The Number Of Countable Models In Categories of Sheaves

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Abstract. The goal of this paper is to extend Morley's results in [**10**] to categories of sheaves. We show that in this context there are four natural notions of "a countable model". We then show $\mathbf{\Pi}^1_3$ determinacy implies that for any of these four notions and for any sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$ either there is a perfect set of non-isomorphic models of T in our category of sheaves or else there are at most $\aleph_1$ many non-isomorphic models of T in our category of sheaves. We also show that for one of the four notions of countable we can remove the determinacy assumption.

1. Introduction

1.1. Summary. At the end of his seminal paper [**12**], Vaught asked the question "Can it be proved, without the use of the continuum hypothesis, that there exists a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?". This question has stood to this day as one of the oldest open problems in model theory. The statement that it has a negative answer, i.e. that no countable first order theory has exactly $\aleph_1$ many countable models (assuming $\neg CH$), has become known as "Vaught's Conjecture".

Over the years much work has been done on Vaught's conjecture and there have been several special situations where it has been shown to hold (such as for ω-stable theories in [**11**] by Shelah, Harrington and Makkai). However the general case remains elusive. In his paper [**10**], Morley took one of the most significant steps towards resolving Vaught's conjecture. He proved that any sentence of $\mathcal{L}_{\omega_1,\omega}(L)$ which does not have a perfect set of countable models must have either countably many or $\aleph_1$ many countable models.

Among the many reasons why Morley's paper represented significant progress towards a resolution of Vaught's conjecture was that it extended the scope of the conjecture from first order theories to sentences of $\mathcal{L}_{\omega_1,\omega}(L)$. By extending the scope of Vaught's conjecture, Morley brought the conjecture into the realm of infinitary logic as well as descriptive set theory and thereby opened up the use of techniques from these areas to its study. This has led to such important discoveries as the existence of a minimal counterexample to Vaught's conjecture by Harnik and Makkai (in [**3**]) as well as several other results about the nature of counterexamples.

1991 *Mathematics Subject Classification.* Primary 03C15, 03C90, 03G30; Secondary 03E15, 18F20.

Key words and phrases. countable model, Vaught's conjecture, category theory, sheaf, site.

One of the most significant discoveries of categorical logic is that the operations of $\mathcal{L}_{\omega_1,\omega}(L)$ can be described categorically. This allows us, as is done by Makkai and Reyes in [**9**], to consider models of sentences of $\mathcal{L}_{\omega_1,\omega}(L)$ in categories other than the category of sets and functions.

The goal of this paper is to extend Morley's result to the context where the underlying category is a category of sheaves on a weak site. In the course of this paper we will see that there are four reasonable notions of "a countable model". We call these notions being purely countable, countably generated, monic countable and epi countable.

The main results of this paper show that for any of our four notions of a countable mode, any countable language L, any sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$ and any countable weak site (C, J_C), if there isn't a perfect set of countable models of T in $\mathrm{Sh}(C, J_C)$[1] then, under $\mathbf{\Pi}^1_3$ determinacy, there are no more than ω_1 many countable models of T in $\mathrm{Sh}(C, J_C)$. Further, in the case of purely countable models, we can remove the determinacy assumption.

1.2. Outline. We prove the main results of this paper by showing there is an equivalence (of the appropriate type) between the collections of countable models we care about in our category of sheaves and specific $\mathbf{\Sigma}^1_2$ subsets of the reals. This reduces the question of counting the number of countable models in a category of sheaves to counting the number of isomorphism classes under an appropriate isomorphism relation on our $\mathbf{\Sigma}^1_2$ set. These isomorphism relations turn out to be either $\mathbf{\Sigma}^1_1$ or $\mathbf{\Sigma}^1_2$ depending on the notion of countability.

We start this paper in Section 2 by reviewing some set theory and category theory which we will need in order to count the number of these isomorpihsm classes of models.

Next, in Section 3, we move onto a discussion of models in a category of sheaves. One of the most important ideas in this section is that when counting the number of models in a category of sheaves, it suffices to restrict ourselves to a particular collection of models in the category of separated presheaves. We further show that these separated presheaf models can be thought of as ordinary set models of a sentence of $\mathcal{L}_{\infty,\omega}(L')$ (for some specific language L').

In Section 4 we introduce the concept of Σ_1-definable and Δ_1-definable classes of models. Σ_1-definable classes of models are particularly well-behaved and most collections of models we consider will be Σ_1-definable classes. Specifically, in Section 4.2, we show that the collection of models of a sentence of $\mathcal{L}_{\infty,\omega}(L)$ in a category of sheaves forms a Δ_1-definable class of models.

Finally, in Section 5, we introduce our four notions of countability and show that the models of each notion form a Σ_1-definable class of models. We then prove, in Section 6, our various results concerning the number of countable models in any Σ_1-definable class (and in particular the number of countable models which satisfy a sentence of $\mathcal{L}_{\omega_1,\omega}(L)$).

We end in Section 7 with some conjectures, including a generalization of Vaught's conjecture and a simple example of a category of sheaves in which Vaught's conjecture holds.

1.3. Acknowledgement. The author would like to thank Gerald Sacks for introducing him to Vaught's conjecture and Morley's paper [**10**] as well as for many

[1] $\mathrm{Sh}(C, J_C)$ is defined in Section 2.2. It is equivalent to the category of sheaves on (C, J_C).

interesting conversations. The author would also like to thank Leo Harrington, Jan Reimann and Dana Scott for useful conversations during the development of this paper.

2. Background

2.1. Set Theory. We begin this section with a brief discussion of the set theoretic notations and concepts used in this paper. We refer the reader to such standard texts as [**6**] for any set theoretic results or definitions not explicitly discussed.

In this paper we will assume Zermelo-Fraenkel Set Theory (ZF) as our ambient theory and we will assume all results take place in a fixed model of ZF which we refer to as SET. By a "standard model" we mean a pair (M, E) where M is a transitive (not necessarily proper) class and E is a well-founded relation. Unless otherwise stated all standard models will satisfy ZF. While we will not necessarily assume this of all standard models of set theory, we will assume that SET $\models [\bigwedge_{\alpha<\omega_1} |X_\alpha| \leq \omega_1] \rightarrow |\bigcup_{\alpha<\omega_1} X_\alpha| \leq \omega_1$.[2]

There is one important situation when we will be interested in standard models which may not satisfy ZF. There are times when we will need to prove the existence of certain standard (countable) models of set theory. However, as ZF doesn't prove such models exist, we will instead use models of a finite fragment which we call ZF^*. We want to think of ZF^* as a fragment of ZF "large enough to prove every result which came before it in this paper". In particular this means that while in each result ZF^* is a fixed finite fragment of ZF, ZF^* is not necessarily fixed between results.

Many of our results depend on results from descriptive set theory which we collect here. As the reader considers these results, it is worth keeping in mind how they will be used. We will show that for our purposes it suffices to consider collections of models where the collection is encoded by a $\mathbf{\Sigma}^1_2$ set of reals. The question of counting the number of models then becomes a question of counting the number of equivalence classes under model isomorphism (in the appropriate category). The notion of model isomorphism will in general be a $\mathbf{\Sigma}^1_2$ relation, except in special situations when we can reduce it to a $\mathbf{\Sigma}^1_1$ relation.

If the reader wishes to take the descriptive set theory results on faith, he can browse Proposition 2.3, Proposition 2.8, Proposition 2.9 and Proposition 2.12 which are the fundamental results in this section and move on to Section 2.2.

We now give an important connection between $\mathbf{\Sigma}^1_2$ sets of reals and Σ_1 classes.

Definition 2.1. Suppose L is a first order language. We say a formula $\varphi(x)$ of set theory (possibly with parameters) is an *L-formula* if

- φ is only satisfied by models of L in SET.
- φ is closed under isomorphism (i.e. if $\varphi(x)$ holds and x is isomorphic to y (as models in SET) then $\varphi(y)$ holds).

[2]This follows from the axiom of choice, but it is consistent with ZF that it does not hold.

Definition 2.2. Suppose L is a countable language and let $Mod_L(\omega) \subseteq 2^\omega$ be the collection of reals which encode (via fixed encoding[3]) models of L. We give $Mod_L(\omega)$ the standard topology (see [**2**]). For each element $x \in Mod_L(\omega)$ let x_M be the model of L on ω which x encodes.

Proposition 2.3. *Suppose L is a countable language and φ is an L formula which is Σ_1 and has only a hereditarily countable parameter P. Also suppose P^* is a real encoding the graph*[4] *of P. Then there is a $\Sigma^1_2(P^*)$ formula, $\varphi^*(x)$, such that $\varphi^*(x)$ holds if and only if $\varphi(x_M)$ holds. Further the choice of φ^* is independent of the standard model of set theory we are working in.*

Proof. Let $\varphi(X) = (\exists y)\psi(X, y, P)$ where ψ is a Δ_1 formula. We define $\varphi^*(x)$ to be the formula satisfying:

- $x \in Mod_L(\omega)$.
- There exists $E \subseteq \omega \times \omega$ such that
  - E is well-founded.
  - $\{x, P\} \subseteq tc(\omega, E)$
  - $tc(\omega, E) \models ZF^* \wedge (\exists y)\psi(x, y, P)$.

It is clear that φ^* is $\Sigma^1_2(P^*)$. Further $\varphi^*(x)$ holds if and only if there is a countable standard model of ZF^* containing x where $\varphi(x_M)$ holds. So by the absoluteness of Δ_1 formulas, if $\varphi^*(x)$ holds then so does $\varphi(x_M)$. Now suppose SET $\models \psi(x_M, y, P)$ for some set y. Let $(K, \in)$ be a transitive model of ZF^* which contains $\{x, y, P\}$ and satisfies $\psi(x, y, P)$. Let $(H, \in)$ be a countable, transitive, elementary substructure of $(K, \in)$ containing $\{x, P\}$ (one exists as the transitive closure of $\{x, P\}$ is countable). So $H \models (\exists y)\psi(x, y, P)$ and satisfies ZF^*. But, as H is countable there is a bijection $f : \omega \to H$. So if $xEy \leftrightarrow f(x) \in f(y)$ then E witnesses that $\varphi^*(x)$ holds. □

We will need the following Theorem due to Harrington and Shelah ([**5**]).

Definition 2.4. A set $X \subseteq \omega^\omega$ is *κ-Suslin* if there is a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ such that $x \in X \leftrightarrow (\exists y \in \kappa^\omega)(x, y) \in [T]$[5]. We say $X = p[T]$. We say X is *co-κ-Suslin* if $(\omega^\omega - X)$ is κ-Suslin.

Definition 2.5. Suppose E is a co-κ-Suslin equivalence relation on ω^ω with $\omega^\omega - E = p[T]$ where $T \subseteq (\omega \times \kappa)^{<\omega}$. E is *stongly thick* if

($*_t$) There is a perfect set P and $t \subseteq T$ with $|t| = \omega$ such that $P \times P \subseteq p[T]$.

Notice that if an equivalence relation is strongly thick then it contains a perfect set of inequivalent reals.

Proposition 2.6 ([**5**]). *Suppose*

- *E is a co-κ-Suslin equivalence relation on ω^ω with $\omega^\omega - E = p[T]$ where $T \subseteq (\omega \times \kappa)^{<\omega}$.*

[3]By a "fixed encoding" we mean any Δ_1 formula d (without parameters) such that for any standard models V of ZF^* and any model M of the language L in V, $V \models |M| = \omega \to [(\exists x \in Mod_L(\omega))(\forall N \in V)d(x, N) \leftrightarrow N = M]$ and $V \models (\forall x \in Mod_L(\omega))(\exists y)d(x, y)$.

