Relativized Grothendieck Topoi

Nathanael Leedom Ackerman
University of California at Berkeley, Berkeley, CA, United States

Abstract

In this paper we define a notion of relativization for higher order logic. We then show that there is a higher order theory of Grothendieck topoi such that all Grothendieck topoi relativizes to all models of set theory with choice.

Key words: Grothendieck Topos, Relativize, Sheaves, Site
2000 MSC: 03B15, 03C55, 03E47, 18B25, 18F10, 18F20

1. Introduction

One of the most important properties of first order logic is that the satisfaction relation between formulas and models is absolute. That is, given two standard set theoretic universes V_0 and V_1, a model M, and a formula φ of first order logic where $M, \varphi \in V_0 \cap V_1$, we have $(M \models \varphi)^{V_0}$ if and only if $(M \models \varphi)^{V_1}$. Unfortunately though when we move to the realm of higher order logic we often have to leave behind absoluteness of the satisfaction relation. This is because, unlike first order logic, higher order logic is able to talk about the ambient set theoretic universe. Hence, if we change the ambient set theoretic universe, we may change the models which satisfy a given higher order formula.

In particular, given a model M and a higher order formula φ such that $(M \models \varphi)^{V_0}$ we often won't have $(M \models \varphi)^{V_1}$ (where $V_0 \subseteq V_1$ are standard models of set theory). But, for certain φ, even if $\neg(M \models \varphi)^{V_1}$ there will be models which contain M as a subset which do satisfy φ in V_1. If there is a smallest such model, M_1, it makes sense to consider M_1 as the "relativization of M to V_1 (as a model of φ)".

In this paper we will make precise this notion of relativization and show that there is a second order theory GT, whose models are exactly the definable expansions of Grothendieck topoi, such that every model of GT has a relativization to every standard model of set theory (assuming the Axiom of Choice). In the process we will also show that every model of the theory of sites as well as every model of the theory of subcanonical sites has a relativization to every model of set theory.

2. Outline

We begin this paper in Section 3 with a discussion of material needed to understand our main results. In this section we introduce our background model of set theory, review some basic notions from category theory, and discuss what we mean by the relativization of a higher order model.

After we have introduced these concepts, we begin our study of the relativization of Grothendieck topoi. Specifically we begin by looking at sites. In Section 4 we show that both sites and subcanonical sites relativize. In the process of doing this we introduce two important notions, that of a weak site and that of an almost subcanonical weak site. These are the absolute analogs of sites and subcanonical sites respectively.

Once we have the notion of an almost subcanonical weak site we move onto the study of categories of sheaves. In Section 5 we define a higher order theory whose sole model is equivalent to category of sheaves on a weak site. We further show that this unique model relativizes. We then show that if two almost subcanonical weak sites have equivalent categories of sheaves in one standard model of set theory then they have equivalent categories of sheaves in any model of set theory.

Finally, in Section 6, we expand the theory of Grothendieck topoi so that each model records the almost subcanonical weak sites whose categories of sheaves it is equivalent to. This will then allow us, using the Axiom of Choice, to explicitly construct a relativization of a model of our expanded theory of Grothendieck topoi.

3. Background

3.1. Set Theory

When one tries to do naive category theory in the language of sets and classes one runs into a problem. This problem arises from the need not only to deal with large categories (i.e. those which have a proper class of objects), but also to deal with categories whose objects are large categories.

In what follows this issue will arise when we need to consider a certain category of Grothendieck topoi in Theorem 6.4. Fortunately for us this category of Grothendieck topoi will be definable over the universe and so a basic theory of hyperclasses is all that will be needed. We present one such theory now.

