] in categories of sheaves there are four natural notion of countable. In [1] we show that Morley's theorem holds for each of these four notions.

sets. This will then also allow us to show that the isomorphism relation in our realizability topos can be described by a PC_{ω_1} class on the collection of models in the category of sets. Finally we will use these results and various theorems of descriptive set theory to prove Morley's theorem in our realizability topos.

1.2. Outline

The structure of the paper is as follows. In Section 2 we review terminology and standard results from infinitary logic, set theory and category theory. In Section 3 we will review the basic notions concerning hyperdoctrines. It is in this section that we introduce the notion of an interpretation of a language in a hyperdoctrine as well as our notion of a model in a hyperdoctrine. This will lead to the most important result of this section, Proposition 3.24, which says there are natural maps between the models of a language in a hyperdoctrine and the models in the category of partial equivalence relations on the hyperdoctrine. Most of the material in Section 2 and Section 3 is either well known or closely related to well known material and we reproduce it here for completeness and to make precise what we are discussing.

In Section 4.1 we introduce the central new notion of this paper, that of a definable hyperdoctrine. The definition introduces added structure which will allow us to capture (some of) the underlying hyperdoctrine operations using models in the category of sets. As we will see this is a versatile notion and many of the hyperdoctrines associated with realizability (and not just those associated with realizability over a PCA) are definable. In Section 4.2 we show that for each language σ there is a theory $T(\sigma)$ and an isomorphism of categories between interpretations of σ and models of $T(\sigma)$. In Section 4.3 and Section 4.4 we show, for a realizability topos over a PCA, how to define the collection interpretations which satisfying a given theory, using those models of $T(\sigma)$ in the category of sets that satisfy a sentence of $\mathcal{L}_{\infty,\omega}$.

In Section 5 we introduce the two notions of countable which we will consider. The first of these notions, being countably generated, comes from an analysis of the partial equivalence relation construction over a definable hyperdoctrine. The second notion, being monic bound means you are isomorphic to a subobject of your bound. In this way if we have an object, like say the natural numbers object, which is intrinsically countable, then we can consider those objects which are bound by it.

Finally in Section 6 we prove the main theorems of this paper. In Section 6.1 we prove a generalization of Morley's theorem for countably generated models. Specifically we show that given any first order theory T, there are either continuum many or at most $\aleph_1$ many countably generated models in the category of partial equivalence relations over a countably presented simple definable hyperdoctrine. In Section 6.2 we then prove the same result for models monic bound by a countably generated object.

We end this paper in Section 7 with two related results. First in Section 7.1, we show that in the case of countably generated models whether or not there is a perfect set of models is independent of the model of set theory we work in. In Section 7.2 we show that for a topological space T with open sets $\mathcal{O}(T)$, the hyperdoctrine from which $\mathcal{O}(T)$-valued sets are constructed is definable and simple. From this we deduce that if T is countable and second countable then Morley's theorem holds for $\mathcal{O}(T)$-valued sets. As the notion of countably generated for $\mathcal{O}(T)$-valued sets in this paper and the notion of countably generated for sheaves on $\mathcal{O}(T)$ in [1] coincide this allows us to remove the determinacy result from the main theorem in [1] (when T is countable and second countable).

2. Background

2.1. Logics

In this paper we only deal with finitary multi-sorted languages of which σ and τ (and their variants) will always be instances. We denote the collection of sorts of σ by $\mathcal{S}_\sigma$, the collection of relation symbols in σ by $\mathcal{R}_\sigma$ and the collection of function symbols in σ by $\mathcal{F}_\sigma$. Further, to simplify the presentation, we will assume that the collection of sorts of any languages is closed under finite sequences[2] and that the corresponding projection functions are always part of the language.

We say a formula $\varphi(x)$ is of type X if x is a variable[3] of sort X. Similarly we say a function $f(x)$ is of type $X \to Y$ if x is a variable of sort X and f takes values in the sort Y. Note that then empty sequence of sorts, $\langle\rangle$, is a sort and as such we allow relations and functions with $\langle\rangle$ as their domain.

We denote by Mod_σ the category of σ structures (in the category of sets) along with maps which preserve and reflect all (non-equality) relations and commute with all functions. For any sentence φ in the language σ we let $\mathrm{Mod}_\sigma(\varphi)$ be the full subcategory of Mod_σ consisting of those models which satisfy φ.

By $\mathcal{L}_{\kappa,\omega}(\sigma)$ we mean the logic where we allow $< \kappa$ sized disjunctions and conjunctions along with finite quantification (see [5] for a more detained introduction). We say a class $\mathcal{K}$ of models of σ is in $\mathrm{PC}_\kappa(\sigma)$ if there is a language σ^* and a formula $\varphi \in \mathcal{L}_{\kappa,\omega}(\sigma^*)$ such that $\sigma \subseteq \sigma^*$, $|\sigma^* - \sigma| < \kappa$, and $\mathcal{M} \in \mathcal{K}$ if and only if $[(\exists \mathcal{M}^* \in \mathrm{Mod}_\sigma(\varphi))\mathcal{M}^*|_\sigma = \mathcal{M}$ and $|\mathcal{M}^*| = |\mathcal{M}|]$. We will let $\mathcal{L}_{\infty,\omega}(\sigma) = \bigcup_{\kappa \in \mathrm{ORD}} \mathcal{L}_{\kappa,\omega}(\sigma)$.

We say an equivalence relation on σ structures, $\equiv_\sigma$, is $\mathrm{PC}_\kappa(\sigma)$ if there is a language σ^* such that σ^* contains two disjoint copies σ_0, σ_1, of σ where $\mathcal{S}_{\sigma^*} = \mathcal{S}_{\sigma_0} \cup \mathcal{S}_{\sigma_1}$, and there is a sentence $\psi_{\equiv_\sigma} \in \mathcal{L}_{\infty,\omega}(\sigma^*)$ such that the following are equivalent for models $\mathcal{M}_0, \mathcal{M}_1 \in \mathrm{Mod}_\sigma$:

- $\mathcal{M}_0 \equiv_\sigma \mathcal{M}_1$.
- $(\exists \mathcal{M}^* \in \mathrm{Mod}_{\sigma^*}(\psi_{\equiv_\sigma}))\mathcal{M}^*|_{\sigma_0} \cong \mathcal{M}_0$ and $\mathcal{M}^*|_{\sigma_1} \cong \mathcal{M}_1$ (where these isomorphisms are under the association of σ_0, σ_1 with σ in the obvious way).

[2] This will allow us to treat X as a sort when $X = \langle X_1, \cdots, X_n\rangle$. This will make it possible to (mostly) avoid dealing with sequences of sorts.

[3] If $X = \langle X_1, \ldots, X_n\rangle$ then a variable of type X is the same as a sequence of variables $(x_1, \ldots, x_n)$ where x_i is of type X_i.

2.2. Set Theory

We begin this section with a brief discussion of notions from set theory which will be important in this paper. We refer the reader to such standard texts as [3] for any set theoretic results or definitions not explicitly discussed.

In this paper we use Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC) as our ambient theory and we will assume all results take place in a fixed model of ZFC which we refer to as SET. In this paper κ (along with its variants) will always be cardinals.

There are a few results of descriptive set theory which will be important later. We mention them here:

Definition 2.1 ([4] Definition 16.5)**.** Let $\sigma = \langle R_i : i \in I\rangle$ be a one sorted countable relational language where R_i has arity n_i. Let $\mathrm{Mod}_\sigma(\mathbb{N}) = \prod_{i\in I} 2^{\mathbb{N}^{n_i}}$. We can view $\mathrm{Mod}_\sigma(\mathbb{N})$ as the space of models with underlying set $\mathbb{N}$, the natural numbers (with the obvious topology). As such there is a natural action of $S_\mathbb{N}$ (the permutation group of $\mathbb{N}$) on $\mathrm{Mod}_\sigma(\mathbb{N})$.

If σ is a countable multi-sorted language then there is a one sorted language σ_1 which contains a relation symbol for each sort of σ and there is a sentence $\phi_\sigma \in \mathcal{L}_{\omega_1,\omega}(\sigma_1)$ such that models of ϕ_σ represent models of σ in the obvious way. In this context $\mathrm{Mod}_\sigma(\mathbb{N})$ then be the subset of $\mathrm{Mod}_{\sigma_1}(\mathbb{N})$ consisting of those elements satisfying ϕ_σ.

Theorem 2.2 ([4] Theorem 16.8)**.** *The $S_\mathbb{N}$ invariant Borel subsets of $Mod_\sigma(\mathbb{N})$ are exactly those subsets of the form $\{x \in Mod_\sigma(\mathbb{N}) : x \models \varphi\}$ for some sentence $\varphi \in \mathcal{L}_{\omega_1,\omega}(\sigma)$.*

Corollary 2.3. *For any PC_{ω_1} formula Φ, the set $\{x : x \in Mod_\sigma(\mathbb{N})$ and $x \models \Phi\}$ is a $\mathbf{\Sigma}^1_1$ invariant subset of $Mod_\sigma(\mathbb{N})$.*

Proof. Φ is of the form $(\exists \mathcal{M}^* \in \mathrm{Mod}_{\sigma^*}(\mathbb{N}))\mathcal{M}^* \models \varphi$ and $\mathcal{M}^*|_\sigma = \mathcal{M}$ because our definition of PC_{ω_1} bounds the size of the expansions we need to consider. □

Corollary 2.4. *For any PC_{ω_1} equivalence relation $\equiv$, the set $\{(x, y) : x, y \in Mod_\sigma(\mathbb{N}),\ x \equiv y\}$ is a $\mathbf{\Sigma}^1_1$ invariant subset of $Mod_\sigma(\mathbb{N}) \times Mod_\sigma(\mathbb{N})$.*

Theorem 2.5 ([3] Theorem 32.9)**.** *If E is a $\mathbf{\Sigma}^1_1$ equivalence relation on 2^ω then one of the following holds:*

- *There is a perfect set of reals in E-inequivalent reals in X.*
- *There are at most $\aleph_1$ many E-inequivalent reals in X.*

Further, whether there is a perfect set of E-inequivalent reals is absolute between models of set theory.

Corollary 2.6. *Suppose X is a $\mathbf{\Sigma}^1_2$ subset of 2^ω and E is a $\mathbf{\Sigma}^1_1$ equivalence relation on elements of 2^ω. Then one of the following holds:*

- *There is a perfect set of reals in E-inequivalent reals in X.*
- *There are at most $\aleph_1$ many E-inequivalent reals in X.*

Proof. This follows in a straight forward way from an analysis of Theorem 25.19 of [3]. For a (slightly) more detailed explanation see [1] Proposition 2.12. □

Corollary 2.7. *Suppose σ is a countable language, $\varphi \in \mathcal{L}_{\omega_1,\omega}(\sigma)$ is a sentence and $\equiv$ is a $PC_{\omega_1}(\sigma)$ equivalence relation. Then one of the following holds:*

- *There is is a perfect set of reals each encoding a countable σ-structures that models φ such that no two models are $\equiv$-equivalent.*
- *There can be at most $\aleph_1$ reals each encoding a countable σ-structures that models φ such that no two models are $\equiv$-equivalent.*

Further which one is independent of the model of set theory we are working in.

Corollary 2.8. *Suppose σ is a countable language, $\varphi \in PC_{\omega_1}(\sigma)$ is a sentence and $\equiv$ is a $PC_{\omega_1}(\sigma)$ equivalence relation. Then one of the following holds:*

- *There is is a perfect set of reals each encoding a countable σ-structures that models φ such that no two models are $\equiv$-equivalent.*
- *There can be at most $\aleph_1$ reals each encoding a countable σ-structures that models φ such that no two models are $\equiv$-equivalent.*

2.3. Category Theory

In this section we review some of the categorical notions which we will need. For more information on the general category theory in this paper the reader is referred to such standard texts as [6].

All categories in this paper will be locally small and we will use the convention that if C is a category with objects A and B, C$[A, B]$ is the set of morphisms whose domain is A and whose codomain is B. We also abuse notation by using $x \in$ C to mean x is an object of C and, when no confusion can arise, by using SET to refer to the category of sets and functions in our ambient model SET of ZFC.

We let Preorder be the partially order enriched category of preorders and Heyting be the (non full) subcategory of Heyting prealgebras. For a definition of these notions (as well as the important notion of a psuedofunctor) we refer the reader to [11] Chapter 2.1.1. We will also assume that for each Heyting prealgebra we have chosen a representative from each equivalence class.[4]

If HA is a Heyting prealgebra we will refer to $\top_{HA}$ as a top element of HA (i.e. $\top_{HA} = \sup\{x : x \in HA\}$) and $\bot_{HA}$ as a bottom element of HA (i.e. $\bot_{HA} = \inf\{x : x \in HA\}$).

3. Hyperdoctrines

In this section we review the notion of a first order hyperdoctrine and the partial equivalence relation construction. The partial equivalence relation construction is important as every realizability topos over a PCA can be constructed using the partial equivalence class construction applied to a certain hyperdoctrine. As most of the ideas in this section can be easily constructed from well known results we refer the reader to the standard text [11] (especially Chapter 2) for a more thorough explanation of the concepts mentioned here.

[4]This will make statements like $a \wedge b = c$ have meaning.

3.1. Definitions and Internal Language

For the rest of the paper C will always be a category with finite products. Further we will assume that a specific choice of products has been made (so statements of the form $X = X_1 \times X_2$ are well defined).

Definition 3.1. A **hyperdoctrine** consists of a pair $(\mathrm{C}, \mathcal{H})$ where C is a category with finite products and $\mathcal{H}$ is a contravariant psuedofunctor $\mathcal{H} : \mathrm{C}^{op} \to$ Heyting from C into the category of Heyting prealgebras which satisfies:

- For each product projection $\pi_X : X \times Y \to X$ in C the functor $\mathcal{H}(\pi_X) : \mathcal{H}(X) \to \mathcal{H}(X \times Y)$ has both a left adjoint $(\exists Y)_X$ and a right adjoint $(\forall Y)_X$ in the category of preorders[5].