[4]A set P is hereditarily countable if its transitive closure, $tc(P)$, is countable. For each transitive set T, $(T, \in)$ is a tree. The graph of a set P is the pair $((tc(P), \in), P)$ treated as a model of the language of graphs with a distinguished predicate.

[5]For a tree T on $(\omega \times \kappa)^{<\omega}$, $[T]$ is the collection $(x, y) \in (\omega \times \kappa)^\omega$ such that $(x|_n, y|_n) \in T$ for all $n \in \omega$.

- *In some Cohen forcing extension of $L[T]$, $\omega^\omega - p[T]$ is an equivalence relation.*
- *E is not strongly thick.*

Then there are at most κ many E-inequivalent reals.

There are two conditions on our ambient set theoretic universe which we would like to consider.

Definition 2.7. We define the following statements:

($*_\#$) Let ($*_\#$) be the statement:
  - $(\forall a \in \omega^\omega)$ $a^\#$ exists.
  - There is a real which is Cohen generic over $L(H(\omega_3))$[6].

($*_B$) Let B be the closure of the $\mathbf{\Sigma}^1_2$ sets under finite boolean combinations and continuous preimage. Let ($*_B$) be the statement
  - All sets in B are determined.
  - Dependent Choice holds.

Proposition 2.8. *Suppose X is a $\mathbf{\Sigma}^1_2$ set of reals and E is a $\mathbf{\Sigma}^1_2$-equivalence relation on ω^ω which does not contain a perfect set of E-inequivalent reals in X. Further suppose ($*_\#$) holds. Then there are at most $\aleph_2$ many E-inequivalent reals in X.*

Proof. Let $E'(x,y) := E(x,y) \vee (\neg X(x) \wedge \neg X(y))$. E' is then $\mathbf{\Pi}^1_3$ and so E' is co-ω_2-Suslin by a theorem of Martin (see [**6**] Theorem 32.15). If there does not exists a perfect set of E'-inequivalent reals, then E' is not strongly thick. Hence, by Proposition 2.6, there are at most $\aleph_2$ many E'-inequivalent reals and so at most $\aleph_2$ many E-inequivalent reals in X. □

The following uses, in a fundamental way, a theorem of Harrington and Sami (from [**4**]).

Proposition 2.9. *Suppose X is a $\mathbf{\Sigma}^1_2$ set of reals and E is a $\mathbf{\Sigma}^1_2$-equivalence relation on ω^ω. If ($*_B$) holds and there is not a perfect set of E-inequivalent reals in X then there are at most $\aleph_1$ many E-inequivalent reals in X.*

Proof. Lets assume there is not a perfect set of E-inequivalent reals in X. We know by a result of Sierpinski (see [**6**] Theorem 25.19) that $X = \bigcup_{\alpha<\omega_1} X_\alpha$ where each X_α is Borel and disjoint. Define $xE^\alpha y \Leftrightarrow xEy \vee [\neg X_\alpha(x) \wedge \neg X_\alpha(y)]$.

Claim 2.10. *For each $\alpha \in \omega_1$, E^α is a $\mathbf{\Delta}^1_2$ equivalence relation such that there does not exists a perfect set of E^α-inequivalent reals.*

Proof. For each $\alpha \in \omega_1$ it is immediate that E^α is a $\mathbf{\Sigma}^1_2$ equivalence relation. If there existed a perfect set of E^α-inequivalent reals, then there would be a perfect set of E-inequivalent reals in X (and we have assumed there isn't). So there must not be a perfect set of E^α-inequivalent reals. Further, by Corollary 3 in [**4**], E^α is a $\mathbf{\Delta}^1_2$ set. □

Claim 2.11. *For each $\alpha < \omega_1$, there are at most ω_1 many E^α-inequivalent reals.*

[6] $H(\kappa)$ consist of all sets whose transitive closure has size less than κ.

Proof. First note E^α is a $\Delta^1_2(e_\alpha)$ equivalence relation for some $e_\alpha \in \omega^\omega$. Because $(*_B)$ holds all $\Sigma^1_1(e_\alpha)$ sets are determined and so, by Theorem 33.19 in [**6**], $e_\alpha^\#$ exists. In particular this means that $|(\aleph_1)^{L[e_\alpha]}| = \omega$ and so, by a (relativized) Corollary 3 of [**5**] there are at most $\aleph_1$ many E^α-inequivalent reals. □

For each α, E and E^α agree on X_α. So in each X_α there are at most $\aleph_1$ many E-inequivalent reals. Hence in $X = \bigcup_{\alpha<\omega_1} X_\alpha$ there are at most $\aleph_1$ many E-inequivalent reals. □

The results above concerning $\mathbf{\Sigma}^1_2$ equvialence relations use large cardinal and determinacy assumptions in a fundamental way. However, if we restrict ourselves to $\mathbf{\Sigma}^1_1$ relations instead of $\mathbf{\Sigma}^1_2$ relations, then we are able to remove the large cardinal assumptions.

Proposition 2.12. *Suppose X is a $\mathbf{\Sigma}^1_2$ set of reals and E is a $\mathbf{\Sigma}^1_1$-equivalence relation ω^ω. If there is not a perfect set of E-inequivalent reals in X, then there are at most $\aleph_1$ many E-inequivalent reals in X.*

Proof. Lets assume there is not a perfect set of E-inequivalent reals in X. We know, by Theorem 25.19 of [**6**], that $X = \bigcup_{\alpha<\omega_1} X_\alpha$ where each X_α is Borel and the X_α are disjoint. Define $xE^\alpha y \Leftrightarrow xEy \vee [\neg X_\alpha(x) \wedge \neg X_\alpha(y)]$.

Each E^α is then a $\mathbf{\Sigma}^1_1$ equivalence relation on ω^ω such that there is not a perfect set of E^α-inequivalet reals (because if there were then there would be a perfect set of E-inequivalent reals on X). Hence for each $\alpha < \omega_1$ there are at most $\aleph_1$ many E^α-inequivalent reals. But for each α, E and E^α agree on X_α. So in each X_α there are at most $\aleph_1$ many E-inequivalent reals. Hence in $X = \bigcup_{\alpha<\omega_1} X_\alpha$ there are at most $\aleph_1$ many E-inequivalent reals. □

2.2. Category Theory. In this section we review some of the categorical notions which we will need. For more information on the category theory in this paper the reader is referred to such standard texts as [**7**]. For more information on the sheaf theoretic ideas used in this paper the reader is referred to such standard texts as [**8**]. In this section we also mention several results concerning the category $\mathrm{Sh}(C, J_C)$ and the notion of a weak site. The proofs of these results we leave to the reader (as they are not difficult). However, for a precise treatment of these results we refer the reader to [**1**].

All categories in this paper will be locally small and we will use the convention that if C is a category with objects A and B, $C[A, B]$ is the set of morphisms whose domain is A and whose codomain is B. If C is a small category we also use $C[-, B]$ for the set of all morphisms whose codomain is B. We will abuse notion (when no confusion can arise) and let SET be the category of sets and functions in our ambient model of ZF.

Definition 2.13. A *weak site* is a pair (C, J_C) where C is a small category and J_C a function from the objects of C to collections of sieves such that for any $A \in \mathrm{obj}(C)$:

- (Identity) $C[-, A] \in J_C(A)$
- (Base Change) If $S \in J_C(A)$ and $f \in C[B, A]$ then $f^*S = \{g \in C[-, B] : f \circ g \in S\} \in J_C(B)$

If in addition (C, J_C) satisfies:

- (Local Character) For all sieves T on A, if $S \in J_C(A)$ and $(\forall f \in S) f^*T \in J_C(\mathrm{dom}(f))$ then $T \in J_C(A)$.

then we say (C, J_C) is a *site*.

The notion of a weak site is the absolute analog of a site and for every weak site there is a minimal site containing it.

DEFINITION 2.14. Let (C, J_C) be a weak site. Define $J_C^\alpha(A)$ on $A \in \mathrm{obj}(C)$ as follows

- $J_C^0(A) = J_C(A)$.
- $J_C^{\alpha+1}(A) = \{T \text{ a sieve on } A : (\exists S \in J_C(A))(\forall f \in S(B)) f^*T \in J_C^\alpha(B)\}$.
- $J_C^{\omega\cdot\gamma}(A) = \bigcup_{\beta<\omega\cdot\gamma} J_C^\beta(A)$.

Define $J_C^{\mathrm{ORD}} = \bigcup_{\alpha\in\mathrm{ORD}} J_C^\alpha$.

We can think of the structure (C, J_C^{ORD}) as the site obtained by closing the weak site (C, J_C) under local character.

LEMMA 2.15. *For any weak site (C, J_C), (C, J_C^{ORD}) is a site.*

LEMMA 2.16. *For any weak site (C, J_C), the set $\{(S, A) : S \in J_C^{ORD}(A)\}$ is Σ_1 definable with parameter (C, J_C).*

DEFINITION 2.17. Suppose (C, J_C) is a weak site and $F : C^{op} \to \mathrm{SET}$ is a presheaf.

- We say F is *separated for* (C, J_C) if for every compatible collection of elements $\langle (a_i, i) : i \in S\rangle$ on A there is a most one $a \in F(A)$ covered by $\langle (a_i, i) : i \in S\rangle$.
- We say F is a *sheaf for* (C, J_C) if for every compatible collection of elements $\langle (a_i, i) : i \in S\rangle$ on A there is exactly one $a \in F(A)$ covered by $\langle (a_i, i) : i \in S\rangle$.

We denote by $\mathrm{Sep}(C, J_C)$ and $\mathrm{Sheaf}(C, J_C)$ the full subcategories of presheaves on C consisting of the separated presheaves and the sheaves for (C, J_C) respectively.

LEMMA 2.18. *We have the following equalities:*

- *$\mathrm{Sheaf}(C, J_C) = \mathrm{Sheaf}(C, J_C^{ORD})$.*
- *$\mathrm{Sep}(C, J_C) = \mathrm{Sep}(C, J_C^{ORD})$.*

This theorem says, for the purpose of considering sheaves and separated presheaves we can restrict our attention to weak sites. This is important for two reasons. First, being a site is not an absolute property. However Lemma 2.18 shows that it suffices to restrict our attention to weak sites which are absolute. Second, as the proof of our main result will rely on facts about definable subsets of reals, it is important that all parameters under consideration be countable. However, there are countable weak sites (C, J_C) such that in any standard model of set theory any site containing (C, J_C) is uncountable (and in fact has size of the continuum).

The following abuse of notation will be useful.

DEFINITION 2.19. If $F : C^{op} \to \mathrm{SET}$ is a functor we say b *is an element of* F ($b \in F$), if $(\exists U \in \mathrm{obj}(C)) b \in F(U)$. We define *the size of* F ($|F|$) to be the size of the collection of elements, i.e. $|\{b : b \in F\}|$. If $F, G : C^{op} \to \mathrm{SET}$ then we say $F \subseteq G$ if $(\forall U \in \mathrm{obj}(C)) F(U) \subseteq G(U)$.

Lemma 2.20. *Suppose $V_0 \subseteq V_1$ are standard models of set theory, A and B are separated presheaves (in V_0) for a weak site (C, J_C) and $V_0 \models A \subseteq B$. For $b \in B(U)$ let $Cov_A(b) \Leftrightarrow \{f \in C[-,U] : B(f)(b) \in A\} \in J_C^{ORD}(U)$. Then for any $b \in B$*

$$V_0 \models Cov_A(b) \Leftrightarrow V_1 \models Cov_A(b).$$

Proof. This is by induction on the least α such that $\{f \in C[-,U] : B(f)(b) \in A\} \in J_C^\alpha$. □

If we are given a separated presheaf B, a subpresheaf A, and an element b of B then whether b is covered by elements of A is independent of the standard model of set theory we are in. This will be important later on.