Definition 3.1. *Let $L_{ST} = \{\in, \mathcal{S}, \mathcal{C}\}$ where $\mathcal{S}$ and $\mathcal{C}$ are constants and $\in$ is a binary relation. Let ST be the theory*

- $\langle \mathcal{C}, \in, \mathcal{S} \rangle \models$ *Bernays-Gödel Set Theory ([6])*
- *(Extensionality)* $(\forall X, Y)[(\forall u) u \in X \leftrightarrow u \in Y] \rightarrow X = Y$
- *(Regularity)* $(\forall X)(X \neq \emptyset \rightarrow (\exists x \in X) X \cap x = \emptyset)$
- *(Definition of Class)* $(\forall X, Y) X \in Y \rightarrow X \in \mathcal{C}$
- *(Hyperclass Comprehension) If φ is a formula where every quantifier is bound by $\mathcal{S}$, then*

$$(\forall X_1, \ldots X_n)(\exists Y) Y = \{x : \varphi(x, X_1, \ldots, X_n)\}$$

We let STC be $ST + \mathcal{S} \models$ "Global Axiom of Choice".

We call the elements a model of ST hyperclasses, those elements which are also elements of $\mathcal{C}$ classes, and those elements which are also elements of $\mathcal{S}$ sets. We also define a *powerset* relation for any hyperclass x by $\mathcal{P}(x) = \{y \in \mathcal{S} : y \subseteq x\}$.

In order to prove our results in maximum generality we will also avoid using arguments which require the Axiom of Choice whenever possible. However when a theorem does use the Axiom of Choice we will mark it with (*).

For the rest of this paper we will fix a model $Set \models ST$ and will assume that all standard models are with respect to this background model of set theory.

Definition 3.2. *If $Set \models ST$ then a standard model is a triple of formulas formula $\varphi_{hc}(x, z), \varphi_c(x, z), \varphi_s(y, z)$ in L_{ST} along with a hyperclass $A \in Set$ such that*

$$\langle \{x \in Set : \varphi_{hc}(x, A)\}, \in, \{y \in Set : \varphi_s(y, A)\}, \{y \in Set : \varphi_c(y, A)\} \rangle \models ST$$

Definition 3.3. *If V is a standard model and $\varphi(x, A)$ is a formula of set theory with A a hyperclass in V then $(\varphi(x, A))^V$ is the formula of set theory obtained by uniformly bounding all quantifiers by V (i.e. replacing $(\forall x)\psi(x, y)$ with $(\forall x)x \in V \to \psi(x, y)$ and replacing $(\exists x)\psi(x, y)$ with $(\exists x)x \in V \wedge \psi(x, y)$). We say $(\varphi(x, A))^V$ is the relativization of $\varphi(x, A)$ to V.*

Before we continue, it is worth saying a few words about our system of set theory ST. The specific axioms we have chosen are not important beyond the fact that the collection of sets are a model of Zermelo-Fraenkel Set Theory[1] (ZF) and that the classes and hyperclasses satisfy comprehension for formulas with quantification restricted to sets. As such there are many other versions of set theory, such as Zermelo-Fraenkel Set Theory with Atoms ([6]), Ackermann Set Theory ([1]), Feferman's Set Theory ([5]) or variants of algebraic set theory[2] ([7], [2]) for which an adequate hyperclass theory could easily be developed and which would serve equally well as an underlying foundation.

We will end this section with an important observation relating our set theory to those mentioned above.

Proposition 3.4. *ST is equiconsistent with ZF.*

Proof. First notice that the collection of sets and classes of ST satisfies Bernay-Gödel Set Theory and hence implies the consistency of ZF ([6]). In the other direction it is known (see [5]) that the set theory ZF/s of [5] is equiconsistent with ZF. However if

$$(M, \in, s^M) \models ZF/s \wedge C = \mathcal{P}(s) \wedge H = \mathcal{P}(C)$$

[1]There are two natural weakenings of ZF which one might consider. The first is Kripke-Platek Set Theory (KP) ([4]) and the second is of Zermelo Set Theory (Z) ([10]). The difficulty with KP is that it lacks a powerset axiom and as such there has no canonical way to deal with the process of sheafification. On the other hand Z, while it has the powerset axiom, does not have replacement. As such operations which may require definition by induction through the ordinals (such as Definition 4.2) can not be carried out in Z.