For a more complete definition of a hyperdoctrine we refer the interested reader to [10]. From now on $(\mathrm{C}, \mathcal{H})$ will always be a first order hyperdoctrine.

One can associate to $(\mathrm{C}, \mathcal{H})$ a language $\sigma_{\mathcal{H}}$ where

- For each object $X \in \mathrm{C}$ there is a sort X^S of $\sigma_{\mathcal{H}}$ and where the sort associated to $\langle X_1, \dots, X_n \rangle^S = X_1^S \times \cdots \times X_n^S$.
- For each function $f \in \mathrm{C}[X, Y]$ there is a function symbol F_f of type $X^S \to Y^S$.
- For each element $Q \in \mathcal{H}(X)$ there is a relation symbol R_Q of type X^S.

For each term t in this language we can associate a map $[t]_{\mathcal{H}}$ in C by induction in the obvious way (where $[F_f]_{\mathcal{H}} = f$). We can also give a interpretation $[\varphi]_{\mathcal{H}}$ of formulas $\varphi(\mathbf{x}) \in \mathcal{L}_{\omega,\omega}(\sigma_{\mathcal{H}})$[6] in the standard way[7].

[5]It is worth stressing that we do not require $(\forall Y)_X$ and $(\exists Y)_X$ to be maps of Heyting prealgebras, but only of preorders.

[6]In general we are not able to extend this interpretation to all of $\mathcal{L}_{\infty,\omega}(\sigma_{\mathcal{H}})$ because we have not assumed the Heyting prealgebras $\mathcal{H}(X)$ are complete (and in the case of realizability toposes they usually won't be).

[7]Notice that these are only unique up to isomorphism as $\mathcal{H}(X)$ is only a Heyting prealgebra and not necessarily a Heyting algebra.

We say that a sentence φ in this language is **satisfied** by a hyperdoctrine $(\mathrm{C}, \mathcal{H})$, written $(\mathrm{C}, \mathcal{H}) \models \varphi$, if $[\varphi]_{\mathcal{H}} \cong \top_{\mathcal{H}(1_{\mathrm{C}})}$ (where 1_{C} is a terminal object in C).

We then have the following important result concerning hyperdoctrines

Theorem 3.2 (Soundness of Hyperdoctrines[8] ([11] Theorem 2.1.6))**.** *Suppose φ is a sentence which is provable in first order intuitionistic logic without equality. Then for every hyperdoctrine $(\mathrm{C}, \mathcal{H})$, $(\mathrm{C}, \mathcal{H}) \models \varphi$.*

For a more thorough treatment see [11] Section 2.1.3.

3.2. Partial Equivalence Relation Construction

In this section we review the partial equivalence relation (PER) construction. Hyperdoctrines are sound for intuitionistic first order logic without equality and the PER construction is a way to allow us to interpret equality. In particular the PER construction on a hyperdoctrine is one way in which we can construct realizability toposes based on a PCA.

Definition 3.3. Given a hyperdoctrine $(\mathrm{C}, \mathcal{H})$ we define the **category of partial equivalence relations** on $(\mathrm{C}, \mathcal{H})$, $\mathrm{C}[\mathcal{H}]$, where

- The objects of $\mathrm{C}[\mathcal{H}]$ are pairs $(X, \sim_X)$ where
  - $X \in \mathrm{obj}(C)$ and $\sim_X \in \mathcal{H}(X \times X)$.
  - The following sentences[9] are satisfied by $(\mathrm{C}, \mathcal{H})$:
    - $(\forall x, x' : X)(x \sim_X x') \to (x' \sim_X x)$.
    - $(\forall x, x', x" : X)(x \sim_X x') \wedge (x' \sim_X x") \to (x \sim_X x")$.
- A morphism from $(X, \sim_X)$ to $(Y, \sim_Y)$ is given by an $F \in \mathcal{H}(X \times Y)$ such that the following sentences[10] are satisfied by $(\mathrm{C}, \mathcal{H})$:

[8] There are other notions of hyperdoctrine than the one presented here and each notion captures a fragment fragment of logic.

[9] These express in the internal language of $(C, \mathcal{H})$ that $\sim_X$ is a partial equivalence relation.

[10] These express in the internal language of $(C, \mathcal{H})$ that F respects the partial equivalence relations $\sim_X$ and $\sim_Y$ and that F is single valued and total with respect to them.

– $(\forall x : X)(\forall y : X)F(x, y) \to (x \sim_X x) \wedge (y \sim_Y y)$.

– $(\forall x_0, x_1 : X)(\forall y_0, y_1 : Y)(x_0 \sim_X x_1) \wedge (y_0 \sim_Y y_1) \wedge F(x_0, y_0) \to F(x_1, y_1)$.

– $(\forall x : X)(\forall y_0, y_1 : Y)F(x, y_0) \wedge F(x, y_1) \to (y_0 \sim_Y y_1)$.

– $(\forall x : X)(x \sim_X x) \to (\exists y : Y)F(x, y)$.

- The identity morphism on $(X, \sim_X)$ is given by $\sim_X$ itself.

- Composition of $F : (X, \sim_X) \to (Y, \sim_Y)$ and $G : (Y, \sim_Y) \to (Z, \sim_Z)$ is the element $[(\exists y : Y)\mathcal{H}[\pi_1]F(x, y) \wedge \mathcal{H}[\pi_2]G(y, z)]$ of $\mathcal{H}(X \times Z)$ (where π_1, π_2 are the projections from $X \times Y \times Z$ onto $X \times Y$ and $Y \times Z$ respectively.)

That composition is associative and that the indicated morphisms are identities follows from the soundness of hyperdoctrines for first order intuitionistic logic.

It is not hard to show that $\mathrm{C}[\mathcal{H}]$ will always be a Heyting category (for a proof of this see [11] Theorem 2.2.1 along with the Remark immediately afterwards). Further, in many cases, like in the case of the construction of a realizability topos on a PCA from a hyperdoctrine, the resulting category will be a topos. But in this paper we will never need anything beyond the structure of a Heyting category.

Lemma 3.4 ([10] Lemma 3.2)**.** $\mathrm{C}[\mathcal{H}]$ *has finite products where:*

- *A terminal object of* $\mathrm{C}[\mathcal{H}]$ *is* $(1_{\mathrm{C}}, \top_{\mathcal{H}(1_{\mathrm{C}} \times 1_{\mathrm{C}})})$ *where* 1_{C} *is a terminal object of* C.

- *A product for* $(X, \sim_X)$ *and* $(Y, \sim_Y)$ *is given by* $(X \times Y, \sim_{X \times Y})$ *where* $(x, y) \sim_{X \times Y} (x', y') \Leftrightarrow (x \sim_X x') \wedge (y \sim_Y y')$ *with the evident projections.*

Lemma 3.5 ([10] Lemma 3.3)**.** *A function* $f : (X, \sim_X) \to (Y, \sim_Y)$ *is a monomorphism if and only if* $(\mathrm{C}, \mathcal{H}) \models (\forall x, x', y)F(x, y) \wedge F(x', y) \to x \sim_X x'$

Definition 3.6. We say an element $P \in \mathcal{H}(Y)$ **strict** on $(Y, \sim_Y)$ if

- $(\mathrm{C}, \mathcal{H}) \models (\forall y : Y)P(y) \Rightarrow y \sim_Y y$.

- $(\mathrm{C}, \mathcal{H}) \models (\forall y, y' : Y) P(y) \wedge y \sim_Y y' \Rightarrow P(y')$.

Lemma 3.7 ([10] p. 271)**.** *Every monomorphism $m : (X, \sim_X) \rightarrowtail (Y, \sim_Y)$ is in the same subobject as one of the form $(Y, \approx)$ where $x \approx x' \leftrightarrow (x \sim_Y x' \wedge St_m(x))$ for a strict relation St_m. We say that St_m **represents** m. Further there is a bijection between subobjects of $(Y, \sim_Y)$ and isomorphism classes of strict relations.*

The point of the previous three lemmas is that the properties of being a product, of being a monomorphism and of being a strict relation all can be described by first order formulas.

3.3. Models and Interpretations

An important feature of the partial equivalence relation construction is that for each model $\mathcal{M}$ in $\mathrm{C}[\mathcal{H}]$ and each formula $\varphi \in \mathcal{L}_{\omega,\omega}(\sigma)$ of type X we can associate a subobject $\{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathcal{M}}$ of $X^{\mathcal{M}}$ in a canonical way. Further this subobject can be characterized in the internal language of $(\mathrm{C}, \mathcal{H})$. We reproduce here the basic ideas which we will need. For a more thorough treatment see [11] Chapter 2.

3.3.1. Models in $\mathrm{C}[\mathcal{H}]$

We now define the notion of a model in a category. For more information on the notion of a model in a category we refer the reader to [8] or [7].

Definition 3.8. A *σ-structure $\mathcal{M}$ in* $\mathrm{C}[\mathcal{H}]$ consist of:

- For every sort $X \in \mathcal{S}_\sigma$ an object $X^{\mathcal{M}}$ of $\mathrm{C}[\mathcal{H}]$ such that for every sequence $\langle X_1, \ldots, X_n \rangle$ of sorts $\langle X_1, \ldots, X_n \rangle^{\mathcal{M}} = X_1^{\mathcal{M}} \times \cdots \times X_n^{\mathcal{M}}$ and the $\langle\rangle^{\mathcal{M}} = 1_{\mathrm{C}[\mathcal{H}]}$ (a terminal object).

- For every function $f \in \mathcal{F}_\sigma$ of type $X \to Y$ we have a map in $\mathrm{C}[\mathcal{H}]$, $f^{\mathcal{M}} : X^{\mathcal{M}} \to Y^{\mathcal{M}}$.

- For every relation $R \in \mathcal{R}_\sigma$ of type X we have an subobject $R^{\mathcal{M}}$ of $X^{\mathcal{M}}$.

We let Str_σ be the collection of σ structures in $\mathrm{C}[\mathcal{H}]$.

Given a model $\mathcal{M} \in \mathrm{Str}_\sigma$ and a formula $\varphi(\mathbf{x}) \in \mathcal{L}_{\omega,\omega}(\sigma)$ of type X we can define a subobject $\{\mathbf{x} : \varphi(\mathbf{x})\}^{\mathcal{M}}$ of $X^{\mathcal{M}}$ in the standard way. From now on $\mathcal{M}$ (and its variants) will always represent models in $\mathrm{C}[\mathcal{H}]$.

Definition 3.9. Suppose that $\mathcal{M}_0, \mathcal{M}_1$ are σ-structures in $\mathrm{C}[\mathcal{H}]$. An **isomorphism** of models $\alpha : \mathcal{M}_0 \to \mathcal{M}_1$ is a set of maps $\langle \alpha_X : X \in \mathcal{S}_\sigma \rangle$ such that:

- For each sort $X \in \mathcal{S}_\sigma$, $\alpha_X \in \mathrm{C}[\mathcal{H}][X^{\mathcal{M}_0}, X^{\mathcal{M}_1}]$ is an isomorphism and for any sequence $X = \langle X_1, \dots, X_n \rangle$ we have $\alpha_X = \alpha_{X_1} \times \cdots \times \alpha_{X_n}$.
- For any function $f \in \mathcal{F}_\sigma$ with $f : X \to Y$, $\alpha_Y \circ f^{\mathcal{M}_0} = f^{\mathcal{M}_1} \circ \alpha_X$.
- For any relation $R \in \mathcal{R}_\sigma$ of type X and any monomorphism r in the subobject $R^{\mathcal{M}_0}$, the subobject containing $\alpha_X \circ r$ is the same subobject as $R^{\mathcal{M}_1}$.

We denote the existence of an isomorphism between $\mathcal{M}_0$ and $\mathcal{M}_1$ by $\mathcal{M}_0 \cong \mathcal{M}_1$.

We now observe that the notion that two models are isomorphic is definable. More specifically we have the following:

Definition 3.10. Let σ_0, σ_1 be two disjoint copies of σ, let $\sigma_{iso} = \sigma_0 \cup \sigma_1 \cup \{\alpha_X : X_0 \to X_1 \text{ for any } X \in \mathcal{S}_\sigma\} \cup \{\beta_X : X_1 \to X_0 \text{ for } X \in \mathcal{S}_\sigma\}$ and let $Th_{iso}(\sigma) \subseteq \mathcal{L}_{\omega,\omega}(\sigma_{iso})$ be the theory which says

- For each sort $X \in \mathcal{S}_\sigma$
  - $(\forall x : X_1)\alpha_X \circ \beta_X(x) =_{X_1} x$.
  - $(\forall x : X_0)\beta_X \circ \alpha_X(x) =_{X_0} x$.
- For each function $f : X \to Y$ in $\mathcal{F}_\sigma$:
  - $(\forall x : X_0)\alpha_Y \circ f_0(x) =_{Y_1} f_1 \circ \beta_X(x)$.
  - $(\forall x : X_1)\beta_Y \circ f_1(x) =_{Y_0} f_0 \circ \alpha_X(x)$.
- For each relation R of type X in $\mathcal{R}_\sigma$:

– $(\forall x : X_0) R_1(\alpha_X(x)) \leftrightarrow R_0(x)$.

– $(\forall x : X_1) R_0(\beta_X(x)) \leftrightarrow R_1(x)$.

Proposition 3.11. *For any two models $\mathcal{M}_0, \mathcal{M}_1$ of σ in $\mathrm{C}[\mathcal{H}]$ the following are equivalent:*

- $\mathcal{M}_0 \cong \mathcal{M}_1$.
- *There is a model I of $Th_{iso}(\sigma)$ where $I|_{\sigma_0} \cong \mathcal{M}_0$ and $I|_{\sigma_1} \cong \mathcal{M}_1$ (where here the isomorphism is after associating σ_0 and σ_1 with σ in the obvious way.)*

Proof. This follows immediately from the soundness of hyperdoctrines for first order logic . □

This tells us that determining if two models are isomorphic can be reduced to determining the existence of a model of a first order theory.