Definition 2.21. Suppose (C, J_C) is a weak site. Let $\mathrm{Sh}^*(C, J_C)$ be the category such that:

(a) The objects of $\mathrm{Sh}^*(C, J_C)$ are the separated preheaves for (C, J_C).
(b) The morphisms of $\mathrm{Sh}^*(C, J_C)[D, R]$ are the tuples $\langle d, r, f\rangle$ where
 - d, r are separated presheaves for (C, J_C).
 - $f : d \Rightarrow r$ is a f is a natural transformation.
 - $d \subseteq D$, $r \subseteq R$ and d, r cover D, R respectively in (C, J_C^{ORD})[7].
(c) $\langle d, r, f\rangle \circ \langle d', r', g\rangle = \langle g^{-1}[r' \cap d], r, f \circ g\rangle$.
(d) If $X \in Obj$ then $Id(X) = \langle X, X, id_X\rangle$.

Definition 2.22. For all $\langle d_f, r_f, f\rangle, \langle d_g, r_g, g\rangle \in \mathrm{Sh}^*(C, J_C)[D, R]$ we define $\langle d_f, r_f, f\rangle \equiv \langle d_g, r_g, g\rangle$ if and only if $(\forall x \in d_f \cap d_g) f(x) = g(x)$. We then define $\mathrm{Sh}(C, J_C) = \mathrm{Sh}^*(C, J_C)/\equiv$.

Lemma 2.23. *If $V_0 \subseteq V_1$ are standard models then $Sh^*(C, J_C)^{V_0} \subseteq Sh^*(C, J_C)^{V_1}$ and for all $f, g \in Sh^*(C, J_C)^{V_0}[A, B]$, $V_0 \models f \equiv g$ if and only if $V_1 \models f \equiv g$.*

Lemma 2.24. *$Sh(C, J_C)$ is definable by a Σ_1 formula whose only parameter is (C, J_C).*

Proof. First notice that $\equiv$ and the property of being a separated presheaf for (C, J_C) are Δ_1 with parameter (C, J_C). So it suffices to show that there is a Σ_1 formulas $S(F, x, y, C, J_C)$ such that if x and y are separated presheaves for (C, J_C) then $S(F, x, y, C, J_C)$ holds if and only if $F = \langle d, r, f\rangle \in \mathrm{Sh}^*(C, J_C)[x, y]$ (i.e F is a map from x to y in $\mathrm{Sh}^*(C, J_C)$). However, to show that S is Σ_1 it suffices to show that the statement "d is a cover of D", i.e. $(\forall a \in D) Cov_d(a)$, is Σ_1. But this follows from Lemma 2.16. □

Notice that $\mathrm{Sh}(C, J_C)$ is a category and

Lemma 2.25. *There is an equivalence of categories $\boldsymbol{j} : Sh(C, J_C)/\!\!\equiv\, \to Sheaf(C, J_C)$.*

Theorem 2.26. *Let $\boldsymbol{b} : Sep(C, J_C) \subseteq Sh(C, J_C)$ be the inclusion map on objects where $\boldsymbol{b}(f) = \langle A, B, f\rangle$ if $f : Sep(C, J_C)[A, B]$. Then $\boldsymbol{j} \circ \boldsymbol{b}$ is right adjoint to the inclusion map $\boldsymbol{i} : Sep(C, J_C) \to Sheaf(C, J_C)$. In particular $\boldsymbol{j} \circ \boldsymbol{b}$ is isomorphic to the sheafification functor $\boldsymbol{a} : Sep(C, J_C) \to Sheaf(C, J_C)$.*

These results tell us that $\mathrm{Sh}(C, J_C)$ is a version of the category of sheaves which is, in some sense, absolute.[8]

[7] We say d covers D if for every $a \in D$, $Cov_d(a)$ holds.

[8] Notice that the category of sheaves is not absolute as the property of a functor being a sheaf is not absolute.

LEMMA 2.27. *Suppose* $A \in obj(Sh(C, J_C))$ *and* $S \in Sub_{Sh(C,J_C)}(A)$ *is a subobject of* A *in* $Sh(C, J_C)$[9]. *Then there is a (unique) subpresheaf* $\lfloor S \rfloor \subseteq A$ *such that* $\langle \lfloor S \rfloor, A, id\rangle \in S$ *and whenever* $\langle d, r, f\rangle \in S$ *then* $f[d]$[10] $\subseteq \lfloor S \rfloor$ *as separated presheaves. Further* $\lfloor S \rfloor$ *is independent of the standard model of set theory we are working in.*

PROOF. Suppose $\langle d, r, f\rangle \in S$. Let $\lfloor S \rfloor_f(U) = \{a \in A : Cov_{f[d]}(a)\} = \mathbf{a}(f[d]) \cap A$. It is immediate that $\lfloor S \rfloor_f \subseteq A$ and hence $\langle \lfloor S \rfloor_f, A, id\rangle$ is a subobject of A. Further, $\langle \lfloor S \rfloor_f, A, id\rangle \circ \langle d, r \cap \lfloor S \rfloor_f, f\rangle \equiv \langle d, r, f\rangle$ and so $\langle \lfloor S \rfloor_f, A, id\rangle \in S$ (as $\langle d, r \cap \lfloor S \rfloor_f, f\rangle$ is an isomorphism).

Finally as $f[d] \subseteq \lfloor S \rfloor_f$ it suffices to show that if $\langle d, r, f\rangle, \langle d', r', f'\rangle \in S$ then $\lfloor S \rfloor_f = \lfloor S \rfloor_{f'}$. Now $\mathbf{j}(\langle d, r, f\rangle)$ and $\mathbf{j}(\langle d', r', f'\rangle)$ are elements of the same subobject of $\mathbf{j}(A)$ so $image(\mathbf{a}(\langle d', r', f'\rangle)) = image(\mathbf{a}(\langle d, r, f\rangle))$ and hence $\lfloor S \rfloor_{f'} = image(\mathbf{a}(\langle d', r', f'\rangle)) \cap A = image(\mathbf{a}(\langle d, r, f\rangle)) \cap A = \lfloor S \rfloor_f$.

It is also immediate from Lemma 2.20 that $\lfloor S \rfloor$ is absolute between standard models of set theory. □

We know that in the category of separated presheaves or in the category of sheaves a subobject is determined by the image of one of its elements. Lemma 2.27 tells us is that the same holds for $\mathrm{Sh}(C, J_C)$. I.e. that a subobject is determined by a maximal subseparated presheaf covered by the range of one of its elements.

3. Model Theory in a Category of Sheaves

In this section we begin our discussion of the models in a category of sheaves. For more on the basics of categorical model theory in categories of sheaves we refer the reader to [**8**] and [**9**].

For simplicity of notation we will fix a first order language $L = \langle \mathcal{S}_L, \mathcal{R}_L, \mathcal{F}_L\rangle$ with sorts $\mathcal{S}_L$, relations $\mathcal{R}_L$ and functions $\mathcal{F}_L$. We will also fix a weak site (C, J_C).

3.1. Models and Languages. In this section we discuss the relationships between models in $\mathrm{Sep}(C, J_C)$ and $\mathrm{Sh}(C, J_C)$.

DEFINITION 3.1. Suppose D is a category which is has finite products. A *model* M *of* L *in* D consist of:

- For every sort $S \in \mathcal{S}_L$ an object S^M of D.
- For every function $f \in \mathcal{F}_L$ of sort $((S_1, \dots, S_n), S)$ a map in D
$$f^M : S_1^M \times \cdots \times S_n^M \to S^M.$$
- For every relation $R \in \mathcal{R}_L$ of sort $(S_1, \dots, S_n)$ an object R_*^M and a monic in D
$$R^M : R_*^M \to S_1^M \times \cdots \times S_n^M.$$

We let $Mod_L(D)$ be the collection of all models of the language L in the category D.

An important observation is that for any language L we can find an expansion L' and a first order theory T' in L' such that:

[9]Recall that a subobject S of an object A is a collection of monomorphisms whose codomain is A such that if $s, s' \in S$ then there is an isomorphism $\alpha : \mathrm{dom}(s) \to \mathrm{dom}(s')$ with $s' \circ \alpha = s$.

[10]$f[d]$ is the image of d under f.

- Every finite product of sorts in L' is isomorphic to an actual sort.[11]
- Every model of L in any category with finite products has a unique expansion to L' making it a model of T'.

This observation allows us to assume, without loss of generality, that our models are such that each finite product of sorts is isomorphic to an actual sort. While this will not add anything substantively it will make the presentation simpler.

DEFINITION 3.2. Suppose that $M, N \in Mod_L(D)$. A map of models $f : M \to N$ is a set of maps $\langle \alpha_S : S \in \mathcal{S}_L \cup \mathcal{R}_L \rangle$ such that:

- $\alpha_S \in D[S^M, S^N]$ if $S \in \mathcal{S}_L$ and $\alpha_R \in D[R^M_*, R^N_*]$ if $R \in \mathcal{R}_L$.
- For any function $f \in \mathcal{F}_L$ with $f : A \to B$, $\alpha_B \circ f^M = f^N \circ \alpha_A$.
- For any relation $R \in \mathcal{R}_L$ of sort S, $\alpha_S \circ R^M = R^N \circ \alpha_R$.

This makes $Mod_L(D)$ into a category.

LEMMA 3.3. *If D is Σ_1-definable then $Mod_L(D)$ is Σ_1-definable.*

LEMMA 3.4. *If $f : D \to E$ is a functor which preserves monomorphisms, then f induces a functor from $Mod_L(D)$ to $Mod_L(E)$ obtained by applying f to every sort, function and relation.*

When no confusion can arise we will use the same symbol for a functor between categories and for the functor between categories of models that it induces.

COROLLARY 3.5. *$\boldsymbol{b} : Sep(C, J_C) \to Sh(C, J_C)$ extends to a functor*

$$\boldsymbol{b} : Mod_L(Sep(C, J_C)) \to Mod_L(Sh(C, J_C)).$$

PROOF. Immediate. □

Corollary 3.5 is important because models in $\mathrm{Sep}(C, J_C)$ are much easier to handle from a set theoretic point of view than models in $\mathrm{Sh}(C, J_C)$. Further, as we will see, every model in $\mathrm{Sh}(C, J_C)$ is isomorphic (in $\mathrm{Sh}(C, J_C)$) to a model which is in the image of $\mathbf{b}$. So, when counting the number of isomorphism classes of models, it will suffice to restrict our attention to models in $\mathrm{Sh}(C, J_C)$ which are in the image of $\mathbf{b}$.

There is a special class of separated presheaf models which are of particular interest.

DEFINITION 3.6. A model M in $Mod_L(\mathrm{Sep}(C, J_C))$ is *sheaf like* if for every relation R in $\mathcal{R}_L$, $R^M_* = \lfloor R^M \rfloor$ and R^M is the identity.