[2]Algebraic set theory has shown there is a close relationship between specific set theories and specific theories of categories. Using the methods of algebraic set theory it has also been shown ([3]) that the notion of sheaf and of sheafifcation makes sense in a much wider context than that of ZF (such as in weak systems of intuitionistic set theory). However, for the results of this paper the full strength of ZF is needed. As such while a foundation based on algebraic set theory is possible, for the proofs in this paper one would need to add sufficient axioms to ensure that the system could interpret (classical) ZF.

then $(H, \in, s^M, C) \models ST$. Hence the consistency of ZF implies the consistency of ST. □

3.2. Category Theory

In this section we will review some of the categorical definitions which will be needed. For more information on the category theory in this section the reader is referred to such standard texts as [8]. For more information on the sheaf theoretic ideas presented in this section the reader is referred to such standard works such as [9].

Definition 3.5. *Let $L_{Cat} = \{Obj, Morph, Dom, Codom, Id\}$ and let $Th_{Cat}(X)$ be the formula which says*

- *$X \in obj(Mod(L_{Cat}))$ and X is a category.*
- *$X \models Obj(A)$ if and only A is an object of X.*
- *$X \models Morph(f)$ if and only if f is a morphism of X.*
- *$Dom, Codom : Morph \to Obj$ are the domain and codomain maps respectively.*
- *$Id : Obj \to Morph$ is the map which takes an object to its identity morphism.*

We call a category small if it has a set of objects and a set of morphisms.

Definition 3.6. *Let **SET** be the category whose objects are sets in Set and whose morphisms are functions in Set.*

Definition 3.7. *Let C be a small category with $F : C^{op} \to \boldsymbol{SET}$ a presheaf on C. An element of F is an element of $\bigcup_{A \in obj(C)} F(A)$. If $f \in C[A, B], x \in F(B)$ we will use the shorthand $x|_f$ for $F(f)(x)$.*

Definition 3.8. *For any small category C let $y_C : C \to \boldsymbol{SET}^{C^{op}}$ be the Yoneda Embedding. i.e. $y_C(A) = C[-, A]$ for all $A \in obj(C)$ and $y_C(f) = C[-, f]$ for all $f \in morph(C)$.*

Definition 3.9. *Let C be a small category and $A \in obj(C)$. A sieve S on A is a subfunctor of $C[-, A]$. If S is a sieve on A and $f : B \to A$ then the pullback of S along f is the sieve on B given by $f^*S(D) = \{g \in C[D, B] : f \circ g \in S\}$. If $X \subseteq C[-, A]$ we define $Gen_{sieve}(X) = \{x \circ f : x \in X, dom(x) = codom(f), f \in morph(C)\}$ to be the sieve generated by X.*

3.3. Models

In this section we give our definition of a higher order model as well as our definition of a map between higher order models. A higher order model will consist of a class along with a collection of relations between tuples of the class and tuples of subsets[3] of the class. These relations are what will be preserved as we relativize our model to a larger standard model of set theory.

In particular, suppose we have some (possibly higher order) property of tuples/subsets, R, which we want to be preserved (either by maps of our models or in the relativization of our model). We can achieve this by adding a predicate $R'(-)$ to our language and adding an axiom to our theory saying "$R'(\mathbf{x})$ holds if and only if R holds of $\mathbf{x}$". (Notice though that this will not necessarily mean that $\neg R$ is preserved unless we explicitly add a relation R'' and an axiom $R''(\mathbf{x}) \leftrightarrow \neg R(\mathbf{x})$).

Definition 3.10. *Suppose $L = \{\equiv_j, \not\equiv_j : j \in \{0,1\}\} \cup \{R_i : i \in I\}$ where R_i is a relation with arity $m_i + n_i$ and $\equiv_j, \not\equiv_j$ are binary relations. Mod(L) is the category such that*