3.3.2. Interpretations

In this section we introduce the notion of an interpretation of a language in a hyperdoctrine.

Definition 3.12. Suppose σ is a relational language. An **interpretation**, $[\cdot]$, of σ in a hyperdoctrine $(\mathrm{C}, \mathcal{H})$ consists of the following:

- For each sort $X \in \mathcal{S}_\sigma$ an object $[X] \in \mathrm{C}$ where $[\langle X_1, \dots, X_n \rangle] = [X_1] \times \dots \times [X_n]$ [11].
- For each relation $R \in \mathcal{R}_\sigma$ of type X we assign an element $[R] \in \mathcal{H}([X])$.

A **morphism of interpretations** $f : [\cdot]_0 \to [\cdot]_1$ consists of a map $f_X : [X]_0 \to [X]_1$ for each sort X such that for each relation R of type X, $\mathcal{H}(f_X)([R]_1) = [R]_0$. We will denote the **category of interpretations on** σ by $\mathrm{Int}(\sigma)$.

[11] In particular, we have $[\langle\rangle]$ is the empty product or a terminal object 1_C.

While the notion of morphism of interpretation will give us a notion of isomorphism, there are times when it will be useful to have a stronger notion.

Definition 3.13. We say that two interpretations $[\cdot]_0, [\cdot]_1$ of σ are **strongly isomorphic** if for every sort $X \in \mathcal{S}_\sigma$, $[X]_0 = [X]_1$ and for every relation $R \in \mathcal{R}_\sigma$, $[R]_0 \cong [R]_1$ in $\mathcal{H}([X]_0)$. In this case we say $[\cdot]_0 \cong_{st} [\cdot]_1$.

We can extend an interpretation $[\cdot]$ to an assignment on $\mathcal{L}_{\omega,\omega}(\sigma)$[12] which takes a formula without equality, φ of type X, to an element $[\varphi] \in \mathcal{H}([X])$. We do this in the obvious way using the internal structure of the hyperdoctrine[13]. We say that an interpretation **satisfies** a sentence φ, $[\cdot] \models \varphi$, if $[\varphi] \cong \top_{\mathcal{H}(1_C)}$.

We will mainly care about interpretations only up to strong isomorphism. Hence the following lemma will be very useful.

Lemma 3.14. *For any formulas ψ_0, ψ_1 (without equality) of type X the following are equivalent for any interpretation* $[\cdot]$

- $[\psi_0] \cong_{st} [\psi_1]$
- $[\cdot] \models (\forall x : X)\psi_0(x) \leftrightarrow \psi_1(x)$.

Proof. This follows from the fact that hyperdoctrines are sound for first order logic. □

The simplest example of an interpretation is the map $[\cdot]_{\mathcal{H}}$ (restricted to the relations of the language $\sigma_{\mathcal{H}}$). We can also define a restriction relation on interpretations.

Definition 3.15. Suppose $\tau \subseteq \sigma$ are relational languages and $[\cdot]$ is an interpretation of σ. We then define $[\cdot]|_\tau$ to be the interpretation of τ which agrees with σ.

[12]If our underlying Heyting prealgebras are complete, we are able to extend this to all of $\mathcal{L}_{\infty,\omega}(\sigma)$.

[13]Notice that this assignment is only unique up to isomorphism in $\mathcal{H}([X])$. As such we consider the particular choice of $[\varphi]$ as part of the structure of $[\cdot]$.

Along with the notion of a restriction we have the notion of a conservative expansion.

Definition 3.16. Suppose $\tau \subseteq \sigma$ and $T \subseteq \mathcal{L}_{\omega,\omega}(\tau)$ is a theory. A theory $T^* \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$ is a **conservative expansion (for interpretations)** of T if

- For all $[\cdot] \in \mathrm{Int}(\sigma)$, if $[\cdot] \models T^*$ then $[\cdot]|_\tau \models T$.
- For all $[\cdot] \in \mathrm{Int}(\tau)$ with $[\cdot] \models T$, there is an $[\cdot]_* \in \mathrm{Int}(\sigma)$ such that $[\cdot]_* \models T^*$ and $[\cdot]_*|_\tau = [\cdot]$.
- For all $[\cdot]_0, [\cdot]_1 \in \mathrm{Int}(\sigma)$, if $[\cdot]_0, [\cdot]_1 \models \psi$ and $[\cdot]_0|_\tau \cong_{st} [\cdot]_1|_\tau$ then $[\cdot]_0 \cong_{st} [\cdot]_1$.

In other words T^* is a conservative expansion of T if there is a map, Ex_{T,T^*}, from interpretations of τ which satisfy T to interpretations of σ which satisfy T^* where $\mathrm{Ex}_{T,T^*}([\cdot])|_\tau \cong_{st} [\cdot]$ (i.e. Ex_{T,T^*} is the inverse to the restriction relation up to strong isomorphism).

The simplest example is the following (which will be important later).

Definition 3.17. Suppose σ is a relational language. We define the **Morleyization** of σ to consist of a language σ_{Mor} defined by:

- Let $\sigma_0 = \sigma \cup \{R_\top, R_\bot\}$ where $R_\top, R_\bot$ are relations on the empty sort $\langle\rangle$.
- Let $\sigma_{i+1} = \sigma_i \cup \{R_\varphi : \varphi \in \mathcal{L}_{\omega,\omega}(\sigma_i)\}$ where R_φ is a relation of the same type as φ.
- $\sigma_{Mor} = \bigcup_{i\in\omega} \sigma_i$.

along with a theory $Mor_\sigma \subseteq \mathcal{L}_{\omega,\omega}(\sigma_{Mor})$ consisting of:

- $R_\top \leftrightarrow (\forall x : X)\top$ and $R_\bot \leftrightarrow (\exists x : X)\bot$.
- For each relation $Q \in \mathcal{R}_\sigma$ of type X we have $(\forall x : X)R_Q(x) \leftrightarrow Q(x)$.
- If $\varphi, \psi \in \mathcal{L}_{\omega,\omega}(\sigma_{Mor})$ are of type X then we have
  - $(\forall x : X)R_{\varphi\wedge\psi}(x) \leftrightarrow [R_\varphi(x) \wedge R_\psi(x)]$.
  - $(\forall x : X)R_{\varphi\vee\psi}(x) \leftrightarrow [R_\varphi(x) \vee R_\psi(x)]$.

– $(\forall x : X)R_{\varphi\to\psi}(x) \leftrightarrow [R_\varphi(x) \to R_\psi(x)]$.

- If $\varphi \in \mathcal{L}_{\omega,\omega}(\sigma_{Mor})$ is of type $\langle X, Y\rangle$ then we have

  – $(\forall x : X)R_{(\exists y:Y)\varphi(x,y)} \leftrightarrow (\exists y : Y)R_\varphi(x, y)$.

  – $(\forall x : X)R_{(\forall y:Y)\varphi(x,y)} \leftrightarrow (\forall y : Y)R_\varphi(x, y)$.

The following are then immediate.

Lemma 3.18. *For $\varphi \in \mathcal{L}_{\omega,\omega}(\sigma_{Mor})$ of type X, $Mor_\sigma \vdash (\forall x : X)\varphi(x) \leftrightarrow R_\varphi(x)$.*

Lemma 3.19. *For any theory $T \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$, then $Mor_\sigma(T) = Mor_\sigma \cup \{R_\varphi : \varphi \in T\}$ is a conservative expansion (for interpretations) of T.*

Lemma 3.20. *If $\tau \subseteq \sigma$ then $\tau_{Mor} \subseteq \sigma_{Mor}$ and $Mor_\tau = Mor_\sigma \cap \mathcal{L}_{\omega,\omega}(\tau_{Mor})$.*

The Morleyization of a language are important because, as we will see in Proposition 4.23, we can define those interpretations (using models in SET) which satisfy the Morleyization of a language.

3.3.3. Models in $(\mathrm{C}, \mathcal{H})$

With the notion of an interpretation in hand we are ready to define the notion of a model in a hyperdoctrine. The goal is to capture the structure of a model in $\mathrm{C}[\mathcal{H}]$ using the structure of the hyperdoctrine $(\mathrm{C}, \mathcal{H})$.

Definition 3.21. If σ is a language let σ_{Rel} be the relational language where:

- $\mathcal{S}_{\sigma_{Rel}} = \mathcal{S}_\sigma$
- $\mathcal{R}_\sigma \subseteq \mathcal{R}_{\sigma_{Rel}}$.
- For every $f \in \mathcal{F}_\sigma$ of type $X \to Y$ there is a $f_g \in \mathcal{R}_{\sigma_{Rel}}$ of type $\langle X, Y\rangle$.
- For each sort $X \in \mathcal{S}_\sigma$ there is a relation $\sim_X \in \mathcal{R}_{\sigma_{Rel}}$ of type $\langle X, X\rangle$.

We define a **model** $\mathcal{N}$ of σ in a hyperdoctrine $(\mathrm{C}, \mathcal{H})$ to be an interpretation $[|\cdot|]_\mathcal{N} \in \mathrm{Int}(\sigma_{Rel})$ where:

- For each sort $X \in \mathcal{S}_\sigma$, $([|X|]_\mathcal{N}, [| \sim_X |]_\mathcal{N})$ is an object in $\mathrm{C}[\mathcal{H}]$ and for each sequence of sorts $\langle X_1, \dots, X_n\rangle$ we have $([|\langle X_1, \dots, X_n\rangle|]_\mathcal{N}, [| \sim_{\langle X_1,\dots,X_n\rangle} |]_\mathcal{N}) \cong ([|X_1|]_\mathcal{N}, [| \sim_{X_1} |]_\mathcal{N}) \times \dots \times ([|X_n|]_\mathcal{N}, [| \sim_{X_n} |]_\mathcal{N})$.

- For each relation $R \in \mathcal{R}_\sigma$ of type X, $[|R|]_\mathcal{N} \in \mathcal{H}([|X|]_\mathcal{N})$ is a strict element on $([|X|]_\mathcal{N}, [| \sim_X |]_\mathcal{N})$ (and hence represents a subobject of $([|X|]_\mathcal{N}, [| \sim_X |]_\mathcal{N})$).

- For each function $f \in \mathcal{F}_\sigma$ of type $X \to Y$ the relation $[|f|]_\mathcal{N}$ is a map in $\mathrm{C}[\mathcal{H}]$ from $([|X|]_\mathcal{N}, [| \sim_X |]_\mathcal{N})$ to $([|Y|]_\mathcal{N}, [| \sim_Y |]_\mathcal{N})$.

Notice by Lemma 3.4 and Lemma 3.5 there is a theory $Th_{Int,\sigma} \subseteq \mathcal{L}_{\omega,\omega}(\sigma_{Rel})$ such that $[| \cdot |]_\mathcal{N}$ is an interpretation associated to a model if and only if $[| \cdot |]_\mathcal{N} \models Th_{Int,\sigma}$. From now on $\mathcal{N}$ (and its variants) will always represent models in $(\mathrm{C}, \mathcal{H})$ with corresponding interpretation $[||]_\mathcal{N}$.

If $\tau \subseteq \sigma$ and $\mathcal{N}$ is a model of σ in $(\mathrm{C}, \mathcal{H})$ we can define $\mathcal{N}|_\tau$ to be the model of τ such that $[| \cdot |]_{\mathcal{N}|_\tau} = [| \cdot |]_\mathcal{N}|_{\tau_{Rel}}$ (i.e. the interpretation associated with $\mathcal{N}|_\tau$ agrees with the interpretation associated to $\mathcal{N}$ on τ_{Rel})

There is a close relationship between σ structures in $\mathrm{C}[\mathcal{H}]$ and models of σ in $(\mathrm{C}, \mathcal{H})$. This is given by the following.

Definition 3.22. For each model $\mathcal{N}$ of σ in $(\mathrm{C}, \mathcal{H})$ there is a σ structure $p(\mathcal{N})$ in $\mathrm{C}[\mathcal{H}]$ where

- For each sort $X \in \mathcal{S}_\sigma$ the object $X^{p(\mathcal{N})} = ([|X|]_\mathcal{N}, [| \sim_X |]_\mathcal{N})$.

- For each relation $R \in \mathcal{R}_\sigma$, $R^{p(\mathcal{N})}$ is the subobject associated to $[|R|]_\mathcal{N}$ (i.e. if $St_m \cong [|R|]_\mathcal{N}$ then $R^{p(\mathcal{N})}$ is the subobject containing m).

- For each function $f \in \mathcal{F}_\sigma$, $f^{p(\mathcal{N})} = [|f|]_\mathcal{N}$.

Definition 3.23. For each σ structure $\mathcal{M}$ in $\mathrm{C}[\mathcal{H}]$ there is a model of σ, $q(\mathcal{M})$, in $(\mathrm{C}, \mathcal{H})$ where:

- For each sort $X \in \mathcal{S}_\sigma$, if $X^\mathcal{M} = (A, E_A)$ then $[|X|]_{q(\mathcal{M})} = A$ and $[| \sim_X |]_{q(\mathcal{M})} = E_A$.

- For each relation $R \in \mathcal{R}_\sigma$ we have $[|R|]_{q(\mathcal{M})}$ is a strict relation which represents the subobject $R^{\mathcal{M}}$.

- For each function $f \in \mathcal{F}_\sigma$ we have $f^{\mathcal{M}} = [|f|]_{q(\mathcal{M})}$.

The following is then immediate.

Proposition 3.24. *(1) For every $\mathcal{M} \in Str_\sigma$ we have $\mathcal{M} = p(q(\mathcal{M}))$.*

(2) For every model $\mathcal{N}$ of σ in $(\mathrm{C}, \mathcal{H})$ we have $\mathcal{N} \cong_{st} q(p(\mathcal{N}))$.