LEMMA 3.7. *The collection of sheaf like models of a language L in $Sep(C, J_C)$ is a Δ_1 class (with parameters are (C, J_C) and L).*

PROOF. Immediate. □

LEMMA 3.8. *For every model M in $Mod_L(Sh(C, J_C))$ there is a sheaf like model M' in $Mod_L(Sep(C, J_C))$ such that $M \cong \boldsymbol{b}(M')$ in $Mod_L(Sh(C, J_C))$ and*

$$|\omega \times L \times \coprod_{S \in \mathcal{S}_L} |S^M|| = |\omega \times L \times \coprod_{S \in \mathcal{S}_L} |S^{M'}||.$$

PROOF. By Lemma 2.25 every map $g \in \mathrm{Sh}(C, J_C)[A, B]$ is taken via $\mathbf{j}$ to a map $\mathbf{j}(g) \in \mathrm{Sheaf}[\mathbf{a}(A), \mathbf{a}(B)]$. For each sort S and $n \leq \omega$ we define S_n as follows:

---

[11]This includes the empty sort. In particular we have not mentioned constants because they can be thought of as functions whose domain is the empty sort.

- $S_0 = S^M$.
- $S_{n+1} = S_n \cup \bigcup_{g:P\to S\in\mathcal{F}_L} \mathbf{j}(g^M)[P_n]$.
- $S_\omega = \bigcup_{n\in\omega} S_n$.

Notice that $|\omega \times L \times \coprod_{S\in\mathcal{S}_L} |S^M|| = |\omega \times L \times \coprod_{S\in\mathcal{S}_L} |S_\omega||$.

So it suffices to create the desired model with sorts S_ω. In order to do this we first observe that for each sort S, $S^M \subseteq S_\omega \subseteq \mathbf{a}(S^M)$. For each sort S let $S^{M'} = S_\omega$ and for each function $f : S \to P$ let $f^{M'} = \mathbf{j}(f^M)|_{S^{M'}}$. Note that by construction $f^{M'}[S^{M'}] \subseteq P^{M'}$. Let $i^S_{M,M'} : S^M \to S^{M'}$ be the inclusion map from S^M into S_ω. For each relation $R \subseteq S$ suppose $R^M = \langle a^M, b^M, r^M\rangle$ and let $r^* = r[a^M]$. Then let $R^{M'}_* = \lfloor i^S_{M,M'}[r^*]\rfloor$ and $R^{M'}$ be the identity. It is then immediate from the definition that $M' \in Mod_L(\mathrm{Sep}(C, J_C))$ and is sheaf like. Further the map of models $\{\langle S^M, S^{M'}, i^S_{M,M'}\rangle : S \in \mathcal{S}_L\} \cup \{\langle a^M, R^{M'}_*, r^M\rangle : R \in \mathcal{R}_L\}$ is an isomorphism between M and $\mathbf{b}(M')$ in $Mod_L(\mathrm{Sh}(C, J_C))$ with inverse $\{\langle S^M, S^M, id\rangle : S \in \mathcal{S}_L\} \cup \{\langle r^M[a^M], a^M, (r^M)^{-1}\rangle : R \in \mathcal{R}_L\}$. □

Lemma 3.8 shows that, for the purposes of counting isomorphism classes of models in $Mod_L(\mathrm{Sh}(C, J_C))$, it suffices to restrict our attention to models of the form $\mathbf{b}(M)$ where M is sheaf like.

Lemma 3.9. *There is a Σ_1 formula $Iso_{(C,J_C),L}(x, y)$ (with parameters (C, J_C) and L) which holds if and only if*

- *$x, y \in Mod_L(Sep(C, J_C))$ are sheaf like.*
- *$\boldsymbol{b}(x)$ is isomorphic to $\boldsymbol{b}(y)$ in $Mod_L(Sh(C, J_C))$.*

Proof. The formula $\varphi(f, g, x, y)$ which says

- $x, y \in Mod_L(\mathrm{Sep}(C, J_C))$ and are sheaf like.
- f is a map from x to y in $Mod_L(\mathrm{Sh}(C, J_C))$.
- g is a map from y to x in $Mod_L(\mathrm{Sh}(C, J_C))$.

is Σ_1 by Lemma 2.24. So $Iso_{(C,J_C),L}(x, y) := [(\exists f, g)\varphi(f, g, x, y) \wedge f\circ g \equiv id_y \wedge g\circ f \equiv id_x]$ is Σ_1. □

3.2. Separated Presheaves as Models. In this section we show the collection of models in $\mathrm{Sep}(C, J_C)$ which are sheaf like is easily described from a descriptive set theory point of view.

Proposition 3.10. *For any language L there is a language L^S_L and a sentence T^S_L of $\mathcal{L}_{\infty,\omega}(L^S_L)$ such that:*

1. *$|L^S_L| = |L|$*
2. *$Mod_{L^S_L}(T^S_L, SET)$ is equivalent to the category of sheaf like models in $Mod_L(Sep(C, J_C))$.*
3. *If $k_0 : Mod_{L^S_L}(T^S_L, SET) \to Mod_L(Sep(C, J_C))$ is the equivalence of categories from (2) then k_0 is Δ_1 with parameters (C, J_C) and L.*
4. *For every model M of T^S_L in SET, $|\omega\times\coprod_{S\in\mathcal{S}_{L^S_L}} S^M| = |\omega\times\coprod_{S\in\mathcal{S}_L} |S^{k_0(M)}||$ (i.e. a model of T^S_L in SET is the same size as the corresponding model in $Sep(C, J_C)$)*

Proof. Let L^S_L be the language consisting of the following:

- A sort S^X for each $X \in \mathrm{obj}(C)$ and each sort S of L.
- A function symbols $S^f : S^X \to S^Y$ for each $f \in C[X, Y]$ and each sort S.

- Function symbols $g_X : S^X \to P^X$ for each function symbol $g : S \to P$ in L and $X \in \mathrm{obj}(C)$.
- Relations R^X of sort S^X for each relation R of sort S.

The theory T_L^S then consists of axioms which say:

(1a) For each sort $S \in \mathcal{S}_L$, if $S^*(X) = S^X$ for each $X \in \mathrm{obj}(C)$ and $S^*(f) = S^f$ for each $f \in C[X,Y]$ then S^* is a presheaf.

(1b) For each relation $R \in \mathcal{R}_L$, if R^* is defined similarly to S^* then R^* is a presheaf and $R^* \subseteq S^*$.

(2a) If $I \in J_C(A)$ and $S \in \mathcal{S}_L$, then $(\forall x, y \in S^A) x = y \leftrightarrow \bigwedge_{f \in I} S^f(x) = S^f(y)$.

(2b) For each relation $R \in \mathcal{R}_L$ and each $I \in J_C(A)$, $(\forall x \in S^A) R^A(x) \leftrightarrow \bigwedge_{f \in I} R^{\mathrm{dom}(f)}(S^f(x))$.

(3) Each collection $\langle g_X : X \in \mathrm{obj}(C)\rangle$ is a natural transformation with respect to the presheaf structure given in (1).

Conditions $(1a)$ and $(1b)$ guarantee that models of T_L^S in SET have presheaves for sorts and relations. $(2a)$ guarantees that these presheaves are separated. (3) further guarantees that every model in $Mod_{L_L^S}(T_L^S, \mathrm{SET})$ gives rise to a model in $Mod_L(\mathrm{Sep}(C, J_C))$. Lastly $(2b)$ ensures that any such model is sheaf like.

It is also immediate that any sheaf like model in $Mod_L(\mathrm{Sep}(C, J_C))$ gives rise to a model in $Mod_{L_L^S}(T_L^S, \mathrm{SET})$, that this operation of transforming a model in $Mod_L(\mathrm{Sep}(C, J_C))$ into models in $Mod_{L_L^S}(T_L^S, \mathrm{SET})$ preserves maps, and this operation induces the required equivalence of categories. □

Corollary 3.11. *If (C, J_C) is countable then $T_L^S \in \mathcal{L}_{\omega_1,\omega}(L_L^S)$.*

Proposition 3.10 allows us to translate results about models of $Mod_{L_L^S}(T_L^S, \mathrm{SET})$ into results about sheaf like models in $Mod_L(\mathrm{Sep}(C, J_C))$. But by Theorem 3.8 we can then also translate results about models of $Mod_{L_L^S}(T_L^S, \mathrm{SET})$ into results about models in $Mod_L(\mathrm{Sh}(C, J_C))$.

For simplicity we will assume from now on that L is a fixed countable language and (C, J_C) is a fixed countable weak site.

4. Definable Classes of Models

4.1. Σ_1-Definable Classes of Models.

Definition 4.1. φ_X is a *definable class of models* if it is a formula (possibly with parameters) such that

- If V is any standard model of ZF^* then $V \models$"$\varphi_X(M)$ implies M is a sheaf like model in $Mod_L(\mathrm{Sep}(C, J_C))$".
- For all sheaf like models $M, N \in Mod_L(\mathrm{Sep}(C, J_C))$, if $\mathbf{b}(M) \cong \mathbf{b}(N)$ then $\varphi_X(M)$ if and only if $\varphi_X(N)$.

We say a definable class is Σ_1 (or Δ_1) if φ_X is Σ_1 (or Δ_1).

Lemma 4.2. *Suppose φ_X is a Σ_1 definable class of models. Then the collection of $x \in Mod_{L_L^S}(\omega)$ such that*

- *$x_M \in Mod_{L_L^S}(T_L^S, SET)$.*
- *$\varphi(k_0(x_M))$ holds.*

is a $\mathbf{\Sigma}_2^1$ set of reals.

PROOF. By Proposition 2.3 and Proposition 3.10. □

DEFINITION 4.3. If φ_X is a Σ_1 definable class of models let $\widehat{\varphi_X}$ be the $\mathbf{\Sigma}^1_2$ formula defined in Lemma 4.2.

There are a few simple results about Σ_1-definable classes of models which are worth stating explicitly.

LEMMA 4.4. *Σ_1-definable classes of models are closed under finite intersection.*

DEFINITION 4.5. Suppose $\varphi(x)$ is a formulas which is only satisfied by sheaf like objects of $Mod_L(\text{Sep}(C, J_C))$. Let $\overline{\varphi}(x)$ be the formula

$$(\exists M \in Mod_L(\text{Sep}(C, J_C)))Iso_{(C,J_C)}(x, M) \wedge \varphi(M).$$

LEMMA 4.6. *If $\varphi(x)$ is a Σ_1-formula then so is $\overline{\varphi}(x)$. Further $\overline{\varphi}(x)$ is a Σ_1-definable class of models.*

PROOF. It is immediate by Lemma 3.9. □

This tells us if we have a Σ_1-collection of sheaf like models, the class of sheaf like models isomorphic to some element of our collection is a Σ_1-definable class of models.

DEFINITION 4.7. Suppose L_1 is the language with no function symbols or relations and where $\mathcal{S}_{L_1} = \{S_1\}$ has only one sort. A *definable class of sheaves* is a definable class of models in L_1. I.e. a class of separated presheaves closed under isomorphism in $\text{Sh}(C, J_C)$. We similarly define Σ_1-definable classes of sheaves and Δ_1-definable classes of sheaves.

LEMMA 4.8. *If X is a Σ_1 (Δ_1)-definable class of sheaves and $Mod_L(X)$ is the formula describing those sheaf like models of L all of whose sorts are in X, then $Mod_L(X)$ is a Σ_1 (Δ_1)-definable class of models.*

PROOF. Immediate. □

4.2. Models of $\mathcal{L}_{\infty,\omega}$. In this section we show that for any sentence $T \in \mathcal{L}_{\infty,\omega}(L)$ the collection of models which satisfy T in $Mod_L(\text{Sh}(C, J_C))$ forms a Δ_1-definable class of models.