- *The objects of Mod(L) are sequences $\mathcal{M} = \langle M, \mathcal{P}(M), \equiv_j^M, \not\equiv_j^M, R_i^M \rangle$ where*
  - *$\equiv_0^M, \not\equiv_0^M \subseteq M \times M$, $\equiv_1^M, \not\equiv_1^M \subseteq \mathcal{P}(M) \times \mathcal{P}(M)$ and $R_i^M \subseteq M^{n_i} \times \mathcal{P}(M)^{m_i}$. We will use the shorthand $\mathcal{M} \models R_i(\boldsymbol{a})$ for $\boldsymbol{a} \in R_i^M$, $a \equiv_j b$ for $\equiv_j^M (a,b)$ and $a \not\equiv_j b$ for $\not\equiv_j^M (a,b)$.*
  - *$\equiv_j$ is an equivalence relation and $a \not\equiv_j b \leftrightarrow \neg(a \equiv_j b)$ for all $a, b \in dom(\equiv_j)$.*
  - *$(\forall m, m' \in \mathcal{P}(M)) m \equiv_1 m' \leftrightarrow (\forall a \in M)(\exists b, b' \in M)(a \in m \to b \in m' \wedge a \equiv_0 b) \wedge (a \in m' \to b' \in m \wedge a \equiv_0 b')$.*
  - *If $M \models \bigwedge_{k \leq m_i} a_k \equiv_0 b_k \wedge \bigwedge_{l \leq n_i} A_l \equiv_1 B_l$ then*
    $$M \models R_i(a_1, \ldots, a_{m_i}, A_1, \ldots, A_{n_i}) \leftrightarrow R_i(b_1, \ldots, b_{m_i}, B_1, \ldots, B_{n_i}).$$
- *Suppose $\mathcal{M} = \langle M, \mathcal{P}(M), \equiv_j^M, \not\equiv_j^M, R_i^M \rangle$ and $\mathcal{N} = \langle N, \mathcal{P}(N) \equiv_j^N, \not\equiv_j^N, R_i^N \rangle$. Further suppose $f : M \to N$ and $\overline{f} : \mathcal{P}(M) \to \mathcal{P}(N)$ is defined by $\overline{f}(X) = \{f(x) : x \in X\}$. Then $f \in$ Mod$(L)[\mathcal{M}, \mathcal{N}]$ if and only if*

[3] Our definition will only consider subsets and not subclasses.

- *For all* $\boldsymbol{a} \in M$, $\boldsymbol{b} \in \mathcal{P}(M)$ *if* $\mathcal{M} \models R_i(\boldsymbol{a}, \boldsymbol{b})$ *then* $\mathcal{N} \models R_i(f[\boldsymbol{a}], \overline{f}[\boldsymbol{b}])$.[4]
- *For all* $a, b \in M$, *if* $\mathcal{M} \models a \equiv_0 b$ *then* $\mathcal{N} \models f(a) \equiv_0 f(b)$ *and if* $\mathcal{M} \models a \not\equiv_0 b$ *then* $\mathcal{N} \models f(a) \not\equiv_0 f(b)$.
- $(\forall m \in \mathcal{P}(M))(\forall a \in M)[(\exists b \in M) a \equiv_0^M b \wedge b \in m] \Leftrightarrow [(\exists b' \in N) f(a) \equiv_0^N b' \wedge b' \in \overline{f}(m)]$.

We call a model small if it's underlying class is a set.

The astute reader will notice that while we have been talking about higher order models, the structures which we have defined are only second order. Restricting to the second order case does not in general limit our expressive power as we can always add a new relation $PS \subseteq M \times \mathcal{P}(M)$ with an axiom which says "$PS(a, A)$ implies a is a name for the set A". By then considering subsets of names we can represent third order structure in our second order model. This procedure can be generalized to any αth-order structure we want (for a set sized ordinal α).

It is also worth mentioning that while we want to think of elements of our model as $\equiv$-equivalence classes we can't quite do this if our set theory doesn't satisfy the Axiom of Choice. The reason is that there may be a map between the equivalence classes of models which doesn't come from an actual map of models (because we are unable to choose a representative of our equivalence classes.)

Definition 3.11. *Let* $Mod_=(L)$ *be the full subcategory of* $Mod(L)$ *consisting of those models* $\mathcal{M} \models (\forall x, y) x \equiv_0 y \rightarrow x = y$

Lemma 3.12. *There is a functor* $F : Mod(L) \rightarrow Mod_=(L)$.