(3) For all model $\mathcal{N}_0, \mathcal{N}_1$ of σ in $(\mathrm{C}, \mathcal{H})$, $[| \cdot |]_{\mathcal{N}_0} \cong_{st} [| \cdot |]_{\mathcal{N}_1}$ if and only if $p(\mathcal{N}_0) = p(\mathcal{N}_1)$.

Proof. This follows immediately from the definition and Lemma 3.7. □

Proposition 3.24 says that there is a bijection between σ structures in $\mathrm{C}[\mathcal{H}]$ and strong isomorphism classes of models of σ in $(\mathrm{C}, \mathcal{H})$.

This bijection between σ structures in $\mathrm{C}[\mathcal{H}]$ and strong isomorphism classes of models of σ in $(\mathrm{C}, \mathcal{H})$ goes even further and (in some sense) preserves formulas.

Proposition 3.25. *For every formula $\varphi \in \mathcal{L}_{\omega,\omega}(\sigma)$ of sort X there is a formula $\hat{\varphi} \in \mathcal{L}_{\omega,\omega}(\sigma_{Rel})$ of sort X such that for any σ structure $\mathcal{M}$ in $\mathrm{C}[\mathcal{H}]$ we have $[|\hat{\varphi}|]_{q(\mathcal{M})} \in \mathcal{H}(X^{\mathcal{M}})$ is a strict relation which represents the subobject $\{\boldsymbol{x} : \varphi(\boldsymbol{x})\}^{\mathcal{M}}$ of $X^{\mathcal{M}}$.*

Proof. We will only give the definition of $\hat{\varphi}$ (by induction) and we leave it to the reader to check that $[|\hat{\varphi}|]_{q(\mathcal{M})}$ has the desired properties. For a more detailed proof we refer the reader to [11] Chapter 2.2.

First for each function symbol F in $\mathcal{F}_\sigma$ we let $\hat{F}$ be the relation $F_g \in \mathcal{R}_{\sigma_{Rel}}$ and for each pair of function symbols F and G of σ where F is of type $X \to Y$ and G is of type $Y \to Z$ we let $\widehat{G \circ F}$ be the formula $(\exists y : Y)F(x, y) \wedge G(y, z)$. In this way we can define, by induction, for each term t in σ of type $X \to Y$, a formula $\hat{t}$ of type $\langle X, Y \rangle$.

- For each sort $X \in \mathcal{S}_\sigma$ we let $(\widehat{x =_X y})$ be $\sim_X (x, y)$.

- If $R \in \mathcal{R}_\sigma$ is of type Y and t is a term of type $X \to Y$ then $\widehat{R(t(x))}$ is $(\exists y : Y)\hat{t}(x, y) \wedge R(y)$.
- If F, G are terms of type $X \to Y$. Then $\widehat{F = G}$ is $(\exists y : Y)(\hat{F}(x, y) \wedge \hat{G}(x, y))$
- If φ and ψ are formulas of type X then
  - $\widehat{\varphi \wedge \psi}(x)$ is $\hat{\varphi}(x) \wedge \hat{\psi}(x)$
  - $\widehat{\varphi \vee \psi}(x)$ is $\hat{\varphi}(x) \vee \hat{\psi}(x)$
  - $\widehat{\varphi \Rightarrow \psi}(x)$ is $\hat{\varphi}(x) \to \hat{\psi}(x) \wedge x \sim_X x$.
- If $\varphi(x_1, \ldots, x_n)$ is a formula of type $\langle X_1, \ldots, X_n \rangle$ then
  - $\widehat{(\exists x_i)\varphi}(x_1, \ldots, x_n)$ is $(\exists x_i : X_i)\hat{\varphi}(x_1, \ldots, x_n)$
  - $\widehat{(\forall x_i)\varphi}(x_1, \ldots, x_n)$ is $(\bigwedge_{j \leq n, j \neq i} x_j \sim_{X_j} x_j) \wedge (\forall x_i : X_i)(x_i \sim_{X_i} x_i \to \hat{\varphi}(x_1, \ldots, x_n))$.

□

For a theory $T \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$ we define $\hat{T} = \{\hat{\varphi} : \varphi \in T\} \subseteq \mathcal{L}_{\omega,\omega}(\sigma_{Rel})$. In particular we have:

Corollary 3.26. *For every theory $T \in \mathcal{L}_{\omega,\omega}(\sigma)$ there is a theory $\hat{T} \in \mathcal{L}_{\omega,\omega}(\sigma_{Rel})$ such that for all σ structures $\mathcal{M}$ in $\mathrm{C}[\mathcal{H}]$ the following are equivalent:*

- $\mathcal{M} \models T$.
- $[|\hat{\varphi}|]_{p(\mathcal{M})} = \top_{\mathcal{H}(1_{\mathrm{C}})}$ *for all* $\varphi \in T$.

Corollary 3.26 reduces the problem of deciding whether a σ structure in $\mathrm{C}[\mathcal{H}]$ satisfies a sentence of $\mathcal{L}_{\omega,\omega}(\sigma)$ to deciding whether a model of σ in $(\mathrm{C}, \mathcal{H})$ satisfies a (closely) related sentence. Hence this reduces the job of counting the number of σ structures in $\mathrm{C}[\mathcal{H}]$ which satisfy a theory to the job of counting the number of interpretations of σ in $(\mathrm{C}, \mathcal{H})$ which satisfy a (closely related) theory.

4. Definability

In this section we give the extra structure we will need to place on a hyperdoctrine in order to be able to characterize, using models in SET, those collections of interpretations satisfying a theory.

4.1. Definable Hyperdoctrines

The notion of a definable hyperdoctrine was chosen to serve two purposes. First the definition was chosen to allow us to describe interpretations in SET[$\mathcal{H}$] via models in SET. Second, the definition was chosen so that many of the examples of hyperdoctrines which arise naturally in the study of realizability (including those associated to realizability toposes over PCAs) are definable.

Definition 4.1. We say a psuedofunctor $\mathcal{H} : \text{SET}^{op} \to \text{Preorder}$ is **definable** if there exists:

- Sets $A_\mathcal{H}, E_\mathcal{H}, S_\mathcal{H}$.
- A partial function[14] $O_\mathcal{H} : E_\mathcal{H} \times A_\mathcal{H} \to A_\mathcal{H}$ along with an element $id \in E_\mathcal{H}$.
- A set $P_\mathcal{H} = \langle p_i : i \in I\rangle$ of **conditions** where
  - $p_i = \langle B_{p_i}, D_{p_i}\rangle$ and $B_{p_i}, D_{p_i} \subseteq A_\mathcal{H}$.
- A set $\Sigma_\mathcal{H} \subseteq \mathfrak{P}(A)$[15].

such that

(i) $S_\mathcal{H} \subseteq \mathfrak{P}(A_\mathcal{H})$.

(ii) $\Sigma_\mathcal{H}$ is the collection of subsets B of $A_\mathcal{H}$ satisfying:

- For some $J \subseteq S_\mathcal{H}$, $B = \bigcup J$.
- For all p_i, if $B_{p_i} \subseteq B$ then $D_{p_i} \subseteq B$.

[14]We will use the standard notation $O_\mathcal{H}(e, a) \downarrow$ to mean $(\exists b)O_\mathcal{H}(e, a) = b$ and $O_\mathcal{H}(e, a) \uparrow$ to mean $\neg O_\mathcal{H}(e, a) \downarrow$.

[15]$\mathfrak{P}(A)$ is the powerset of A.

(iii) $\emptyset, A_{\mathcal{H}} \in \Sigma_{\mathcal{H}}$ (i.e. the empty set and the total set are in $\Sigma_{\mathcal{H}}$).

(iv) $(\forall a \in A_{\mathcal{H}})O_{\mathcal{H}}(id, a) = a$.

(v) For any $X \in$ SET, $\mathcal{H}(X)$ is the preorder $(\text{SET}[X, \Sigma_{\mathcal{H}}], \leq_{\mathcal{H}(X)})$ where $\alpha \leq_{\mathcal{H}(X)} \beta$ if and only if

$$(\exists e \in E)(\forall x \in X)(\forall a \in \alpha(x))O_{\mathcal{H}}(e, a) \downarrow \wedge O_{\mathcal{H}}(e, a) \in \beta(x).$$

(vi) For $f : X \to Y$ a function between sets, $\mathcal{H}(f) : \text{SET}[Y, \Sigma] \to \text{SET}[X, \Sigma]$ is precomposition with f. I.e. $\mathcal{H}(f)(\alpha) = \alpha \circ f$ for any $\alpha \in \text{SET}[Y, \Sigma]$.

Notice that $\mathcal{H}$ is completely determined by the sets $A_{\mathcal{H}}, E_{\mathcal{H}}, S_{\mathcal{H}}, P_{\mathcal{H}}$ and $O_{\mathcal{H}}$ and any such sets satisfying the above determine a psudofunctor. However, for an arbitrary collection of $A_{\mathcal{H}}, E_{\mathcal{H}}, S_{\mathcal{H}}, P_{\mathcal{H}}$ and $O_{\mathcal{H}}$ the corresponding psudofuctore might not be a hyperdoctrine.

Definition 4.2. We say $\mathcal{H} : \text{SET}^{op} \to \text{Heyting}$ is a **definable hyperdoctrine** if $(\text{SET}, \mathcal{H})$ is a first order hyperdoctrine and the composition of $\mathcal{H}$ with the inclusion functor from Heyting to Preorder is a definable psuedofunctor.

The idea behind $A_{\mathcal{H}}, E_{\mathcal{H}}, S_{\mathcal{H}}, O_{\mathcal{H}}$ and $P_{\mathcal{H}}$ are as follows. By (v) of Definition 4.1, in order to define our psudofunctor all we need is to determine $\Sigma_{\mathcal{H}} \subseteq \mathfrak{P}(A_{\mathcal{H}})$ and $\leq_{\mathcal{H}(X)}$. We want to define $\Sigma_{\mathcal{H}}$ so that there is a collection, $S_{\mathcal{H}}$, of subsets of $A_{\mathcal{H}}$ where every element of $\Sigma_{\mathcal{H}}$ is a union of these subsets. However, it turns out that in some circumstances (see Example 4.6 and Example 4.8) we don't want to allow arbitrary unions. Rather we want to allow arbitrary unions subject to some conditions. In particular the conditions which we will require will always be of the form "If every element of B_p is in our set then so must be every element of D_p".

This then just leaves the definition of the ordering $\leq_{\mathcal{H}(X)}$. We want to think of $E_{\mathcal{H}}$ as a collection of transformations of $A_{\mathcal{H}}$ (via $O_{\mathcal{H}}$). For $\alpha, \beta \in \mathcal{H}(X)$ we say $\alpha \leq_{\mathcal{H}(X)} \beta$ if there is a single transformation $e \in E_{\mathcal{H}}$ such that for all $x \in X$, e transforms all elements of $\alpha(x)$ to elements of $\beta(x)$.

Before we move onto several examples there is one more notion we need. We say that an definable hyperdoctrine $(\mathrm{SET}^{op}, \mathcal{H})$ is **κ-presentable** if $|A_\mathcal{H}|, |E_\mathcal{H}|, |S_\mathcal{H}|$ and $|P_\mathcal{H}| \leq \kappa$. We call an $\aleph_0$-presentable definable hyperdoctrines, **countably presentable**. From now on we will let $\mathrm{C} = \mathrm{SET}$ and $(\mathrm{SET}, \mathcal{H})$ will be a definable $\kappa_\mathcal{H}$-presentable hyperdoctrines.

4.1.1. Examples of Definable Hyperdoctrines

In this section we show that several well known examples of first order hyperdoctrines associated with realizability are all definable. We will begin by reviewing the definition the hyperdoctrine associated to a realizability topos over a PCA and then giving the structure that witnesses that it is definable. As the main focus of this paper is realizability toposes over PCAs, for the other examples of definable hyperdoctrines we will simply give the structure that witnesses the hyperdoctines are definable and refer the reader to [11] for the definitions of the hyperdocrines (and a proof that they are hyperdoctrines). Also, as it is routine to check that the below structures actually witness that our hyperdoctrines are definable, we leave their verification to the enthusiastic reader.

Example 4.3 (Realizability over a PCA)**.** *Suppose* $(A, \cdot)$ *is a partial combinatorial algebra and let* $\mathrm{RH}(A)$ *be the hyperdoctrine on SET given by*

- $\mathrm{RH}(A)(X) = (SET[X, \mathfrak{P}(A)], \leq_X)$ *where* $\alpha \leq_X \beta$ *if and only if* $(\exists a \in A)(\forall x \in X)(\forall b \in \alpha(x)) a \cdot b \downarrow$ *and* $a \cdot b \in \beta(x)$.
- *For a map of sets* $f : X \to Y$, $\mathrm{RH}(A)(f)(x) = x \circ f$.

This is the hyperdoctrine where SET$[\mathrm{RH}(A)] = \mathrm{RT}(A)$*, the realizability topos on* A*.*

$\mathrm{RH}(A)$ *is then definable with:*

- $A_{\mathrm{RH}(A)} = E_{\mathrm{RH}(A)} = A$ *and* $O_{\mathrm{RH}(A)}(a, b) \simeq a \cdot b$[16].

[16]Here $\simeq$ is the relation where either both sides are undefined or both sides exist and are equal.

- $S_{\mathrm{RH}(A)} = \{\{a\} : a \in A\}$ *and* $P_{\mathrm{RH}(A)} = \emptyset$.

For a more thorough introduction to this hyperdoctrine see [10].