DEFINITION 4.9. Let $\mathcal{L}_{\infty,\omega}(L)$ be the smallest collection of formulas containing all atomic formulas of L and closed under: finite quantification, negation, implication and set sized conjunctions and disjunctions with the condition that every subformula of a formula of $\mathcal{L}_{\infty,\omega}(L)$ can have at most finitely many free variables. We let $\mathcal{L}_{\omega_1,\omega}(L)$ be the subset of $\mathcal{L}_{\infty,\omega}(L)$ consisting of those formulas all of whose conjunctions and disjunctions are countable.

DEFINITION 4.10. If $\varphi(\mathbf{x}) \in \mathcal{L}_{\infty,\omega}(L)$ with a free variable of sort S and if $M \in Mod_L(D)$ then we define $\{\mathbf{x} : D \models \varphi(\mathbf{x})\}^M \subseteq S^M$ by induction in the standard way (see [**8**]). If φ is a sentence in $\mathcal{L}_{\infty,\omega}(L)$ then we we say that $M \models \varphi$ if $\{\mathbf{x} : D \models \varphi\}^M \cong 1$.

DEFINITION 4.11. If $T \in \mathcal{L}_{\infty,\omega}(L)$ we then define $Mod_L(T, D)$ to be the full subcategory of $Mod_L(D)$ consisting of those models which satisfy T.

It is worth stressing that for a formula $\varphi \in \mathcal{L}_{\infty,\omega}(L)$ and a sheaf like model M in $Mod_L(\mathrm{Sep}(C, J_C))$, it is not necessarily the case that $\mathbf{b}(\{\mathbf{x} : \mathrm{Sep}(C, J_C) \models \varphi(\mathbf{x})\}^M)$ and $\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models \varphi(\mathbf{x})\}^{\mathbf{b}(M)}$ are the same subobject. In particular it is not necessarily the case that M satisfies the same sentences of $\mathcal{L}_{\infty,\omega}(L)$ as $\mathbf{b}(M)$ does.

However, as we will see, for any model $M \in Mod_L(\mathrm{Sh}(C, J_C))$, the satisfaction relation between M and sentences of $\mathcal{L}_{\infty,\omega}(L)$ is absolute.

Proposition 4.12. *Suppose $V_0 \subseteq V_1$ are standard models of ZF^*, $\varphi(\boldsymbol{x}) \in \mathcal{L}_{\infty,\omega}(L)^{V_0}$ is a formula of sort S, and $M \in Mod_L(Sh(C, J_C))^{V_0}$. Further suppose that $M = \boldsymbol{b}(M')$ where $M' \in Mod_L(Sep(C, J_C))$ and is sheaf like. Then $\lfloor\{\boldsymbol{x} : Sh(C, J_C) \models \varphi(\boldsymbol{x})\}^M\rfloor^{V_0} = \lfloor\{\boldsymbol{x} : Sh(C, J_C) \models \varphi(\boldsymbol{x})\}^M\rfloor^{V_1}$. I.e. the maximal subpresheaf in the subobject $\{\boldsymbol{x} : Sh(C, J_C) \models \varphi(\boldsymbol{x})\}^M$ is independent of the model of set theory we are working in.*

Proof. We will prove this by induction on the complexity of the formula.

Base Case:
If $\varphi(\mathbf{x})$ is a relation, $R(\mathbf{x})$, then this follows by Lemma 2.27 and because M is the image of a sheaf like model. If $\varphi(\mathbf{x}) :=$ "$f(\mathbf{x}) = g(\mathbf{x})$" for functions $f, g \in \mathcal{F}_L$, then $\lfloor\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models \varphi(\mathbf{x})\}\rfloor^M$ is the equalizer of f^M and g^M in $\mathrm{Sh}(C, J_C)$. However as f^M, g^M are in the image of $\mathbf{b}$, $\lfloor\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models \varphi(\mathbf{x})\}\rfloor^M$ is also $\lfloor\{\mathbf{x} : \mathrm{Sep}(C, J_C) \models \varphi(\mathbf{x})\}\rfloor^M$ which is absolute.

Inductive Case:
$\varphi(\mathbf{x}) = \bigvee_{i\in I} \psi_i(\mathbf{x})$: Let $X_i = \lfloor\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models \psi_i(\mathbf{x})\}\rfloor$ (which is independent of the model of set theory by the inductive hypothesis) and let $X = \{\mathbf{x} : \mathrm{Sep}(C, J_C) \models \bigvee_{i\in I} X_i\}$. For any $A \in \mathrm{obj}(C)$, $X(A) = \bigcup_{i\in I} X_i(A)$ (which is also independent of the model of set theory). We then have $\lfloor\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models \bigvee_{i\in I} X_i\}\rfloor = \mathbf{a}(X) \cap S^M = \lfloor X\rfloor$ (in either V_0 or V_1). But $\lfloor X\rfloor^{V_0} = \lfloor X\rfloor^{V_1}$ by Lemma 2.27 and so $\lfloor\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models \bigvee_{i\in I} X_i\}\rfloor^{V_0} = \lfloor\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models \bigvee_{i\in I} X_i\}\rfloor^{V_1}$.

$\varphi(\mathbf{x}) = \bigwedge_{i\in I} \psi_i(\mathbf{x})$: Identical to the case where $\varphi(\mathbf{x}) = \bigvee_{i\in I} \psi_i(\mathbf{x})$.

$\varphi(\mathbf{x}) = (\exists_f)\psi_i(\mathbf{y})$: Because M is the image of a sheaf like model (from $\mathbf{b}$), f is a map in $\mathrm{Sep}(C, J_C)$. Let $Y = \lfloor\{\mathbf{y} : \mathrm{Sh}(C, J_C) \models \psi(\mathbf{y})\}\rfloor$ and let $X = f[Y]$ in $\mathrm{Sep}(C, J_C)$. Then $\lfloor\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models (\exists_f)\psi(\mathbf{x})\}\rfloor = \mathbf{a}(X) \cap S^M = \lfloor X\rfloor$. But $\lfloor X\rfloor^{V_0} = \lfloor X\rfloor^{V_1}$ by Lemma 2.27.

$\varphi(\mathbf{x}) = (\forall_f)\psi_i(\mathbf{y})$: Suppose $f : S_0 \to S_1$ and $A \in \mathrm{obj}(C)$. Further suppose $a \in S_1^M(A)$. Let $X_a = f^{-1}(a)$ in $\mathrm{Sep}(C, J_C)$. Then $a \in \lfloor\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models (\forall_f)\psi(\mathbf{x})\}\rfloor$ if and only if $\lfloor X_a\rfloor \subseteq \lfloor\{\mathbf{y} : \mathrm{Sh}(C, J_C) \models \psi(\mathbf{y})\}\rfloor$. But this holds if and only if $X_a \subseteq \lfloor\{\mathbf{y} : \mathrm{Sh}(C, J_C) \models \psi(\mathbf{y})\}\rfloor$. Notice X_a is absolute and $\lfloor\{\mathbf{y} : \mathrm{Sh}(C, J_C) \models \psi(\mathbf{y})\}\rfloor$ is absolute by the inductive hypothesis. So the statement $X_a \subseteq \lfloor\{\mathbf{y} : \mathrm{Sh}(C, J_C) \models \psi(\mathbf{y})\}\rfloor$ is absolute. Hence $\lfloor\{\mathbf{x} : \mathrm{Sh}(C, J_C) \models (\forall_f)\psi(\mathbf{x})\}\rfloor$ is absolute.

$\varphi(\mathbf{x}) = \psi(\mathbf{x}) \to \phi(\mathbf{x})$: Because for any subobjects A, B of E we have that $(A \Rightarrow B) = \forall_\alpha(A \wedge B)$ where $\alpha : A \to E$ is a monic and $A \wedge B$ is considered a subobject of A.

$\underline{\varphi(\mathbf{x}) = \neg\psi(\mathbf{x})}$: Because for any subobjects A of E, $\neg A = A \to \bot$ and $\bot$ is absolute. □

Corollary 4.13. *For any model $M \in Mod_L(Sh(C, J_C))^{V_0}$ and any sentence $\varphi \in \mathcal{L}_{\infty,\omega}(L)^{V_0}$, $V_0 \models$ "$M \models \varphi$" if and only if $V_1 \models$ "$M \models \varphi$".*

Proof. Because the satisfaction relation of $\mathcal{L}_{\infty,\omega}(L)$ is closed under isomorphism, by Lemma 3.8 it suffices to restrict our attention to models of the form $M = \mathbf{b}(M')$ where M' is sheaf like. But by Proposition 4.12, the satisfaction relation is absolute for those models. □

Hence if $Th(M) = \{\varphi \in \mathcal{L}_{\infty,\omega}(L) : M \models \varphi\}$ then $Th(M)^{V_0} = Th(M)^{V_1} \cap V_0$.

Definition 4.14. Suppose $T \in \mathcal{L}_{\infty,\omega}(L)$. Let $Mod_L^{SL}(C, J_C)$ be the full subcategory of $Mod_L(\mathrm{Sep}(C, J_C))$ consisting of sheaf like models. Further let $Mod_L^{SL}(T, (C, J_C))$ be the full subcategory of $Mod_L^{SL}(C, J_C)$ consisting of models M such that $\mathbf{b}(M) \models T$ in $\mathrm{Sh}(C, J_C)$.

Lemma 4.15. *$Mod_L^{SL}(T, (C, J_C))$ is a Δ_1-definable class of models.*

Proof. First, as $Mod_L^{SL}(T, (C, J_C))$ is closed under isomorphism in $\mathrm{Sh}(C, J_C)$ (for sheaf like models) it suffices to prove that it is a Δ_1 class. We know, by Proposition 4.12, that for $M \in Mod_L^{SL}(T, (C, J_C))$ the following are equivalent

- SET $\models$ "$\mathbf{b}(M) \models T$" in $Mod_L(\mathrm{Sh}(C, J_C))$.
- $(\exists W \models ZF^*, W \text{ standard}, M \in W) W \models$ "$\mathbf{b}(M) \models T$" in $Mod_L(\mathrm{Sh}(C, J_C))$.
- $(\forall W \models ZF^*, W \text{ standard}, M \in W) W \models$ "$\mathbf{b}(M) \models T$" in $Mod_L(\mathrm{Sh}(C, J_C))$.

□

5. Notions of Countable Sheaves

We are now ready say what we mean by a countable model. We will do this as follows.

Definition 5.1. Suppose $M \in Mod_L(\mathrm{Sh}(C, J_C))$ is a model of L. We say M is *countable* if for each $S \in \mathcal{S}_L$, S^M is a countable sheaf.

The astute reader will notice that this definition isn't quite complete as we still haven't defined what we mean by a countable sheaf. Unfortunately it turns out that there are four notions of countability which we want to consider: purely countable, countably generated, monic countable and epi countable. We will discuss each of these in this section. We will also refer to a model as being purely countable, countably generated, etc. if it satisfies Definition 5.1 with the corresponding notion of countable.

5.1. Purely Countable Sheaves.

Definition 5.2. We say that $A \in \mathrm{obj}(\mathrm{Sh}(C, J_C))$ is *purely countable* if $|\mathbf{a}(A)| = \omega$ (i.e. the sheafification of A is countable as a functor).

Lemma 5.3. *$A \in obj(Sh(C, J_C))$ is purely countable if and only if every separated presheaf which is isomorphic to it (in $Sh(C, J_C)$) is countable as a functor.*

Proof. This is because every presheaf which is isomorphic to A in $\mathrm{Sh}(C, J_C)$ is isomorphic (in $\mathrm{Sep}(C, J_C)$) to a subpresheaf of $\mathbf{a}(A)$. □

From a set theoretic point of view being a purely countable sheaf is a natural notion and is absolute between standard models of set theory.