Proof. For each model $\mathcal{M} = \langle M, \equiv_j^M, \not\equiv_j^M, \mathcal{P}(M), R^M \rangle$ let $F(M) = \langle M/\equiv_0^M, \mathcal{P}(M)/\equiv_1^M, =, \neq, R^M \rangle$. For each $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ and $a \in M$ let $F(\alpha)(a) = [\alpha(a)]_{\equiv_0^N}$. It is then clear that $F(M) \in \text{Mod}_=(L)$ (because R^M is closed under $\equiv_j^M$ and $[\mathcal{P}(M)/\equiv_1^M] \cong \mathcal{P}(M/\equiv_0^M)$). Further each function $F(\alpha)$ is a map between $F(M)$ and $F(N)$. □

Lemma 3.13. *F is full if the global axiom of choice holds.*

[4]Here we use boldface to represent tuples and, if $\mathbf{x} = \langle x_0, \ldots, x_n \rangle$ is a tuple, then we let $g[\mathbf{x}] = \langle g(x_0), \ldots, g(x_n) \rangle$

Proof. Suppose $\alpha : F(A) \to F(B) \in \text{Mod}_=(L)$. For each $x \in F(B)$ choose an element $b_x \in x$ and let $\alpha^*(a) = b_x$ if and only if $\alpha([a]) = x$. Then clearly $F(\alpha^*) = \alpha$. □

We will often associate a model with its underlying class and will think of $\equiv_j$ and $\not\equiv_j$ as representing "equal" and "not equal". As such we will omit explicit mention of $\equiv_j$ and $\not\equiv_j$ as distinguished relations. We will also omit the superscript from relations as well as the subscript from $\equiv_j, \not\equiv_j$ when they are clear from context.

Definition 3.14. *If $\varphi(x, y)$ is a formula of set theory such that $Set \models \varphi(x, A) \to x \in obj(Mod(L))$ then let $Mod_L(\varphi(x, A))$ be the full subcategory of $Mod(L)$ containing those objects M where $Set \models \varphi(M, A)$.*

So $\text{Mod}_L(\varphi(x, A))$ is the category of models which satisfy $\varphi(x, A)$.

3.4. Relativized Models

In this paper we will be interested in models which, as we change models of set theory, have a smallest extension which satisfies a given formula.

Definition 3.15. *Let $\varphi(x, A)$ be as in Definition 3.14 and let M be in $obj(Mod(L))$. We define $Exp(\varphi, M)$ to be the category whose objects are those models N such that*

- *$M \subseteq N$ and $R^M \subseteq R^N$ for all relations.*
- *$Set \models \varphi(N, A)$.*

and whose morphism are those maps $f \in Mod(L)[N, P]$ such that $f(m) = m$ for all $m \in M$.

Definition 3.16. *Suppose, $\varphi(x, A)$, M, and V_0, V_1 are such that*

- *V_0, V_1 are standard models of ST with $V_0 \subseteq V_1$.*
- *$\varphi(x, A)$ is a formula for the language L with $A \in V_0$.*
- *$V_0 \models M \in obj(Mod_L(\varphi(x, A)))$.*

If N is an object of $(Exp(\varphi, M))^{V_1}$[5] such that

[5] The category $Exp(\varphi, M)$ can be described by a formula, $\psi(x)$, of set theory. By $(Exp(\varphi, M))^{V_1}$ we mean the category described by $\psi^{V_1}(x)$. I.e. the category of models in V_1 which satisfy φ and contain M.

(1) For every object Q of $(Exp(\varphi, M))^{V_1}$ there is a map $i : N \to Q$ in $(Exp(\varphi, M))^{V_1}$.

(2) Every endomorphism of N in $(Exp(\varphi, M))^{V_1}$ is an automorphism.

then we say that N is a relativization of M to V_1 for $\varphi(x, A)$.

If $M \in V_0$ and N is a relativization of M to V_1 for $\varphi(x, A)$ then (1) ensures that when Q is any model of $\varphi(x, A)$ in V_1 which contains M then Q must also contain a copy of N. So, N is a "minimal" extension of M to a model of $\varphi(x, A)$ in V_1.