Example 4.4 (Relative Realizability)**.** *Suppose* $(A, \cdot)$ *is a PCA and* $(A^{\#}, \cdot)$ *is an elementary sub-PCA. Let* $\mathrm{RR}(A^{\#}, A)$ *be the hyperdoctrine constructed from* A *and* $A^{\#}$ *(see [11] Chapter 2.6.9 for the definition).* $\mathrm{RR}(A)$ *is then definable with:*

- $A_{\mathrm{RR}(A^{\#},A)} = A$, $E_{\mathrm{RR}(A^{\#},A)} = A^{\#}$ *and* $O_{\mathrm{RR}(A^{\#},A)}(a, b) \simeq a \cdot b$.

- $S_{\mathrm{RR}(A^{\#},A)} = \{\{a\} : a \in A\}$ *and* $P_{\mathrm{RR}(A)} = \emptyset$.

Example 4.5 (Realizability over Ordered PCAs)**.** *Suppose* $(A, \cdot, \leq_A)$ *is an ordered PCA. Let* $\mathrm{OH}(A)$ *be the hyperdoctrine constructed from* A *(See [11] Chapter 2.6.2).* $\mathrm{OH}(A)$ *is then definable with:*

- $A_{\mathrm{OH}(A)} = E_{\mathrm{OH}(A)} = A$ *and* $O_{\mathrm{OH}(A)}(a, b) \simeq a \cdot b$.

- $S_{\mathrm{OH}(A)} = \{\{b : b \leq_A a\} : a \in A\}$ *and* $P_{\mathrm{OH}(A)} = \emptyset$.

Notice, because of how $S_{\mathrm{OH}(A)}$ *is defined,* $\Sigma_{\mathrm{OH}(A)}$ *is the collection of downward closed sets.*

Example 4.6 (Extensional Realizability)**.** *Let* ER *be the hyperdoctrine constructed for extensional realizability (See [11] Chapter 2.6.6).* ER *is then definable with:*

- $A_{\mathrm{ER}} = \{(x, y) : x, y \in \mathbb{N}\}$, $E_{\mathrm{ER}} = \mathbb{N}$ *and* $O_{RR}(e, (x, y)) \simeq (\{e\}(x), \{e\}(y))$[17].

- $S_{\mathrm{ER}} = \{\{(x, y)\} : (x, y) \in A\}$.

- $P_{\mathrm{ER}} = P_{ref} \cup P_{sym} \cup P_{tran}$ *where:*

  - $P_{ref} = \{\langle B_{ref}(x, y), D_{ref}(x, y)\rangle : x, y \in \mathbb{N}\}$ *where* $B_{ref}(x, y) = \{(x, y)\}$ *and* $D_{ref}(x, y) = \{(x, x), (y, y)\}$.

[17] $\{e\}(x)$ is the value of the eth computer run on x. Also $(\{e\}(x), \{e\}(y))$ is defined if and only if both $\{e\}(x)\downarrow$ and $\{e\}(y)\downarrow$.

- $P_{sym} = \{\langle B_{sym}(x,y), D_{sym}(x,y)\rangle : x,y \in \mathbb{N}\}$ *where* $B_{sym}(x,y) = \{(x,y)\}$ *and* $D_{sym}(x,y) = \{(y,x)\}$.
- $P_{tran} = \{\langle B_{tran}(x,y,z), D_{tran}(x,y,z)\rangle : x,y,z \in \mathbb{N}\}$ *where* $B_{tran}(x,y,z) = \{(x,y),(y,z)\}$ *and* $D_{tran}(x,y) = \{(x,z)\}$.

In particular, these conditions guarantee the elements of $E \in \Sigma_{\mathrm{ER}}$ *are exactly the equivalence relations on some subsets of* $\mathbb{N}$ *(the specific subset is* $\{x : (x,x) \in E\} \subseteq \mathbb{N}$*).*

Example 4.7 (Modified Realizability)**.** *Let* MR *be the hyperdoctrine constructed for modified realizability (See [11] Chapter 2.6.7).* MR *is then definable with:*

- $A_{\mathrm{MR}} = \{(n,0),(n,1) : n \in \mathbb{N}\}$, $E_{\mathrm{MR}} = \mathbb{N}$ *and* $O_{\mathrm{MR}}(e,(n,i)) \simeq (\{e\}(n), i)$.
- $S_{\mathrm{MR}} = \{(n,1) : n \in \mathbb{N}\} \cup \{(n,0),(n,1) : n \in \mathbb{N}\}$ *and* $P_{\mathrm{MR}} = \emptyset$.

In particular Σ_{MR} *consists of exactly those subsets* $B \subseteq \mathbb{N} \times 2$ *where if* $B_0 = \{(n,0) \in B : n \in \mathbb{N}\}$ *and* $B_1 = \{(n,1) \in B : n \in \mathbb{N}\}$ *then* $B_0 \subseteq B_1$*. In this way there is a canonical bijection between* Σ_{MR} *and* Σ_2 *from [11] Chapter 2.6.7.*

Example 4.8 (Lifschitz Realizability)**.** *Let* LR *be the hyperdoctrine constructed for Lifschitz realizability (See [11] Chapter 2.6.8).* LR *is then definable with:*

- $A_{\mathrm{LR}} = J$, $E_{\mathrm{LR}} = \mathbb{N}$ *and* $O_{\mathrm{LR}}(e,n) \simeq \{e\}(n)$.
- $S_{\mathrm{LR}} = \{\{j\} : j \in J\}$.
- $P_{\mathrm{LR}} = P_{(i)} \cup P_{(ii)}$ *where*
  - $P_{(i)} = \{\langle \{e\}, \{f\}\rangle : e,f \in J$ *and* $V_f \subseteq V_e\}$.
  - $P_{(ii)} = \{\langle \{e,f\}, \{g\}\rangle : e,f,g \in J$ *and* $V_e \cup V_f = V_g\}$.

4.2. Characterizing Interpretations by Models in SET

Definition 4.9. Let $\sigma_{\mathrm{DS}} = \{\mathrm{DS}\} \cup \{A,E,S,O\} \cup \{c_a : a \in A_{\mathcal{H}}\} \cup \{c_e : e \in E_{\mathcal{H}}\} \cup \{c_s : s \in S_{\mathcal{H}}\}$ where DS is (the only) sort, A,E,S are unary relations, O is a relation of arity three and each c_a, c_e, c_s are constants.

We then let $T_{\mathrm{DS}} \in \mathcal{L}_{\kappa^+_{\mathcal{H}},\omega}(\sigma_{\mathrm{DS}})$ be the theory which says

- $(\forall x : \mathrm{DS}) A(x) \vee E(x) \vee S(x)$.

- $(\forall x : \mathrm{DS}) S(x) \leftrightarrow (\neg A(x) \wedge \neg E(x))$.

- $(\forall x : \mathrm{DS}) A(x) \leftrightarrow \bigvee_{a \in A_{\mathcal{H}}} (x = c_a)$.

- $(\forall x : \mathrm{DS}) E(x) \leftrightarrow \bigvee_{e \in E_{\mathcal{H}}} (x = c_e)$.

- $(\forall x : \mathrm{DS}) S(x) \leftrightarrow \bigvee_{s \in S_{\mathcal{H}}} (x = c_s)$.

- $\bigwedge \{O(c_e, c_a, c_b) : O_{\mathcal{H}}(e, a) \downarrow \text{ and } O_{\mathcal{H}}(e, a) = b\}$.

- $\bigwedge \{\neg O(c_e, c_a, c_b) : O_{\mathcal{H}}(e, a) \uparrow \text{ or } [O_{\mathcal{H}}(e, a) \downarrow \text{ and } O_{\mathcal{H}}(e, a) \neq b]\}$.

The theory T_{DS} has names for each element of $A_{\mathcal{H}}, E_{\mathcal{H}}$ and $S_{\mathcal{H}}$ and captures the action of $E_{\mathcal{H}}$ on $A_{\mathcal{H}}$. Further T_{DS} has a unique model up to isomorphism.

We now show how to characterize an interpretation.

Definition 4.10. If σ_R is a language with a single relation R of type X let $\mathrm{L}(\sigma_R) = \sigma_{\mathrm{DS}} \cup \{H_X : X \in \mathcal{S}_\sigma\} \cup \{i_{\langle X_1, \dots, X_n \rangle} : H_{\langle X_1, \dots, X_n \rangle} \to H_{X_1} \times \cdots \times H_{X_n}\} \cup \{R_S, R_A\}$ where the H_X's are sorts and R_A, R_S are relations on $H_X \times \mathrm{DS}$. We then let $T(\sigma_R)$ be the theory which says:

(i) T_{DS}.

(ii) $i_{\langle X_1, \dots, X_n \rangle}$ is an isomorphism for all $\{X_1, \cdots, X_n\} \subseteq \mathcal{S}_\sigma$.

(iii) $(\forall x \in H_X)(\forall a \in \mathrm{DS}) R_A(x, a) \to A(a)$.

(iv) $(\forall x \in H_X)(\forall s \in \mathrm{DS}) R_S(x, s) \to S(s)$.

(v) $(\forall x \in H_X) \bigwedge_{a \in A_{\mathcal{H}}} R_A(x, c_a) \to \bigvee_{a \in s \in S_{\mathcal{H}}} R_S(x, s)$.

(vi) $(\forall x \in H_X) \bigwedge_{s \in S_{\mathcal{H}}} R_S(x, c_s) \leftrightarrow [\bigwedge_{a \in S_{\mathcal{H}}} R_A(x, c_a)]$.

(vii) For each $p \in P$, $(\forall x \in H_X)[\bigwedge_{a \in B_p} R_A(x, c_a)] \to [\bigwedge_{a \in D_p} R_A(x, c_a)]$.

Notice $T(\sigma_R) \in \mathcal{L}_{\kappa^+, \omega}(\mathrm{L}(\sigma_R))$.

A model $\mathcal{U}$ of $T(\sigma_R)$ gives an interpretation $[\cdot]_{\mathcal{U}}$ of σ_R in the following way. First, by (ii) we can associate every sort of the form $H_{\langle X_1,\dots,X_n\rangle}$ with $H_{X_1} \times \dots \times H_{X_n}$ in a canonical way (which we will do from now on without mention). We then let $[X]_{\mathcal{U}} = H_X^{\mathcal{U}}$ for every sort X and for each $x \in [X]_{\mathcal{U}}$ we let $[R]_{\mathcal{U}}(x) = \{a : \mathcal{U} \models R_A(x, c_a)\} \subseteq A$. It then follows immediately from Definition 4.1 that for each $x \in X$, $[R]_{\mathcal{U}}(x) \in \Sigma_{\mathcal{H}}$ and so $[R]_{\mathcal{U}} \in \mathcal{H}(X)$. Under this association we also have $\mathcal{U} \models R_S(x, s)$ if and only if $s \subseteq [R]_{\mathcal{U}}(x)$. In particular this means if two models of $T(\sigma_R)$ have the same restriction to $T(\sigma_R - \{R_S\})$ then they are actually the same model.

Definition 4.11. Let σ be a relational language. For each relation $R \in \mathcal{R}_\sigma$ of type X let σ_R be the relational language with the same sorts as σ and the single relation R of type X[18]. Let $\mathrm{L}(\sigma) = \bigcup\{\mathrm{L}(\sigma_R) : R \in \mathcal{R}_\sigma\}$ and let $T(\sigma) = \bigcup\{T(\sigma_R) : R \in \mathcal{R}_\sigma\}$.

From now on $\mathcal{U}$ (and its variants) will always represent models in SET which satisfy a theory $T(\sigma)$.

Proposition 4.12. *There is an isomorphism of categories $h_\sigma : Mod_{\mathrm{L}(\sigma)}(T(\sigma)) \to Int(\sigma)$ with inverse $j_\sigma : Int(\sigma) \to Mod_{\mathrm{L}(\sigma)}(T(\sigma))$. Further if $X \in \mathcal{S}_\sigma$ then for each model $\mathcal{U}$ of $T(\sigma)$ if $[\cdot]_{\mathcal{U}} = h_\sigma(\mathcal{U})$ then $H_X^{\mathcal{U}} = [X]_{\mathcal{U}}$.*

Proof. For each $\mathcal{U} \models T(\sigma)$ we define an interpretation $[\cdot]_{\mathcal{U}}$ as follows:

- For each sort $X \in \mathcal{S}_\sigma$ we let $[X]_{\mathcal{U}} = H_X^{\mathcal{U}}$.
- For any relation $R \in \mathcal{R}_\sigma$ of type X we let $[R]_{\mathcal{U}} : H_X^{\mathcal{U}} \to \mathfrak{P}(A_{\mathcal{H}})$ where $[R]_{\mathcal{U}}(x) = \{a \in A_{\mathcal{H}} : \mathcal{M} \models R_A(x, c_a)\}$.

By conditions $(v), (vi)$ of Definition 4.9 for each $x \in X$, $[R]_{\mathcal{U}}(x) = \bigcup\{s : s \in S_{\mathcal{H}}$ and $\mathcal{U} \models S_R(x, c_s)\}$. Further, by condition (vii) of Definition 4.9 $[R]_{\mathcal{U}}(x)$ satisfies all of the conditions in $P_{\mathcal{H}}$ and hence $[R]_{\mathcal{U}}(x) \in \Sigma_{\mathcal{H}}$. In particular this means $[R]_{\mathcal{U}} \in \mathcal{H}([X]_{\mathcal{U}})$ and so $[\cdot]_{\mathcal{U}}$ is an interpretation. We let $h_\sigma(\mathcal{U}) = [\cdot]_{\mathcal{U}}$.

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[18] Notice that if $R \neq R'$ then $\sigma_R \cap \sigma_{R'} = \mathcal{S}_{\sigma_R} = \mathcal{S}_{\sigma_{R'}}$.