PROPOSITION 5.4. *There is a* $\mathbf{\Pi}^1_1$ *statement* $PC(x)$ *which holds of if and only if*

(1) $x \in Mod_{L^S_{L_1}}(\omega)$.
(2) $x_M \models T^S_{L_1}$.
(3) $S_1^{k_0(x_M)}$ *is a sheaf.*

PROOF. (1) and (2) are Borel properties so it suffices to show that (3) is a $\mathbf{\Pi}^1_1$ property. However (3) is equivalent to:

> [For all $\langle y_i : i \in I\rangle$ such that $I \in J_C(A)$ and $(\forall i \in I) y_i \in x$][there exists a unique $y \in x$] such that y is covered by $\langle y_i : i \in S\rangle$.

which is $\mathbf{\Pi}^1_1$ because (C, J_C) is countable and so any compatible collection $\langle y_i : i \in I\rangle$ can be encoded by a real. □

COROLLARY 5.5. *For any countable language* L *there is a* $\mathbf{\Pi}^1_1$ *statement* $PC_L(x)$ *which holds if and only if*

(1) $x \in Mod_{L^S_L}(\omega)$.
(2) $x_M \models T^S_L$.
(3) *Every sort of* $k_0(x_M)$ *is a sheaf.*

COROLLARY 5.6. *The collection of purely countable models is a* Δ_1*-definable collection of models.*

PROOF. As all $\mathbf{\Pi}^1_1$ formulas are absolute between standard models of ZF^* being purely countable is absolute between standard models of ZF^*. So a model M is purely countable if either of the following equivalent statements holds:

- $(\exists W$ transitive$)M \in W$ and $(W, \in) \models$"$ZF^* \wedge M$ is purely countable".
- $(\exists W$ transitive$)M \in W$ and $(W, \in) \models$"$ZF^* \wedge M$ is purely countable".

□

COROLLARY 5.7. *If* (A *is a purely countable sheaf*)^{SET} *then* A *is a sheaf in all standard models of* ZF^* *in which* A *is countable.*

PROOF. Because $\mathbf{\Pi}^1_1$ sentences are absolute between standard models of ZF^*, if x is a real encoding A (i.e. $x_M = A$) then $PC(x)$ holds in any standard model of ZF^*in which x exists. But in any standard model of ZF^* in which A is countable there is some real encoding A. Hence in any standard model of ZF^* where A is countable there is some real x encoding A with $PC(x)$. Therefore in any standard model of set theory where A is countable, A is a sheaf. □

While the notion of being purely countable is absolute from a set theoretic point of view, it is unfortunately very sensitive to the underlying weak site (C, J_C). In particular it is not preserved by passing to an equivalent category of sheaves.

LEMMA 5.8. *If* $A \in Sh(C, J_C)$ *and* A *is not a subobject of* 1 *then there exists a countable* (D, J_D) *such that*

- $C \subseteq D$, $|obj(D) - obj(C)| = 1$, $J_C(A) = J_D(A)$ *if* $A \in obj(C)$, *and* $|J_D(*_x)| = 2$ *if* $*_x \notin obj(C)$.

- *If $i : C \to D$ is the inclusion map, then $i \circ - : Sh(D, J_D) \to Sh(C, J_C)$ is an equivalence of categories (with inverse i').*
- *$|i'(A)|$ has (at least) continuum many elements (and hence isn't purely countable).*

Proof. By assumption there is some $x \in \mathrm{obj}(C)$ where $|A(x)| > 1$. Let D be the free category generated by

- $\mathrm{obj}(D) = \mathrm{obj}(C) \cup \{*_x\}$.
- $D[y, z] = C[y, z]$ if $y, z \in \mathrm{obj}(C)$.
- $D[x, *_x] = \{f_i^x : i \in \omega\}$ (where the f_i^x are new morphisms).
- $D[y, *_x] = \{f_i^x \circ g : g \in D[y, x]\}$.
- $D[*_x, y] = \emptyset$ if $y \in \mathrm{obj}(C)$ and $D[*_x, *_x] = id_{*_x}$.

We then let $J_D(y) = J_C(y)$ if $y \in \mathrm{obj}(C)$ and $J_D(*_x) = \{D[-, *_x], S_x\}$ where S_x is the sieve generated by $D[x, *_x]$. It is then immediate that $i \circ -$ is an equivalence of categories as there is a unique way to extend any sheaf on (C, J_C) to a sheaf on (D, J_D) which preserves all maps. Further, it is clear that the unique (non-total) cover of $*_x$ induces a bijection (for any sheaf B on (D, J_D)) between $B(*_x)$ and $B(x)^\omega$. Hence, as $|A(x)| \geq 2$, we have $i'(A)(*_x)$ has at least 2^ω elements. □

A similar argument also shows

Lemma 5.9. *For all (C, J_C) there is a (D, J_D) with $C \subseteq D$, $|morph(D)| = |\omega \times morph(C)|$, $Sh(C, J_C)$ is equivalent to $Sh(D, J_D)$ but all purely countable objects in $Sh(D, J_D)$ are subobjects of* 1.

Not only is the notion of being purely countable very sensitive to the underlying site, but we also have that, unlike in the case of a purely countable set, there are sheaves which are not purely countable in any extension of the universe.

Corollary 5.10. *There is a countable weak site (D, J_D) such that the natural number object in $Sh(D, J_D)$ is not purely countable in any standard model of set theory.*

5.2. Countably Generated. While the notion of being purely countable is very dependent on the underlying weak site, there is a notion of countability which is much less so. This is what we call being countably generated. While a sheaf is purely countable if every object to which it is isomorphic (in $\mathrm{Sh}(C, J_C)$) is countable (as a functor), a sheaf is countably generated if there is some object in $\mathrm{Sh}(C, J_C)$ to which it is isomorphic and which is countable as a functor.

Definition 5.11. We say that $A \in \mathrm{obj}(\mathrm{Sh}(C, J_C))$ is *countable generated* if there is an $A^* \in \mathrm{obj}(\mathrm{Sh}(C, J_C))$ such that $\langle A^*, A, id\rangle \equiv \langle A, A, id\rangle$ in $\mathrm{Sh}(C, J_C)$ on $|A^*| = \omega$ (i.e. there is a countable A^* which covers A).

Lemma 5.12. *Suppose $V_0 \subseteq V_1$ are standard models of set theory. If A is countably generated in $Sh(C, J_C)^{V_0}$ then A is countably generated in $Sh(C, J_C)^{V_1}$.*

Lemma 5.13. *The collection of countably generated sheaves forms a Σ_1-definable class.*

Proof. Because $\mathrm{Sh}(C, J_C)$ is Σ_1 definable, so is the property $\langle A^*, A, id\rangle \in \mathrm{Sh}(C, J_C)$. Further the collection of $\langle A^*, A, id\rangle$ such that $|A^*| \leq \omega$ is Σ_1 definable. □

Corollary 5.14. *The collection of countably generated models in* $Mod_L^{SL}(C, J_C)$ *is a* Σ_1*-definable class of models.*

Proof. By Lemma 5.13 and because being countably generated is closed under isomorphism (in $Mod_L(\mathrm{Sh}(C, J_C))$). □

However, unlike in the purely countable case, being countably generated is preserved under equivalences of categories, at least as long as we restrict ourselves to nice countable weak sites.

Lemma 5.15. *Suppose* $A \in obj(Sh(C, J_C))$*,* $C \subseteq D$*,* $|morph(D) - morph(C)| \leq \omega$ *and* $i \circ - : Sh(D, J_D) \cong Sh(C, J_C)$ *is an equivalence of categories. Then for any* $A \in obj(Sh(C, J_C))$*,* $i \circ A$ *is countably generated if and only if* A *is.*

Proof. If $\langle A^*, A, id\rangle \equiv \langle A, A, id\rangle$ then $\langle i \circ A^*, i \circ A, id\rangle \equiv \langle i \circ A, i \circ A, id\rangle$ and $|i \circ A^*| \leq |A^*|$. So if A is countably generated, so is $i \circ A$.

In the other direction, if $\langle A_i, i \circ A, id\rangle \equiv \langle i \circ A, i \circ A, id\rangle$ let $A^*(U) = \{x \in A(U) : (\exists V \in \mathrm{obj}(C))(\exists f \in D[U, V])(\exists y \in A_i(V))A(f)(y) = x\}$. Then $|A^*| = \omega \times |A_i|$ and $\langle A^*, A, id\rangle \equiv \langle A, A, id\rangle$. So if $i \circ A$ is countably generated so is A. □

We also have that every sheaf is countably generated in some model of set theory.

Lemma 5.16. *If* A *is an object of* $Sh(C, J_C)$ *then there is a forcing extension where* A *is countably generated.*

Proof. We always have $\langle A, A, id\rangle \equiv \langle A, A, id\rangle$ so if $|A|$ is countable then A is countably generated. But we can always find a generic extension where $|A|$ is countable (as $|A| = |\{a : a \in A\}|$ and $\{a : a \in A\}$ is absolute). □

While the notion of being countably generated is independent of countable changes to the site, it is not in general preserved under arbitrary equivalences of categories.

Lemma 5.17. *There is a* (D, J_D) *such that* $C \subseteq D$*,* (C, J_C) *is equivalent to* (D, J_D)*, but no element of* $Sh(D, J_D)$ *(other than the initial element* $0^{Sh(D,J_D)}$*) is countably generated.*

Proof. Let D be a category equivalent to C but where each object of C is isomorphic to $\kappa > \omega$ many objects. Then if A is any separated presheaves for (D, J_D) (such that $|A| \neq 0$) then $|A| \geq \kappa > \omega$. So if $A \neq 0^{\mathrm{Sh}(D,J_D)}$ is an object of $\mathrm{Sh}(D, J_D)$ then A is not countably generated. □

5.3. Monic Countable. In this section we introduce the notion of monic countability. This is the first of two notions of countability which are determined solely by the existence of certain maps to/from the natural number object (and hence are preserved by equivalence of categories).

Definition 5.18. Let $N = (\coprod_\omega 1)^{\mathrm{Sep}(C,J_C)}$. i.e. $N =$ the coproduct of ω many copies of the terminal object (in $\mathrm{Sep}(C, J_C)$).

Notice that $\mathbf{b}(N)$ is a natural number object in $\mathrm{Sh}(C, J_C)$.

Definition 5.19. A sheaf A is *monic countable* if there is a monomorphism (in $\mathrm{Sh}(C, J_C)$) from A into $\mathbf{b}(N)$.

Lemma 5.20. *The monic countable sheaves form a Σ_1-definable class.*

Proof. Notice that being monic countable is preserved by isomorphism. Further, $\langle d, r, f\rangle : A \to N$ is a monomorphism if and only if f is injective. So, as $\mathrm{Sh}(C, J_C)$ is Σ_1 definable, so are the monic countable sheaves. □

Lemma 5.21. *Every monic countable sheaf is countably generated.*

Proof. Suppose A is monic countable and $\langle d, r, f\rangle : A \to N$ is a monomorphism. Then d covers A and so $\langle d, A, id\rangle \equiv \langle A, A, id\rangle$. But $|d| = |f[d]|$ as f is injective and $|f[d]| \le |r| = |N| = \omega$ (as $|\mathrm{obj}(C)| = \omega$). Hence A is countably generated. □

Corollary 5.22. *The monic countable models form a Σ_1-definable class of countably generated models.*

Lemma 5.23. *Being monic countable is preserved under equivalence of categories.*

5.4. Epi Countable.

Definition 5.24. We say a sheaf A is *epi countable* if there is an epimorphism in $\mathrm{Sh}(C, J_C)$ from N onto A.