Lemma 3.17. *If $M \in V_0$ and N_0, N_1 are relativizations of M to V_1 for $\varphi(x, A)$, then there is an isomorphism $i : N_0 \cong N_1$ which is constant on M.*

Proof. Because N_0 and N_1 are relativizations of M there are maps $i_0 : N_0 \to N_1$ and $i_1 : N_0 \to N_1$ in $(Exp(\varphi, M))^{V_1}$. But, by condition (2) of Definition 3.16 this means that $i_0 \circ i_1$ and $i_1 \circ i_0$ are automorphisms and hence i_0 and i_1 must be isomorphisms. □

This lemma shows that not only is a relativization of M a "minimal" extension, but any two such minimal extensions must be isomorphic (although there need not be a unique isomorphism). Thus a relativization of M for $\varphi(x, A)$ is the unique "smallest" extension of M satisfying $\varphi(x, A)$.

In particular we have, as a special case of relativization, the following lemma.

Lemma 3.18. *If N is an initial object of $(Exp(\varphi, M))^{V_1}$ then N is a relativization of M to V_1 for $\varphi(x, A)$*

4. Sites

Before we begin discussing relativizations of Grothendieck topoi it will be useful to discuss relativizations of sites. We start this discussion by introducing the notions of a weak site and of an almost subcanonical weak site. These are the absolute analogs of the notion of a site and of a subcanonical site respectively. Next we show that all (small) sites and (small) subcanonical sites relativize to all models of set theory (with respect to the appropriate theory). It is worth pointing out, that even if we start with a subcanonical site we may get different results we relativize it with respect to the theory of sites and when we relativize it with respect to the theory of subcanonical sites.

4.1. Weak Site

Definition 4.1. *A weak site is a pair (C, J_C) where C is a small category and J_C is a function which takes objects of C and returns a collection of sieves such that, for any $A \in obj(C)$:*

- *(Identity) $C[-, A] \in J_C(A)$.*
- *(Base Change) If $S \in J_C(A)$ and $f : B \to A$ then $f^*S \in J_C(B)$.*

We call $J_C(A)$ the covering sieves of A.

Definition 4.2. *If (C, J_C) and (D, J_D) are weak sites and $F : C \to D$ is a functor, then we say that F is a map of weak sites if $(\forall S \in J_C(A))\{F(f) \circ x : f \in S, x \in D\} \in J_D(F(A))$.*

Definition 4.3. *A site is a weak site (C, J_C) satisfying*

- *(Local Character) Let $S \in J_C(A)$ and let T be any sieve on A. If $(\forall B \in obj(C))(\forall f \in S(B))f^*T \in J_C(B)$ then $T \in J_C(A)$.*

We want to think of a weak site as an absolute analog of a site in much the same way as a basis for a topological space can be thought of as an absolute analog of a topological space (i.e. there is a unique minimal way to generate a topological space from a basis just as there is a unique minimal way to generate a site from a weak site (as we will see in Definition 4.2)).

One important example of how this closure plays a role is the following lemma.

Lemma 4.4. *If (C, J_C) is a site and $S, S' \in J_C(A)$ then $S \cap S' \in J_C(A)$*

Proof. First if $f \in S \cap S'$ then so is $f \circ x$ for any x. Hence $S \cap S'$ is a sieve if S and S' are. Also, for all $f \in S$ and all x we have $f \circ x \in S$. So if $f \in S$, $f^*S' \subseteq f^*S$ and hence $f^*(S \cap S') = f^*S' \in J_C(A)$. So by (Local Character) $S \cap S' \in J_C(A)$. □

Notice that this does not in general hold if (C, J_C) is just a weak site. Many of the concepts related to sheaves and separated presheaves generalize to the case of weak sites in the obvious way.

Definition 4.5. *Let (C, J_C) be a weak site and $F : C^{op} \to$ **SET** be a presheaf on C. If $A \in obj(C)$ and $S \in J_C(A)$, a compatible collection of elements for S is a collection $\langle (a_i, i) : i \in S \rangle$ such that*

- $(\forall i \in S(B))\ a_i \in F(B)$
- $(\forall i' \in C[B', B]) a_{i \circ i'} = F(i')(a_i)$

If there is an $a \in F(A)$ *such that* $a|_i = a_i$ *for all* $i \in S$ *then we say* $\langle (a_i, i) : i \in S\rangle$ *covers* a. *A compatible collectin of elements for* A *is a compatible collection of elements for some* $S \in J_C(A)$.