Now suppose $f \in \mathrm{Mod}_\sigma[\mathcal{U}_0, \mathcal{U}_1]$ (where $\mathcal{U}_0, \mathcal{U}_1 \models T(\sigma)$) and let $f_X : H_X^{\mathcal{U}_0} \to H_X^{\mathcal{U}_1}$ be the corresponding map of sort X. Further suppose $R \in \mathcal{R}_\sigma$ is of type X. Then for all $a \in A_{\mathcal{H}}$ and for all $x \in H_X^{U_0}$, we have $\mathcal{U}_0 \models R_A(x, c_a) \Leftrightarrow \mathcal{U}_1 \models R_A(f_X(x), c_a)$. Hence for all $x \in X$, $\mathcal{H}(f_X)([R]_{\mathcal{U}_1})(x) = \{a : \mathcal{U} \models R_A(x, c_a)\} = [R]_{U_0}(x)$ and so $\mathcal{H}(f_X)([R]_{\mathcal{U}_1}) = [R]_{U_0}$. But this then means that $\langle f_X : X \in \mathcal{S}_\sigma \rangle \in \mathrm{Int}(h_\sigma(\mathcal{U}_0), h_\sigma(\mathcal{U}_1)]$.

In the other direction if $[\cdot]$ is an interpretation of σ let $j_\sigma([\cdot]) = \mathcal{U}_{[\cdot]}$ be the model of $\mathrm{L}(\sigma)$ where:

- For every sort $X \in \mathcal{S}_\sigma$, $H_X^{\mathcal{U}_{[\cdot]}} = [X]$.
- For every relation $R \in \mathcal{R}_\sigma$, $\mathcal{U}_{[\cdot]} \models R_A(x, c_a)$ if and only if $a \in [R](x)$ and $\mathcal{U}_{[\cdot]} \models R_S(x, c_s)$ if and only if $s \subseteq [R](x)$.

It is then immediate from the definition of $T(\sigma)$ that $\mathcal{U}_{[\cdot]} \models T(\sigma)$. Further it is immediate that if $f : [\cdot]_0 \to [\cdot]_1$ then f extends to a map from $j_\sigma([\cdot]_0)$ to $j_\sigma([\cdot]_1)$ in the obvious way.

It then easy to check that h_σ and j_σ are inverse functors and hence witness the isomorphism of the two categories. $\square$

From now for $\mathcal{U} \in \mathrm{Mod}_\sigma(T(\sigma))$ we will let $[\cdot]_{\mathcal{U}} = h_\sigma(\mathcal{U})$. From the above argument it is also immediate that the isomorphism of categories between $\mathrm{Int}(\sigma)$ and $\mathrm{Mod}_{\mathrm{L}(\sigma)}(T(\sigma))$ commutes with restriction.

Corollary 4.13. *If $\tau \subseteq \sigma$ and $\mathcal{U} \models T(\sigma)$ then $[\cdot]_{\mathcal{U}}|_\tau = [\cdot]_{\mathcal{U}|_{\mathrm{L}(\tau)}}$*

4.3. Definability of the Hyperdoctrine Structure

Now that we have a way of associating to each interpretation of σ a model of $\mathrm{L}(\sigma)$ in SET, the next step will be to define subsets of interpretations on σ using formulas in $\mathcal{L}_{\infty,\omega}(\mathrm{L}(\sigma))$. This suggests the following definition.

Definition 4.14. Suppose V is a collection of interpretations of σ. We say a formula $\varphi \in \mathcal{L}_{\infty,\omega}(\mathrm{L}(\sigma))$ **defines** V if

- $\mathrm{Mod}_{\mathrm{L}(\sigma)}(\varphi) \subseteq \mathrm{Mod}_{\mathrm{L}(\sigma)}(T(\sigma))$.

- For all models $\mathcal{U}$ of $T(\sigma)$, $\mathcal{U} \models \varphi$ if and only if $h_\sigma(\mathcal{U}) \in V$.

The idea is that a formula φ defines V if the models of φ are exactly those models which correspond, under h_σ, to interpretations in V.

Definition 4.15. We say a formula $\varphi \in \mathcal{L}_{\infty,\omega}(\mathrm{L}(\sigma))$ of type $\langle X, \mathrm{DS}\rangle$ **defines a relation on** X if for every $\mathcal{U} \models T(\sigma)$ there is a model $\mathcal{U}' \models T(\sigma \cup \{R\})$ (where R is a new relation of type $\langle X, \mathrm{DS}\rangle$) such that:

- $\mathcal{U}' \models (\forall x : H_X)(\forall a : \mathrm{DS})R(x, a) \leftrightarrow \varphi(x, a)$
- $\mathcal{U}'|_{\mathrm{L}(\sigma)} = \mathcal{U}$.

We will abuse notation and let $[\overline{\varphi}]_{\mathcal{U}} \in \mathcal{H}([X]_{\mathcal{U}})$ where $[\overline{\varphi}]_{\mathcal{U}}(x) = \{a : \mathcal{U} \models R_A(x, c_a)\}$.

When we have a formula which defines a relation, we would like to simply add in a new relation equivalent to the formula and the look at the collection of those interpretations defined by that new theory. However this runs into a slight problem in that this collection will not be closed under strong isomorphism (as the formula only defines a single element of $\mathcal{H}([X]_{\mathcal{U}})$). Fortunately we can fix this.

Definition 4.16. If φ defines a relation let $Cl(\sigma, \varphi, R)$ be the theory extending $T(\sigma)$ which says:

- $\bigvee_{e \in E}(\forall x : X)(\forall a : \mathrm{DS})R(x, a) \rightarrow (\exists b : \mathrm{DS})O(e, a, b) \wedge \varphi(x, b)$
- $\bigvee_{f \in E}(\forall x : X)(\forall a : \mathrm{DS})\varphi(x, a) \rightarrow (\exists b : \mathrm{DS})O(f, a, b) \wedge R(x, b)$

Let $V(\sigma, \varphi, R) = \{[\cdot] : j_\sigma([\cdot]) \models Cl(\sigma, \varphi, R)\}$.

Lemma 4.17. *$V(\sigma, \varphi, R)$ is closed under strong isomorphism and $Cl(\sigma, \varphi, R)$ defines $V(\sigma, \varphi, R)$.*

Proof. This follows because $Cl(\sigma, \varphi, R)$ says that for any model $\mathcal{U}$, $[R]_{\mathcal{U}} \leq [\overline{\varphi}]_{\mathcal{U}}$ and $[\overline{\varphi}]_{\mathcal{U}} \leq [R]_{\mathcal{U}}$. $\square$

The first (and easiest) examples of collections of interpretations which we can define are the following.

Definition 4.18. Let $F_\top, F_\bot$ be formulas of type $\langle X, \mathrm{DS}\rangle$ where $F_\top(x,a) \Leftrightarrow A(a)$ and $F_\bot(x,a) \leftrightarrow \bot$.[19]

That $F_\top, F_\bot$ define relations on X follows immediately from the fact that $\emptyset, A_{\mathcal{H}} \in \Sigma_{\mathcal{H}}$. We then also have:

Lemma 4.19. *If* $T_\top(\sigma, R) = Cl(\sigma, F_\top, R)$ *and* $T_\bot(\sigma, R) = Cl(\sigma, F_\bot, R)$ *then*

- $T_\top(\sigma, R)$ *defines* $\{[\cdot] : [\cdot] \models (\forall x : X)R(x) \leftrightarrow \top\}$.
- $T_\bot(\sigma, R)$ *defines* $\{[\cdot] : [\cdot] \models (\forall x : X)R(x) \leftrightarrow \bot\}$.

Proof. This is because $V(\sigma, F_\top, R) = \{[\cdot] : [\cdot] \models [R] \cong_{st} \top_{\mathcal{H}(X)}\}$ and $V(\sigma, F_\bot, R) = \{[\cdot] : [\cdot][R] \cong_{st} \bot_{\mathcal{H}(X)}\}$. □

Next we show we can define $\leq$.

Definition 4.20. Let $R_0, R_1 \in \mathcal{R}_\sigma$ be relations of type X and let $T_\leq(\sigma, R^0, R^1)$ be the theory extending $T(\sigma)$ which says

- $\bigvee_{e\in E}(\forall x : H_X)(\forall a : \mathrm{DS})R^0_A(x,a) \rightarrow (\exists a' : \mathrm{DS})O(c_e, a, a') \wedge R^1_A(x, a')$.

Lemma 4.21. $T_\leq(\sigma, R_0, R_1)$ *defines* $\{[\cdot] : [R_0] \leq [R_1]\}$.

Proof. This is immediate from the definition $\leq$ in definable hyperdoctrines. □

4.4. Simple definable Hyperdoctrines

Unfortunately being a definable hyperdoctrine does not guarantee enough structure to define all of the internal operations of a hyperdoctrine using formulas of $\mathcal{L}_{\infty,\omega}$. In this section we define the notion of a simple definable hyperdoctrine (i.e. one which does have enough structure) and we show that in the case of realizability toposes over PCAs the corresponding hyperdoctrines are simple.

[19] I.e. it never holds

Definition 4.22. We say a κ-presentable definable hyperdoctrine $\mathcal{H}$ is **simple** if there are sentences of $\mathcal{L}_{\kappa^+,\omega}$ where:.

($\wedge$) $T_\wedge(\sigma, R_0, R_1, R^*)$ defines $\{[\cdot] : [\cdot] \models (\forall x : X)R^*(x) \leftrightarrow R_0(x) \wedge R_1(x)\}$.

($\vee$) $T_\vee(\sigma, R_0, R_1, R^*)$ defines $\{[\cdot] : [\cdot] \models (\forall x : X)R^*(x) \leftrightarrow R_0(x) \vee R_1(x)\}$.

($\rightarrow$) $T_\rightarrow(\sigma, R_0, R_1, R^*)$ defines $\{[\cdot] : [\cdot] \models (\forall x : X)R^*(x) \leftrightarrow [R_0(x) \rightarrow R_1(x)]\}$.

($\exists X$) $T_{(\exists Y)}(\sigma, R_0, R^*)$ defines $\{[\cdot] : [\cdot] \models (\forall x : X)R^*(x) \leftrightarrow [(\exists y : Y)R_0(x, y)]\}$.

($\forall X$) $T_{(\forall Y)}(\sigma, R_0, R^*)$ defines $\{[\cdot] : [\cdot] \models (\forall x : X)R^*(x) \leftrightarrow [(\forall y : Y)R_0(x, y)]\}$.

From now on $(\mathrm{SET}, \mathcal{H})$ will always be a simple definable hyperdoctrine. The following is the most important result of Section 4. It tells us that in simple definable hyperdoctrines we can define those interpretations which satisfy a given sentence.

Lemma 4.23. *Let σ be a relational language. Then the collection of interpretations of σ_{Mor} which satisfy Mor_σ is definable by a $\mathcal{L}_{\kappa^+_{\mathcal{H}},\omega}(\mathrm{L}(\sigma_{Mor}))$ sentence.*

Proof. Let Th^σ_{Mor} say:

- $T_\top(\sigma_{Mor}, R_\top)$ and $T_\bot(\sigma_{Mor}, R_\bot)$.
- $T_\leq(\sigma_{Mor}, Q, R_Q)$ and $T_\leq(\sigma_{Mor}, R_Q, Q)$ for every relation $Q \in R_{\sigma_{Rel}}$.
- For every pair of formulas ψ_0, ψ_1 of type X in Mor_σ:
  - $T_\wedge(\sigma_{Mor}, R_{\psi_0}, R_{\psi_1}, R_{\psi_0\wedge\psi_1})$.
  - $T_\vee(\sigma_{Mor}, R_{\psi_0}, R_{\psi_1}, R_{\psi_0\vee\psi_1})$.
  - $T_\rightarrow(\sigma_{Mor}, R_{\psi_0}, R_{\psi_1}, R_{\psi_0\rightarrow\psi_1})$.
- For every formulas ψ of type $\langle X, Y\rangle$ in Mor_σ:
  - $T_{\exists X}(\sigma_{Mor}, R_\psi, R_{(\exists x:X)\psi})$.
  - $T_{\forall X}(\sigma_{Mor}, R_\psi, R_{(\forall x:X)\psi})$.

Now by Lemma 4.21, Lemma 4.19 and Definition 4.22 we see Th^{σ}_{Mor} classifies $\bigcap_{\varphi \in Mor_\sigma} \{[\cdot] : [\cdot] \models \varphi\} = \{[\cdot] : [\cdot] \models Mor_\sigma\}$. □

Corollary 4.24. *For each $T \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$ there is a theory $Th^{\sigma}_{Mor}(T) \in \mathcal{L}_{\kappa^{+}_{\mathcal{H}},\omega}(\mathrm{L}(\sigma_{Mor}))$ which defines $\{[\cdot] : [\cdot] \models Mor_\sigma \cup T\}$*

Proof. We let $Th^{\sigma}_{Mor}(T) = Th^{\sigma}_{Mor} \cup \{T_\top(\sigma, R_\varphi) : \varphi \in T\}$. □

4.4.1. Examples

Not surprisingly many of the examples in Section 4.1.1 turn out to be simple. However as we will only focus on realizability toposes over a PCA we will prove the hyperdoctrines RH(A) are simple and leave the other examples to the enthusiastic reader.