Lemma 5.25. *The epi countable sheaves form a Σ_1-definable class.*

Proof. Notice that being epi countable is preserved by isomorphism. Further, $\langle d, r, f\rangle : \mathbf{b}(N) \to A$ is an epimorphism in $\mathrm{Sh}(C, J_C)$ if and only if $f[d]$ covers A (i.e. if and only if $\mathbf{j}(\langle d, r, f\rangle)_U : \mathbf{a}(N)(U) \to \mathbf{a}(A)(U)$ is surjective for all $U \in \mathrm{obj}(C)$). If we let $\varphi_{e.c.}(A) := \exists \langle d, f[d], f\rangle \in \mathrm{Sh}(C, J_C)[\mathbf{b}(N), A]$ then $\varphi_{e.c.}(A) \leftrightarrow A$ is epi-countable. Hence, as $\mathrm{Sh}(C, J_C)[\mathbf{b}(N), A]$ is Σ_1 definable we have $\varphi_{e.c.}(A)$ is Σ_1. □

Lemma 5.26. *Epi countable sheaves are countably generated.*

Proof. Suppose A is epi-countable and $\langle d, r, f\rangle \in \mathrm{Sh}(C, J_C)[N, A]$ is an epimorphism. Then $f[d]$ covers A. Hence $\langle f[d], A, id\rangle \equiv \langle A, A, id\rangle$. But $d \subseteq N$ and $|N| = \omega$ (as $|\mathrm{obj}(C)| = \omega$) so $|f[d]| \le \omega$. Hence A is countably generated. □

Corollary 5.27. *The epi countable models of L in $Sh(C, J_C)$ form a Σ_1-definable class of countably generated models.*

Lemma 5.28. *Being epi countable is preserved under equivalence of categories.*

6. Number of Countable Models

We are now ready to prove our bounds on the number of countable models of a sentence of $\mathcal{L}_{\infty,\omega}(L)$. The following definitions will be useful

Definition 6.1. For $x, y \in Mod_{L_L^S}(\omega)$ let $x \cong_{\mathrm{Sh}} y$ if and only if $k_0(x_M) \cong k_0(x_M)$ in $Mod_L(\mathrm{Sh}(C, J_C))$.

Definition 6.2. For $x, y \in Mod_{L_L^S}(\omega)$ let $x \cong_{\mathrm{Sep}} y$ if and only if $k_0(x_M) \cong k_0(x_M)$ in $Mod_L(\mathrm{Sep}(C, J_C))$.

Definition 6.3. Suppose φ_X is a Σ_1-definable class of models. We say φ_X has the *Morley Property* either there is a perfect set of $\cong_{\mathrm{Sh}}$-inequivalent reals in $\widehat{\varphi_X}$ or there are at most ω_1 many $\cong_{\mathrm{Sh}}$-inequivalent reals in $\widehat{\varphi_X}$.

We say φ_X has the *Weak Morley Property* either there is a perfect set of $\cong_{\mathrm{Sh}}$-inequivalent reals in $\widehat{\varphi_X}$ or there are at most ω_2 many $\cong_{\mathrm{Sh}}$-inequivalent reals in $\widehat{\varphi_X}$.

Proposition 6.4. *Suppose φ_X is a Σ_1-definable class of countably generated models. Then there are at most 2^ω many isomorphism classes (for isomorphism in $Mod_L(Sh(C, J_C))$) in φ_X.*

Proof. This follows immediately from Proposition 3.10 and the fact that there are at most 2^ω many countable models in $Mod_{L_L^S}(T, \mathrm{SET})$. □

6.1. Purely Countable.

Lemma 6.5. *For every model $M \in Mod_L(Sh(C, J_C))$ there is a unique (up to isomorphism in $Mod_L(Sep(C, J_C))$) model $M' \in Mod_L(Sep(C, J_C))$ such that*

- *M' is sheaf like.*
- *For all $S \in \mathcal{S}_L$, $S^{M'}$ is a sheaf for (C, J_C).*
- *$M \cong \boldsymbol{b}(M')$ (in $Mod_L(Sh(C, J_C))$).*

Proof. By Lemma 3.8 there is a sheaf like model M^* such that $M \cong \mathbf{b}(M^*)$. If we let $M' = \mathbf{i}(\mathbf{a}(M^*))$, i.e the model obtained by applying the sheafification functor $\mathbf{a}$ to M^*, then M' has the desired properties. Further it is unique up to isomorphism in $Mod_L(\mathrm{Sep}(C, J_C))$ as $\mathbf{i}$ is fully faithful. □

Corollary 6.6. *Two model $M_0, M_1 \in Mod_L(Sh(C, J_C))$ are isomorphic if and only if $\boldsymbol{i}(\boldsymbol{j}(M_0)), \boldsymbol{i}(\boldsymbol{j}(M_1)) \in Mod_L(Sep(C, J_C))$ are isomorphic.*

The following Corollary collects much of what we have shown about purely countable models.

Corollary 6.7. *Suppose $Mod_L^{PCS}(C, J_C) = \{\boldsymbol{i}(\boldsymbol{j}(M)) : M \in Mod_L(Sh(C, J_C))$ and M is purely countable$\} \subseteq Mod_L^{SL}(C, J_C)$ then*

- *$(\forall M \in Mod_L(Sh(C, J_C)),\ M$ purely countable$)(\exists M' \in Mod_L^{PCS}(C, J_C))\boldsymbol{b}(M') \cong M$ (in $Mod_L(Sh(C, J_C))$).*
- *$(\forall M_0, M_1 \in Mod_L^{PCS}(C, J_C))\boldsymbol{b}(M_0) \cong \boldsymbol{b}(M_1)$ in $Mod_L(Sh(C, J_C))$ if and only if $M_0 \cong M_1$ in $Mod_L(Sep(C, J_C))$.*
- *$(\forall M \in Mod_L^{PCS}(C, J_C))(\exists x \in \omega^\omega)PC_L(x)$ and x_M is isomorphic to M in $Mod_L(Sep(C, J_C))$.*

Corollary 6.7 tells us that for the purposes of counting isomorphism classes in $Mod_L(\mathrm{Sh}(C, J_C))$ of purely countable models, it suffices to consider models which are encoded by reals satisfying $PC_L(x)$. However, when considering these models, it also suffices to worry about isomorphism in $Mod_L(\mathrm{Sep}(C, J_C))$. This is important as isomorphism between separated presheaves is, from a set theoretic point of view, much simpler than isomorphism between sheaves.

Theorem 6.8. *Let φ_D be any Σ_1-definable class of purely countable models with only hereditarily countable parameters. Then D has the Morley property.*

PROOF. Let $D^* = \widehat{\varphi_D} \cap Mod_L^{PCS}(C, J_C)$. By Corollary 6.7 we know two things. First, $D^* = \{x \in \widehat{\varphi_D} : PC_L(x)\}$ and second there is a bijection between $\cong_{\text{Sep}}$-isomorphism classes of reals in D^* and $\cong_{\text{Sh}}$-isomorphism classes of reals in $\widehat{\varphi_D}$.

CLAIM 6.9. *$x \cong_{Sep} y$ is a $\mathbf{\Sigma}_1^1$ relation on $Mod_{L_L^S}(\omega)$.*

PROOF. Because $x \cong_{\text{Sep}} y$ if and only if $x_M \cong y_M$ as models of L_L^S (which is a $\mathbf{\Sigma}_1^1$ property). □

CLAIM 6.10. *D^* is a $\mathbf{\Sigma}_2^1$ set of reals.*

PROOF. By Proposition 2.3 and the fact that $PC_L(x)$ is $\mathbf{\Pi}_1^1$. □

But by Proposition 2.12 these claims imply that D^* contains either a perfect set of $\cong_{\text{Sep}}$-inequivalent reals or at most ω_1 many $\cong_{\text{Sep}}$-inequivalent reals. So by Corollary 6.7 $\widehat{\varphi_D}$ contains either a perfect set of $\cong_{\text{Sh}}$-inequivalent reals or at most ω_1 many $\cong_{\text{Sh}}$-inequivalent reals and hence φ_D has the Morley property. □

COROLLARY 6.11. *For any theory $T \in \mathcal{L}_{\omega_1,\omega}(L)$ the collections of purely countable models of T has the Morley property.*

PROOF. This follows from Theorem 6.8 because the purely countable models satisfying T form a Σ_1-definable class of models.

□

6.2. Countably Generated.

LEMMA 6.12. *$\cong_{Sh}$ is a $\mathbf{\Sigma}_2^1$ equivalence relation.*

PROOF. This follows Lemma 3.9 and Proposition 2.3. □

THEOREM 6.13. *Let φ_D be any Σ_1-definable class of countably generated models with only hereditarily countable parameters. We then have the two results:*

(1) *If $(*_\#)$ holds then φ_D has the weak Morley property.*
(2) *If $(*_B)$ holds then φ_D has the Morley property.*

PROOF. We know that $\widehat{\varphi_D}$ is a $\mathbf{\Sigma}_2^1$ set by Lemma 4.2 and that $\cong_{\text{Sh}}$ is $\mathbf{\Sigma}_2^1$ by Lemma 6.12. (1) then follows from Proposition 2.8 and (2) then follows from Proposition 2.9. □

COROLLARY 6.14. *For any theory $T \in \mathcal{L}_{\omega_1,\omega}(L)$ the following hold.*

(1) *If $(*_\#)$ holds then the collection countably generated models which satisfy T has the weak Morley property.*
(2) *If $(*_B)$ holds then the collection countably generated models which satisfy T has the Morley property.*

6.3. Monic and Epi Countable. We know by Corollary 5.27 and Corollary 5.22 that the collection of monic and epi countable models are Σ_1-definable classes of countably generated models. Hence as an immediate consequence of Theorem 6.13 we have:

COROLLARY 6.15. *For any theory $T \in \mathcal{L}_{\omega_1,\omega}(L)$ the following hold.*

(1) *If $(*_\#)$ holds then*

   (a) *The collection monic countable models which satisfy T has the weak Morley property.*
   (b) *The collection epi countable models which satisfy T has the weak Morley property.*
(2) *If $(*_B)$ holds then*
   (a) *The collection monic countable models which satisfy T has the Morley property.*
   (b) *The collection epi countable models which satisfy T has the Morley property.*

7. Vaught's (and other) Conjectures

We end with a few conjectures.

7.1. Absoluteness of Isomorphism. In Lemma 6.12 we showed that the isomorphism relation between reals encoding countably generated models was $\mathbf{\Sigma}^1_2$. As a consequence of this we see, by Shoenfield's Absoluteness Theorem, that the isomorphism relation between two countably generated models is absolute between standard models of set theory which contain ω_1^{SET}[12]. However it is the authors belief that we can remove the condition that ω_1^{SET} be in our standard model of set theory when talking about absoluteness.

CONJECTURE 7.1. *For any two models M and N in $Mod_L(Sh(C, J_C))$ the statement "M is isomorphic to N in $Mod_L(Sh(C, J_C))$" is absolute between all standard models of ZF^*.*

If this conjecture holds then it is possible to remove the large cardinal and determinacy assumptions from Theorem 6.13.