Definition 4.6. *Suppose* $X \subset C[-, A]$ *and* $Gen_{sieve}(X) \in J_C(A)$. *Then we say* $\langle (a_i, i) : i \in X\rangle$ *covers* a *if* $\langle (a_i|_f, i \circ f) : i \circ f \in Gen_{sieve}(X)\rangle$ *is a cover for* a.

Definition 4.7. *Suppose* (C, J_C) *is a weak site. A presheaf* $F : C^{op} \to$ ***SET*** *is separated for* (C, J_C) *if every compatible collection of elements of* F *covers at most one element of* F. *We let* $\boldsymbol{Sep}(C, J_C)$ *be the category whose objects are separated presheaves for* (C, J_C) *and whose morphisms are natural transformations.*

Definition 4.8. *Suppose* (C, J_C) *is a weak site. A presheaf* $F : C^{op} \to$ ***SET*** *is a sheaf for* (C, J_C) *if every compatible collection of elements of* F *covers exactly one element* F. *We let* $\boldsymbol{Sheaf}(C, J_C)$ *be the category whose objects are sheaves for* (C, J_C) *and whose morphisms are natural transformations.*

The following are important facts concerning separated presheaves and sheaves for weak sites. Their proofs, however, are routine (and are left to the enthusiastic reader).

Lemma 4.9. *Suppose* (C, J_C) *is a site,* F *is a separated presheaf for* (C, J_C), *and* $S, S' \in J_C(A)$ *with* $S \subseteq S'$. *If* $\langle (a_i, i) : i \in S'\rangle, \langle (b_i, i) : i \in S'\rangle$ *are compatible collections of elements of* F *such that* $(\forall j \in S) a_j = b_j$ *then* $(\forall j \in S') a_j = b_j$.

Definition 4.10. *Suppose* (C, J_C) *is a weak site and* F *is a presheaf for* (C, J_C). *We say* $X \subseteq \bigcup_{A \in obj(C)} F(A)$ *covers* F *if for all* $A \in obj(C)$ *and for all* $a \in F(A)$ *there exists* $\langle (a_i, i) : i \in S\rangle$ *such that*

- $\langle (a_i, i) : i \in S\rangle$ *covers* a
- $(\forall i \in S(B))(\exists \alpha :\in C[B, B'])(\exists b \in X \cap F(B')) b|_\alpha = a_i$

We say that a subpresheaf $G \rightarrowtail F$ *covers* F *if* $\bigcup_{A \in obj(C)} G(A)$ *covers* F.

A set X covers F if every element of F can be covered by restrictions of elements of X. So in particular F can be recovered from X (and the site).

Lemma 4.11. *Suppose F is a separated presheaf for (C, J_C) and X covers F. Then there is a smallest subpresheaf $F_X \subseteq F$ such that $X \subseteq \bigcup_{A \in obj(C)} F(A)$ and F_X covers F.*

Lemma 4.12. *If (C, J_C) is a site and $F \subseteq G \subseteq H$ are separated presheaves for (C, J_C) such that F covers G and G covers H then F covers H.*

Notice this does not necessarily hold if (C, J_C) is only a weak site.

Lemma 4.13. *Suppose (C, J_C) is a site, F is a separated presheaf for (C, J_C), and F_0, F_1 are covers of F. Then $G(A) = F_0(A) \cap F_1(A)$ is also a covering presheaf for F.*

Proof. By (Local Character) and Lemma 4.12 it suffices to prove that every element of $F_0(A)$ is covered by a collection of elements in $F_0 \cap F_1$. Every element a of F_0 is covered by a collection of elements, $\langle (a_i, i) : i \in S \rangle$, from F_1 (as $F_0 \subseteq F$). But any cover of a must consist of restrictions of a. Hence any cover must consist of elements of F_0. In particular we have the cover $\langle (a_i, i) : i \in S \rangle$ consists of elements of $F_0 \cap F_1$. □

Now we introduce the important notion of being almost subcanonical. This is the absolute analog of being subcanonical.