Before we begin recall that in any PCA, $(A, \cdot)$, there are elements m, m_1, m_2 such that $(a, a') \mapsto (m \cdot a) \cdot a'$ is an injection with left inverse $a \mapsto (m_1 \cdot a, m_2 \cdot a)$. Also recall (or see [10] p. 268)

Lemma 4.25. *For $\Phi_0, \Phi_1 \in \mathcal{H}(X)$,*

- $\Phi_0 \wedge \Phi_1 \cong_{st} \lambda x \in X.\{(m \cdot a) \cdot a' : a \in \Phi_0(x) \wedge a' \in \Phi_1(x)\}$
- $\Phi_0 \vee \Phi_1 \cong_{st} \lambda x \in X.\{(m \cdot m_1) \cdot a : a \in \Phi_0(x)\} \cup \{(m \cdot m_2) \cdot a' : a' \in \Phi_1(x)\}$
- $\Phi_0 \rightarrow \Phi_1 \cong_{st} \lambda x \in X.\{a : (\forall b \in \Phi_0(x)) a \cdot b \downarrow \textit{ and } a \cdot b \in \Phi_1(x)\}$

and for $\Phi \in \mathcal{H}(X \times Y)$

- $(\exists Y)_X \Phi \cong_{st} \lambda x \in X. \bigcup_{y \in Y} \Phi(x, y)$
- $(\forall Y)_X \Phi \cong_{st} \lambda x \in X. \bigcap_{y \in Y} \Phi(x, y)$

Example 4.26. *For $R^0, R^1 \in \mathcal{R}_\sigma$ of type X we define the following formulas:*

- *$D_\wedge(R^0, R^1)(x, a)$ is the formula of type $\langle H_X, DS\rangle$ equivalent to:*

$$(\exists b, b', b" : DS)[O(m, b, b') \wedge O(b', b", a)] \wedge R^0_A(x, b) \wedge R^1_A(x, b")$$

- $D_{\to}(R^0, R^1)(x,a)$ *is the formula of type* $\langle H_X, DS\rangle$ *equivalent to:*

$$(\exists b, b' : DS)[R^0_A(x,b) \wedge O(m, m_1, b') \wedge O(b', b, a)] \vee \\ [R^1_A(x,a') \wedge O(m, m_2, a") \wedge O(a", a', a)]$$

- $D_{\to}(R^0, R^1)(x,a)$ *is the formula of type* $\langle H_X, DS\rangle$ *which is equivalent to:*

$$(\forall b : DS) R^0_A(x,b) \to (\exists b' : DS) O(a,b,b') \wedge R^1_A(x,b')$$

and for $R \in \mathcal{R}_\sigma$ *of type* $\langle X, Y\rangle$ *we define the following formulas:*

- $D_{(\exists Y)}(R)(x,a)$ *is the formula of type* $\langle H_{\langle X,Y\rangle}, DS\rangle$ *equivalent to* $(\exists y : Y) A_R(x,y,a)$

- $D_{(\forall Y)}(R)(x,a)$ *is the formula of type* $\langle H_{\langle X,Y\rangle}, DS\rangle$ *equivalent to* $(\forall y : Y) A_R(x,y,a)$

The following lemma is then immediate as all of these functions were designed to capture the corresponding operations from Lemma 4.25.

Lemma 4.27. *We have*

- $V(\sigma, D_\wedge(R_0,R_1), R^*) = \{[\cdot] : [\cdot] \models (\forall x : X) R^*(x) \leftrightarrow [R_0(x) \wedge R_1(x)]\}$

- $V(\sigma, D_\vee(R_0,R_1), R^*) = \{[\cdot] : [\cdot] \models (\forall x : X) R^*(x) \leftrightarrow [R_0(x) \vee R_1(x)]\}$

- $V(\sigma, D_\to(R_0,R_1), R^*) = \{[\cdot] : [\cdot] \models (\forall x : X) R^*(x) \leftrightarrow [R_0(x) \to R_1(x)]\}$

- $V(\sigma, D_{(\exists X)}(R), R^*) = \{[\cdot] : [\cdot] \models (\forall y : Y) R^*(y) \leftrightarrow (\exists x : X) R(x,y)\}$

- $V(\sigma, D_{(\forall X)}(R), R^*) = \{[\cdot] : [\cdot] \models (\forall y : Y) R^*(y) \leftrightarrow (\forall x : X) R(x,y)\}$

Corollary 4.28. RH(A) *is a simple hyperdoctrine.*

Proof. This follows immediately from 4.17. $\square$

5. Size of Objects

5.1. Size Of The Underlying Set

Looking at the construction of SET[$\mathcal{H}$] from (SET, $\mathcal{H}$) we see that a natural candidate for the notion of the size of an object $(X, \sim_X)$ is just the size of X. Unfortunately this notion of size has the serious drawback that not only is it not closed under isomorphism, but for every $(X, \sim_X)$ there are isomorphic objects with arbitrarily large underlying sets.

Lemma 5.1. *Suppose $(X, \sim_X) \in SET[\mathcal{H}]$. Then for each set κ there is an object $(X \times \kappa, \sim^*_\kappa)$ which is isomorphic to $(X, \sim_X)$.*

Proof. Let $\sim^*_\kappa (\langle x, i\rangle, \langle x', j\rangle) =\sim_X (x, x')$. Then functions $F(\langle x, i\rangle, y) =\sim_X (x, y)$ and $G(y, \langle x, i\rangle) =\sim_X (y, x)$ are then easily seen to be inverses of each other. □

This leads to the following definition.

Definition 5.2. We say $(X, \sim_X) \in$ SET[$\mathcal{H}$] is **κ-generated** if there is an $(X', \sim_{X'})$ such that $|X'| = \kappa$ and $(X, \sim_X) \cong (X', \sim_{X'})$. We say $(X, \sim)$ is **countably generated** if it is $\aleph_0$-generated.

As we will see, being countably generated turns out to be a very natural notion from the point of view of descriptive set theory. It is also worth mentioning that if an object is κ generated it is also κ' generated for all $\kappa' \geq \kappa$ (by Lemma 5.1).

Of course the natural example of an object we would hope would be countably generated is the natural number object in SET[$\mathcal{H}$]. And, at least in the case of realizability toposes over PCAs this is the case.

Lemma 5.3. *Suppose $(A, \cdot)$ is a PCA and $\mathbb{N}_A$ is a natural number object in* RT(A)*. Then $\mathbb{N}_A$ is countably generated.*

Proof. Let $\{\overline{n} : n \in \mathbb{N}\}$ be the Curry numerals in A. Then $(\mathbb{N}, \sim_A)$ is a natural number object where $\sim_A (n, m) = \{\overline{n} : n = m\}$ (see [11] p. 268). □

5.2. *Relative Size*

Our second notion of size is a relative one. Instead of determining the size of an object outright we will instead say when the size of one object is "bound" by the size of another. In this way, if we have an object which is in some sense canonically countable (like say the natural number object) then we can think of all those objects whose size is bound by it as also being countable.

Definition 5.4. Suppose $A \in \mathrm{C}[\mathcal{H}]$. We say an object $B \in \mathrm{C}[\mathcal{H}]$ is **monic bounded** by A if there is a monomorphism $m : B \to A$ in $\mathrm{C}[\mathcal{H}]$.

Lemma 5.5. *If $A \in SET[\mathcal{H}]$ is κ-generated and $B \in SET[\mathcal{H}]$ is monic bounded by A then B is κ-generated.*

Proof. This follows immediately from Lemma 3.7. □

A disadvantage the notion of monic bounded has over the notion of begin κ-generated is that the size of an object is no longer a cardinal. However an advantage of the notion of monic bounded is that it is closed under equivalences of categories (whereas being κ-generated is not).

We end this section with the observation that we could have used epimorphisms to compare the size of objects and obtained a similar notion of A being epi bound by B (i.e. there is an epimorphism from B to A). However we do not consider this notion here as there is no (obvious) way to guarantee an object which is epi bound by a countably generated object will be countably generated (and the techniques used in this paper to count the number of countable models of a theory only work if the models we are looking at are all countably generated).

6. Number of Countable Models

We are now, finally, ready to prove the main results of this paper.

Definition 6.1. We say a σ structure $\mathcal{M}$ in SET[$\mathcal{H}$] is κ-generated if for every sort $X \in \mathcal{S}_\sigma$, $X^{\mathcal{M}}$ is a κ-generated object.

From now on $\mathcal{H}$ will be countably presented and all languages will be countable.

Lemma 6.2. *There is a surjection f_σ from the objects of $Mod_{\mathrm{L}((\sigma_{Rel})_{Mor})}(Th^{\sigma_{Rel}}_{Mor})$ to $Str_\sigma(SET[\mathcal{H}])$.*

Proof. We let $f_\sigma(\mathcal{M}) = p(h_{\sigma_{Mor}}(\mathcal{U})|_{\sigma_{Rel}})$. □

Lemma 6.3. *The equivalence relation "$\mathcal{U}_0 \equiv_{\mathrm{C}[\mathcal{H}]} \mathcal{U}_1$ if and only if $f_\sigma(\mathcal{U}_0) \cong f_\sigma(\mathcal{U}_1)$" is a PC_{ω_1} equivalence relation.*

Proof. By Lemma 3.11 we know $f_\sigma(\mathcal{U}_0) \cong f_\sigma(\mathcal{U}_1)$ if and only if there is a σ_{iso} structure I in SET[$\mathcal{H}$] where $I \models Th_{iso}(\sigma)$, $I_{\sigma_0} \cong f_\sigma(\mathcal{U}_0)$ and $I_{\sigma_1} \cong f_\sigma(\mathcal{U}_1)$. However such an I exists if and only if there is an interpretation of $[\cdot]_I$ of $(\sigma_{iso})_{Rel}$ where $[\cdot]_I \models \widehat{Th_{iso}}$ and $[\cdot]_I|_{(\sigma_0)_{Rel}} = h_{\sigma_{Mor}}(\mathcal{U}_0)|_{\sigma_{Rel}}$ and $[\cdot]_I|_{(\sigma_1)_{Rel}} = h_{\sigma_{Mor}}(\mathcal{U}_1)|_{\sigma_{Rel}}$ (where this equality is after the obvious association of σ_i with σ).

But this last statement holds if and only if there is some $((\sigma_{iso})_{Rel})_{Mor}$ model I_{SET} which satisfies $Mor_{(\sigma_{iso})_{Rel}}(\widehat{Th_{iso}})$ and $I_{\mathrm{SET}}|_{\mathrm{L}((\sigma_0)_{Rel})} \cong \mathcal{U}_0$ and $I_{\mathrm{SET}}|_{\mathrm{L}((\sigma_1)_{Rel})} \cong \mathcal{U}_1$. Hence $\equiv_{\mathrm{C}[\mathcal{H}]}$ is PC_{ω_1}. □

Proposition 6.4. *For any $T \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$ and any $\mathcal{U} \in Mod_{(\sigma_{Rel})_{Mor}}$ the following are equivalent:*

(1) $f_\sigma(\mathcal{U}) \models T$.

(2) $\mathcal{U} \models Th^{\sigma_{Rel}}_{Mor}(\hat{T})$

Proof. By Corollary 4.24 (2) is equivalent to the statement that $h_\sigma(\mathcal{U}) \models Mor_{\sigma_{Rel}} \cup \hat{T}$ which is in turn equivalent to the statement that $h_\sigma(\mathcal{U})|_{\sigma_{Rel}} \models \hat{T}$ (as $Mor_{\sigma_{Rel}}$ is a conservative extension of the empty theory for interpretations). But we also know by Proposition 3.25 that $h_\sigma(\mathcal{U})|_{\sigma_{Rel}} \models \hat{T}$ if and only if $[f_\sigma(\mathcal{U}) =]p(h_\sigma(\mathcal{U})|_{\sigma_{Rel}}) \models T$. □

Lemma 6.5. *For every model $\mathcal{U} \in Mod_{\mathrm{L}((\sigma_{Rel})_{Mor})}(Th^{\sigma}_{Mor})$, f_σ is countably generated if and only if $\mathcal{U}$ is countable.*

Proof. This follows from the fact that the sorts of σ and $(\sigma_{Rel})_{Mor}$ are the same and for every such sort X we have $X^{f_\sigma(\mathcal{U})} = (H^{\mathcal{U}}_X, \sim)$ for some $\sim$. □

This then gives us the following very important corollary.

Corollary 6.6. *There is a bijection between isomorphism classes of countably generated σ-structures in SET[$\mathcal{H}$] that satisfy a theory T and $\equiv_{SET[\mathcal{H}]}$ equivalence classes of countable models of $(\sigma_{Rel})_{Mor}$ which satisfy $Th^{\sigma_{Rel}}_{Mor}(\hat{T})$ in SET.*

Theorem 6.7. *If $(SET, \mathcal{H})$ is a countably presented simple definable hyperdoctrine then for any theory $T \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$ one of the following holds:*

- *There is are continuum many countably generated models of T in SET[$\mathcal{H}$] (up to isomorphism.)*
- *There are at most $\aleph_1$ many countably generated models of T in SET[$\mathcal{H}$] (up to isomorphism).*

Proof. We know that $\equiv_{\mathrm{SET}[\mathcal{H}]}$ is a PC_{ω_1} equivalence relation and $Th^{\sigma_{Rel}}_{Mor}(\hat{T}) \in \mathcal{L}_{\omega_1,\omega}(\mathrm{L}((\sigma_{Rel})_{Mor}))$ so by Corollary 2.7 there is either a perfect set of $\equiv_{\mathrm{SET}[\mathcal{H}]}$ inequivalent models in SET satisfying $Th^{\sigma_{Rel}}_{Mor}(\hat{T})$ or there are at most $\aleph_1$ many $\equiv_{\mathrm{SET}[\mathcal{H}]}$ inequivalent models in SET satisfying $Th^{\sigma_{Rel}}_{Mor}(\hat{T})$.

But by Corollary 6.6 the number of isomorphism classes of countably generated models in SET[$\mathcal{H}$] which satisfy T is the same as the number of $\equiv_{\mathrm{SET}[\mathcal{H}]}$ equivalence classes of models satisfying $Th^{\sigma_{Rel}}_{Mor}(T^*)$ in SET. □

Corollary 6.8. *If $(A, \cdot)$ is a countable PCA then for any theory $T \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$ one of the following holds:*

- *There is are continuum many countably generated models of T in* $\mathrm{RT}(A)$.
- *There are at most $\aleph_1$ many countably generated models of T in* $\mathrm{RT}(A)$.

We will now show that Morley's theorem holds for monic bound models as well. In this section let $(\overline{B}, \sim_{\overline{B}})$ be a countably generated element of SET$[\mathcal{H}]$. Our first step will be to characterize $(\overline{B}, \sim_{\overline{B}})$.