THEOREM 7.2. *Let φ_D be a Σ_1-definable class of countably generated models with only a hereditarily countable parameter P. If Conjecture 7.1 holds and there is a real Cohen generic over $L[P]$ then φ_D has the Morley property.*

PROOF. We define $Iso^*(x, y)$ as the formula satisfying
- $x, y \in Mod_{L^S_L}(\omega)$ and $x_M, y_M \models T^S_L$.
- For all $E \subseteq \omega \times \omega$, if
  - (ω, E) is a well-founded model of ZF^*.
  - $\{x, y, P\} \subseteq tc(\omega, E)$

  then $tc(\omega, E) \models x \cong_{\mathrm{Sh}} y$.

CLAIM 7.3. *$Iso^*(x, y)$ holds if and only if $x \cong_{Sh} y$.*

PROOF. The implication from left to right holds because $\cong_{\mathrm{Sh}}$ is upwards absolute. The implication from right to left is immediate from Conjecture 7.1. □

$Iso^*(x, y)$ is clearly a $\mathbf{\Pi}^1_2$ relation (and by Claim 7.3 an equivalence relation). Let $Iso^*_D(x, y) \leftrightarrow Iso^*(x, y) \vee [x \notin \widehat{\varphi_D} \wedge y \notin \widehat{\varphi_D}]$. $Iso^*_D(x, y)$ is then also a $\mathbf{\Pi}^1_2$ equivalence relation as $\widehat{\varphi_D}$ is a $\mathbf{\Sigma}^1_2$ set. So $Iso^*_D(x, y)$ is a co-ω_1-Suslin equivalence relation. By Proposition 2.6 it then follows that there are either a perfect set of Iso^*_D-inequivalent reals or there are at most $\aleph_1$ many Iso^*_D-inequivalent reals.

[12] In fact it can be shown that for any two models M and N (not necessarily countably generated), the statement "M is isomorphic to N" is absolute between standard models which contain the ordinals.

But as $x \cong_{Sh}$ and Iso^*_D agree on $\widehat{\varphi_D}$ (by Claim 7.3) we therefore have φ_D has the Morley property. □

Corollary 7.4. *Suppose $T \in \mathcal{L}_{\omega_1,\omega}(L)$, there is a real Cohen generic over $L[(C, J_C), L, T]$*[13] *and Conjecture 7.1 holds. Then*

(a) *The collection countably generated models which satisfy T has the Morley property.*
(b) *The collection monic countable models which satisfy T has the Morley property.*
(c) *The collection epi countable models which satisfy T has the Morley property.*

7.2. Absoluteness Conjecture. In [**10**] Morley proved more than just that there is a trichotomy of options for the number of countable models of a sentence of $\mathcal{L}_{\omega_1,\omega}(L)$. He showed that, for any particular sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$, which of the trichotomy was realized was independent of the standard model of set theory where it was considered.

It seems reasonable that this would hold for models in a category of sheaves as well. So we make the following conjecture:

Conjecture 7.5. *For any sentence $T \in \mathcal{L}_{\omega_1,\omega}(L)$ and any countable weak site (C, J_C) the following cardinalities are each independent of the standard model of set theory where we consider them:*

- *The number of purely countable models of T in $Sh(C, J_C)$.*
- *The number of countable generated models of T in $Sh(C, J_C)$.*
- *The number of monic countable models of T in $Sh(C, J_C)$.*
- *The number of epi countable models of T in $Sh(C, J_C)$.*

We now prove partial absoluteness result for the number of purely countable models. For the purpose of this result let $(\neg CH)^*$ be the statement:

- There exists an $i : \omega_2 \to 2^\omega$ where i is an injection[14].

Lemma 7.6. *Suppose $E \subseteq \omega^\omega \times \omega^\omega$ is a $\mathbf{\Sigma}^1_1$ equivalence relation on ω^ω and that $X \subseteq \omega^\omega$ is a $\mathbf{\Sigma}^1_2$ set. Then the statement:*

There exists a perfect set of E-inequivalent reals in X.

is absolute between standard models of set theory which satisfies $(\neg CH)^*$ and contains ω_1^{SET}.

Proof. Let $L_\leq$ be the language of linear orders and $LO \subseteq Mod_{L_\leq}(\omega)$ be the collection of linear orders. Let $T(x, y)$ be a tree on $(\omega \times \omega) \times \omega$ such that for all $x, y \in \omega^\omega$

$$E(x, y) \Leftrightarrow T(x, y) \text{ is ill-founded.}$$

There is a $\mathbf{\Sigma}^1_1$ set $E' \subseteq \omega^\omega \times \omega^\omega \times LO$ such that $E'(x, y, \alpha) = \neg(\|T(x, y)\| < \alpha)$ if α is well-founded.

We also know (by Theorem 25.19 of [**6**]) that there is a $\mathbf{\Delta}^1_1$ subset $D_X \subseteq \omega^\omega \times \omega^\omega \times LO$ such that if $D'(x, a) = (\forall y \in \omega^\omega) D_X(x, y, a)$ then

- $X(x) \leftrightarrow (\exists \alpha < \omega_1) D'(x, \alpha)$

[13] The first L represents the constructible universe while the second L is the language we are working in.

[14] This is equivalent to $\neg CH$ in the presence of the axiom of choice.

- For all well-founded $\alpha \in LO$, $D'(x,\alpha)$ is a Borel relation.

Further the Borel code of D_X is dependent only on the formula describing X (i.e. is independent of the standard model of set theory we are working in).

Notice the following statement is $\mathbf{\Sigma}^1_2$:

($\star$) $(\exists \beta,\gamma \in LO)(\exists$ perfect set of reals $S)(\beta,\gamma$ are well-founded) $\wedge$ $(\forall x \in S)D'(x,\gamma) \wedge (\forall x,y \in S)\neg E'(x,y,\beta)$

Hence, by Shoenfield's Absoluteness Theorem (see Theorem 25.20 of [**6**]), ($\star$) is absolute between standard models of set theory which contain the ω_1^{SET}. Hence it suffices to prove the following claim:

Claim 7.7. *Suppose V is a standard model of set theory satisfying $(\neg CH)^*$. Then ($\star$) is equivalent to the statement "There are a perfect set of E-inequivalent reals in X".*

Proof. The implication from left to right is immediate.

Assume that there are a perfect set of E-inequivalent reals in X in V. Let G be a generic such that $V[G] \models |\aleph_1^V| = \aleph_0$ and $|(2^\omega)^V| \geq |\aleph_2^V| \geq \aleph_1$. We then have $E^{V[G]} \cap V = E^V$. Similarly we have $X^{V[G]} \cap V = X^V$ and $[D'(-,\beta)]^{V[G]} \cap V = [D'(-,\beta)]^V$ for any $\beta \in \omega_1^V$. But in $V[G]$ we have that $E^{\omega_1^V}$ is a Borel equivalence relation (by Lemma 30.10 of [**6**]) and has more than countably many equivalence classes on $D'(-,\omega_1^V)^{V[G]} = \bigcap_{\alpha<\omega_1^V} D'(-,\alpha)^{V[G]}$ (which is also a Borel set). Hence, by a theorem of Silver (see Theorem 32.1 of [**6**]), there is a perfect set of $E^{\omega_1^V}$-inequivalent reals on $D'(-,\omega_1^V)$ in $V[G]$. So $V[G] \models (\star)$ and by absoluteness we then also have that $V \models (\star)$. $\square$

$\square$

The above proof follows closely the proof of Theorem 32.9 of [**6**] (which is a theorem of Burgess).

In general we cannot remove $(\neg CH)^*$ from the assumption. To see this notice that if L is the constructible universe then $\mathbb{R}^L$ is a Σ^1_2 set of reals. However in L, $\mathbb{R}^L$ contains a perfect set (and hence a perfect set of $=$-inequivalent reals) while there are generics G such that $L[G] \models |\mathbb{R}^L| = \omega$.

Corollary 7.8. *Let φ_D be any Σ_1-definable class of purely countable models with only hereditarily countable parameters. Then whether or not $\widehat{\varphi_D}$ has a perfect set of $\cong_{Sh}$-inequivalent reals is absolute between standard model of set theory which satisfies $(\neg CH)^*$ and contain ω_1^{SET}.*

Proof. This follows immediately from the proof of Theorem 6.8 and Lemma 7.6. $\square$

7.3. Topos Vaught's Conjecture. Over the years Vaught's conjecture has been studied in ever widening contexts, such as "Vaught's conjecture for $\mathcal{L}_{\omega_1,\omega}(L)$" and the "Topological Vaught's Conjecture". Each time Vaught's conjecture is extended to a new area, we obtain access to a new collection of techniques for it's study.

In this section we present a new generalization of Vaught's conjecture. It is our hope that the study of Vaught's conjecture in this context will shed light on the original conjecture as well as its variants.

Definition 7.9. Let (C, J_C) be a countable weak site and let L be a countable language. Suppose φ_D is a Σ_1-definable class of models in $Mod_L(C, J_C)$. We then say φ_D has the *Vaught Property* if either:

- There is a perfect set of $\cong_{\mathrm{Sh}}$-inequivalent reals in $\widehat{\varphi_X}$.
- There are at most $\aleph_0$ many $\cong_{\mathrm{Sh}}$-inequivalent reals in $\widehat{\varphi_X}$.

Definition 7.10. We say a countable weak site (C, J_C) has the *Vaught property* if the for every countable language L and every $T \in \mathcal{L}_{\omega_1,\omega}(L)$ the collections of purely countable, countably generated, monic countable and epi countable models of T all have the Vaught property.

In particular, a weak site has the Vaught property if, inside the category of sheaves on the weak site, Vaught's conjecture holds for any of the four notions of countable presented in this paper. It isn't hard to see that there are weak sites which have the Vaught property.

Proposition 7.11. *There is a countable weak site (C, J_C) which satisfies:*

- *J_C has only identity sieves (so $Sh(C, J_C) = SET^{C^{op}}$).*
- *For every sheaf A the following are equivalent:*
  - *A is purely countable.*
  - *A is countably generated.*
  - *A is monic countable.*
  - *A is epi countable.*
- *(C, J_C) has the Vaught property.*

Proof. Let C be the set ω treated as a category. Fix $T \in \mathcal{L}_{\omega_1,\omega}(L)$, for some countable language L, and let φ_T be the collection of countably generated models of T in $\mathrm{SET}^{C^{op}}$. (Note φ_T is also the collection of purely countable, epi countable and monic countable models of T in $\mathrm{SET}^{C^{op}}$.)

A model of T in $\mathrm{SET}^{C^{op}}$ consists of ω many models of T in SET. In particular a model of T in $\mathrm{SET}^{C^{op}}$ is in φ_T if and only if each of the corresponding models in SET are countable.

We now break into two cases. First suppose that T has at least two countable models in SET. Then T must have at least 2^ω many countably generated models in $\mathrm{SET}^{C^{op}}$. And in particular $\widehat{\varphi_T}$ must have a perfect set of $\cong_{\mathrm{Sh}}$-inequivalent reals. On the other hand suppose T has at most one countable model in SET. Then T has at most one countably generated model in $\mathrm{SET}^{C^{op}}$ and so $\widehat{\varphi_T}$ is either empty or has no $\cong_{\mathrm{Sh}}$-inequivalent reals.

Hence, (C, J_C) has the Vaught property. □

In the other direction we then also have:

Proposition 7.12. *The following are equivalent.*

- *Vaught's Conjecture.*
- *The terminal site*[15] *has the Vaught property.*

This suggests the following generalization of Vaught's conjecture (which implies the actual Vaught's conjecture).

Conjecture 7.13 (Topos Vaught's Conjecture). *Every countable weak site (C, J_C) has the Vaught property.*

[15]The terminal site is the unique site on the category with one object and one morphism.

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