Proposition 6.9. *The collection of $\mathcal{U} \in Mod_{\mathrm{L}(\tau)}(Th^{\sigma}_{Mor})$ such that $f_\sigma(\mathcal{U})$ is monic bound by $(B, \sim_B)$ is a PC_{ω_1} class.*

Proof. Let $\sigma^m = \sigma \cup \{B\} \cup \{f_X : X \to B$ for each sort $X \in \mathcal{S}_\sigma\}$ where B is a new sort. Let $Bound_m(\sigma) \subseteq \mathcal{L}_{\omega,\omega}(\sigma^m)$ be the theory which says "Every f_X is monic" (note that this can be expressed in $\mathcal{L}_{\omega,\omega}(\sigma^m)$ by Lemma 3.5). The following are then equivalent for any $\mathcal{U} \in \mathrm{Mod}_{\mathrm{L}(\tau)}(Th^{\sigma}_{Mor})$.

(1) $f_\sigma(\mathcal{U})$ is monic bound by $(\overline{B}, \sim_{\overline{B}})$.

(2) There is a σ^m structure $\mathcal{M}$ in SET$[\mathcal{H}]$ such that $\mathcal{M} \models Bound_m(\sigma)$, $B^{\mathcal{M}} = (\overline{B}, \sim_{\overline{b}})$ and $\mathcal{M}|_\sigma = f_\sigma(\mathcal{U})$.

(3) There is a model $\mathcal{U}^* \models Th^{\sigma^m_{Rel}}_{Mor}(\widehat{Bound_m}(\sigma))$ where $\mathcal{U}^*|_{\mathrm{L}((\sigma_{Rel})_{Mor})} = \mathcal{U}$, $H^{\mathcal{U}^*}_B = \overline{B}$ and $[\sim_B]_{\mathcal{U}^*} = \sim_{\overline{B}}$ (where $h_{(\sigma^{mb}_{Rel})_{Mor}} = [\cdot]_{\mathcal{U}^*}$).

Now consider the language $\tau_{\overline{B}} = (\sigma^m_{Rel})_{Mor} \cup \{c_b : b \in \overline{B}\}$ (where each c_b is a constant) and the sentence φ_B which says

(i) $(\forall x : B) \bigvee_{b \in \overline{B}} (x = c_b)$ and $\bigwedge_{b_0, b_1 \in \overline{B}, b_0 \neq b_1} c_{b_0} \neq c_{b_1}$.

(ii) $\bigwedge \{(\sim_B)_A((c_{b_0}, c_{b_1}), c_a) : a \in A_{\mathcal{H}}, (b_0, b_1) \in \overline{B} \times \overline{B}$ and $a \in \sim_B (b_0, b_1)\}$.

(i) says we have a constant for each element of $\overline{B}$ and the sort B consists of exactly those constants (no two of which are equal). (ii) then characterizes the relation $\sim_B$ so that when B is interpreted as $\overline{B}$ then $\sim_B$ must be $\sim_{\overline{B}}$. In particular this means that (3) above is equivalent to

(4) There is a model $\mathcal{U}^* \models Th^{\sigma^m_{Rel}}_{Mor}(\widehat{Bound_m}(\sigma)) \wedge \varphi_B$ where $\mathcal{U}^*|_{\mathrm{L}((\sigma_{Rel})_{Mor})} = \mathcal{U}$.

But this then implies the collection $\{\mathcal{U} : f_\sigma(\mathcal{U})$ is monic bound by $(\overline{B}, \sim_{\overline{B}})\}$ is PC_{ω_1}. □

We then have the following immediate consequence:

Theorem 6.10. *If $(SET, \mathcal{H})$ is a countably presented simple definable hyperdoctrine and B is a countably generated element of $SET[\mathcal{H}]$ then for any theory $T \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$ one of the following holds:*

- *There is are continuum many models of T in $SET[\mathcal{H}]$ which are monic bounded by B.*
- *There are at most $\aleph_1$ many models of T in $SET[\mathcal{H}]$ which are monic bounded by B.*

Proof. This follows immediately from Lemma 2.8, Lemma 6.3 and Proposition 6.9. □

Corollary 6.11. *If $(A, \cdot)$ is a countable PCA and $\mathrm{RT}(A)$ is the realizability topos over $(A, \cdot)$ then for any theory $T \subseteq \mathcal{L}_{\omega,\omega}(\sigma)$ one of the following holds:*

- *There is are continuum many models of T in $\mathrm{RT}(A)$ which are monic bounded by B.*
- *There are at most $\aleph_1$ many models of T in $\mathrm{RT}(A)$ which are monic bounded by B.*

7. Miscellaneous

7.1. Absoluteness

In this section we observe that in Theorem 6.7 which of the two cases occurs is independent of the model of set theory. Specifically we have:

Proposition 7.1. *Suppose SET_0, SET_1 are standard*[20] *models of ZFC and let $A, E, S, O, P \subseteq SET_0 \cap SET_1$ come from a psuedofunctor. Denote by $\mathcal{H}_0$ and $\mathcal{H}_1$ the corresponding psuedofunctors in SET_0 and SET_1 respectively. Further suppose $(SET_0, \mathcal{H}_0)$ is simple in SET_0, $(SET_1, \mathcal{H}_1)$ is simple in SET_1 and the*

[20] By a standard model we mean a subclass X of SET where $(X, \in) \models$ ZFC

same formulas witness this in both models of set theory. Then the following are equivalent.

- *There is a perfect set of reals in SET_0 each encoding a countably generated model of T in $SET_0[\mathcal{H}_0]$.*
- *There is a perfect set of reals in SET_1 each encoding a countably generated model of T in $SET_1[\mathcal{H}_1]$.*

Proof. Notice the same PC_{ω_1} formula defines $\equiv_{\mathrm{SET}_0[\mathcal{H}_0]}$ and $\equiv_{\mathrm{SET}_1[\mathcal{H}_1]}$ and $Th^{\sigma_{Rel}}_{Mor}(\hat{T})$ is the same formula in both of these contexts. Hence the result follows from Corollary 2.7. □

Corollary 7.2. *If $(A, \cdot)$ is a PCA and SET_0, SET_1 are standard models of ZFC containing A then the following are equivalent for any theory T.*

- *There is a perfect set of reals each encoding a countably generated model of T in $\mathrm{RT}(A)^{SET_0}$.*
- *There is a perfect set of reals each encoding a countably generated model of T in $\mathrm{RT}(A)^{SET_1}$.*

7.2. $\mathcal{O}(T)$-Valued Sets

Up until now all of the hyperdoctrines we have considered have been motivated by some form of realizability. In this section we will consider a different type of hyperdoctrine, one that comes from the H-valued sets for a Heyting algebra H. The category of H-valued sets, which is equivalent to the category of sheaves on H, were originally introduced by Higg and were studied in [2] by Fourman and Scott. Further one of the original motivations for the PER construction was to find a common generalization of the category of H-valued sets and the effective topos (see [10]).

In this section we show that for any topological space T, the hyperdoctrine from which we obtain $\mathcal{O}(T)$-valued sets (where $\mathcal{O}(T)$ is the collection of open sets of T) is both simple and definable. This will then allow us to deduce Morley's theorem for $\mathcal{O}(T)$-valued sets (for certain T).

We now consider the hyperdoctrines, $\mathrm{HS}(\mathcal{O}(T))$, associated with $\mathcal{O}(T)$-valued sets. For a more thorough treatment see [10].

Example 7.3 ($\mathcal{O}(T)$-Valued Sets)**.** *Suppose T is a topological space with open sets $\mathcal{O}(T)$ and a basis $B(T)$. Let $\mathrm{HS}(T)$ be the hyperdoctrine on SET given by*

- $\mathrm{HS}(T)(X) = (SET[X, \mathcal{O}(T)], \leq_X)$ *and* $\alpha \leq_X \beta \Leftrightarrow (\forall x \in X)\alpha(x) \subseteq \beta(x)$.
- *For a map* $f \in SET[X, Y]$, $\mathrm{HS}(T)(f)(x) = x \circ f$.

Then $HS(T)$ is the definable with:

- $A_{\mathrm{HS}(T)} = T$ *and* $E_{\mathrm{HS}(T)} = \{id\}$ *(so* $O_{\mathrm{HS}(T)}$ *is trivial).*
- $S_{\mathrm{HS}(T)} = \{s : s \in B(T)\}$ *and* $P_{\mathrm{HS}(T)} = \emptyset$.

The only non-trivial part to check of the above definition is that $\alpha \leq_X \beta$ if and only if $(\forall x : X)\alpha(x) \subseteq \beta(x)$. But this is because the only element of $E_{\mathrm{HS}(T)}$ is the identity map.

In particular if T is countable and second countable then $\mathrm{HS}(T)$ is countably presentable.

Lemma 7.4. $\mathrm{HS}(T)$ *is simple.*

Proof. For $R^0, R^1 \in \mathcal{R}_\sigma$ of type X we define the following formulas of type $\langle H_X, \mathrm{DS}\rangle$:

- $D_\wedge(R^0, R^1)(x, a) \Leftrightarrow R^0_A(x, a) \wedge R^1_A(x, a)$.
- $D_\vee(R^0, R^1)(x, a) \Leftrightarrow R^0_A(x, a) \vee R^1_A(x, a)$.
- $D_\rightarrow(R^0, R^1)(x, c_a) \Leftrightarrow \bigvee_{a \in s \in S_{HS(T)}} \bigwedge_{b \in s} \neg R^0_A(x, c_b) \wedge R^1_A(x, c_b)$.

and for $R \in \mathcal{R}_\sigma$ of type $\langle X, Y\rangle$ we define the following formulas of type $\langle H_{\langle X,Y\rangle}, \mathrm{DS}\rangle$:

- $D_{(\exists Y)}(R)(x, a) \Leftrightarrow (\exists y : Y)A_R(x, y, a)$.
- $D_{(\forall Y)}(R)(x, c_a) \Leftrightarrow \bigvee_{s: a \in s \in S_{\mathrm{HS}(T)}} \bigwedge_{b \in s} (\forall y : Y) \in A_R(x, y, c_b)$.

The following claim then follows as the Heyting algebra structure on $HS(T)(X)$ is calculated pointwise[21] and for $\Phi \in \mathcal{H}(X \times Y)$

- $(\exists Y)_X \Phi \cong_{st} \lambda x \in X. \bigvee_{y \in Y} \Phi(x, y)$.
- $(\forall Y)_X \Phi \cong_{st} \lambda x \in X. \bigwedge_{y \in Y} \Phi(x, y)$.

Claim 7.5. *We have*

- $V(\sigma, D_\wedge(R_0, R_1), R^*) = \{[\cdot] : [\cdot] \models (\forall x : X) R^*(x) \leftrightarrow [R_0(x) \wedge R_1(x)]\}$
- $V(\sigma, D_\vee(R_0, R_1), R^*) = \{[\cdot] : [\cdot] \models (\forall x : X) R^*(x) \leftrightarrow [R_0(x) \vee R_1(x)]\}$
- $V(\sigma, D_\rightarrow(R_0, R_1), R^*) = \{[\cdot] : [\cdot] \models (\forall x : X) R^*(x) \leftrightarrow [R_0(x) \rightarrow R_1(x)]\}$
- $V(\sigma, D_{(\exists Y)}(R)(x, a), R^*) = \{[\cdot] : [\cdot] \models (\forall x : X) R^*(x) \leftrightarrow (\exists y : Y) R(x, y)\}$
- $V(\sigma, D_{(\forall Y)}(R)(x, a), R^*) = \{[\cdot] : [\cdot] \models (\forall x : x) R^*(x) \leftrightarrow (\forall y : Y) R(x, y)\}$

As a result HS(T) is a simple hyperdoctrine (because by Lemma 4.17 each of the above collections is definable). $\square$

Before we state the main result of this section we will make a couple of observations. First the following follows immediately from [2].

Lemma 7.6. *Let i be the equivalence of categories mentioned in [2] between $\mathcal{O}(T)$-valued sets and sheaves on $B(T)$. Then following are equivalent for any $\mathcal{O}(T)$-valued set $(A, \sim_A)$.*

- *$(A, \sim_A)$ is countably generated.*
- *$i(A, \sim_A)$ has a countable subseparated presheaf. I.e. there is a subseparated presheaf which, when considered as a functor $F : B(T)^{op} \rightarrow SET$ has $|\bigcup_{s \in B(T)} F(S)|$ countable.*

[21] In particular the only non-trivial connective on sets to consider is $A \rightarrow B$ which is the interior of $(T - A) \cup B$.

In [1] we called a sheaf which satisfied the above a "countably generated" sheaf and we called a sheaf which was monic bound by the natural numbers object monic countable.

Corollary 7.7. *If T is a countable topological space with a countable basis $B(T)$. Then the natural number object $\mathbb{N}_T$ is countably generated.*

Proof. This is because in the category of sheaves on T (and hence also in the category of $\mathcal{O}(T)$-valued sets) the natural number object is the colimit of ω many copies of the terminal object. □

The following is then immediate from Theorem 6.7, Theorem 6.10, Corollary 7.7 and Lemma 7.6.

Proposition 7.8. *If T is a countably topological space with a countable basis $B(T)$ then for any theory Th in a countable language we either have:*

- *There is are continuum many countably generated models of Th in the category of sheaves on $B(T)$.*
- *There are at most $\aleph_1$ many countably generated models of Th in the category of sheaves on $B(T)$.*

and we also either have

- *There is are continuum many monic countable models of Th in the category of sheaves on $B(T)$.*
- *There are at most $\aleph_1$ many monic countable models of Th in the category of sheaves on $B(T)$.*

This is an improvement on the main result of [1], in that it allows us to remove the determinacy requirement from the statement of the theorem (at least for topological spaces of the above form).

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