Definition 4.14. *We say a site (C, J_C) is subcanonical if $y_C(A)$ is a sheaf for all $A \in obj(C)$. We say (C, J_C) is almost subcanonical if $y_C(A)$ is a separated presheaf for all $A \in obj(C)$.*

Proposition 4.15. *Suppose V_0, V_1 are standard models of set theory with $V_0 \subseteq V_1$. Further suppose that $(C, J_C) \in V_0$ and $V_0 \models$ "(C, J_C) is an almost subcanonical weak site". Then $V_1 \models$ "(C, J_C) is an almost subcanonical weak site".*

Proof. Suppose (in V_1) $A \in \text{obj}(C)$, $f, g \in y_C(A)(B)$ and $\langle (x_i, i) : i \in S \rangle$ with $S \in J_C(B)$ are such that both f and g are covered by $\langle (x_i, i) : i \in S \rangle$. Then the same holds in V_0 and hence $f = g$. So (in V_1) (C, J_C) is almost subcanonical. □

4.2. Relativized Sites

Definition 4.16. *Let (C, J_C) be a weak site. Define J_C^α on $A \in obj(C)$ as follows*

- $J_C^0(A) = J_C(A)$.
- $J_C^{\alpha+1}(A) = \{T$ *a sieve on* $A : (\exists S \in J_C(A))(\forall f \in S(B)) f^*T \in J_C^\alpha(B)\}$.
- $J_C^{\omega\cdot\gamma}(A) = \bigcup_{\beta<\omega\cdot\gamma} J_C^\beta(A)$.

We define $J_C^{ORD} = \bigcup_{\alpha\in ORD} J_C^\alpha$ and if $T \in J_C^{ORD}(A)$ we say the degree of T is the least ordinal α such that $T \in J_C^\alpha(A)$.

We can think of the structure (C, J_C^{ORD}) as the site we get when we close the weak site (C, J_C) under local character.

Proposition 4.17. (C, J_C^{ORD}) *is a Site.*

Proof. (Identity):
This is immediate because $J_C \subseteq J_C^{\mathrm{ORD}}$.

(Local Character):
Let $S \in J_C^{\mathrm{ORD}}(A)$ and let T be any sieve on A such that $f^*T \in J_C^{\mathrm{ORD}}(B)$ for all $f \in S$. We want to show $T \in J_C^{\mathrm{ORD}}(A)$. If degree$(S) = 0$ then $T \in J_C^\alpha(A)$ where $\alpha = \sup\{degree(f^*T) : f \in S\}$.

Assume that the conclusion in (Local Character) holds if degree$(S) \leq \alpha$ and let degree$(S) = \alpha + 1$. Then there is an $R \in J_C(A)$ such that $(\forall f \in R(B)) f^*S \in J_C^\alpha(B)$. Now if $f \in R$ then f^*T is a sieve on dom(f) and $f^*S \in J_C^\alpha(\mathrm{dom}(f))$. But $(\forall g \in f^*S) f \circ g \in S$ and $g^*f^*T = (f \circ g)^*T$. So $g^*f^*T \in J_C^{\mathrm{ORD}}(\mathrm{dom}(g))$. Hence by the inductive assumption, $f^*T \in J_C^{\mathrm{ORD}}(\mathrm{dom}(f))$ for all $f \in R$, and by the definition of J_C^{ORD}, $T \in J_C^{\mathrm{ORD}}(A)$. So (C, J_C^{ORD}) satisfies (Local Character).

(Change of Base):
Let $g : D \to A$ and $T \in J_C^{\mathrm{ORD}}(A)$. If degree$(T) = 0$ then $g^*T \in J_C^{\mathrm{ORD}}(D)$ because $J_C \subseteq J_C^{\mathrm{ORD}}$.

Assume the conclusion in (Change of Base) holds if degree$(T) \leq \alpha$ and let degree$(T) = \alpha + 1$. Then there is a cover $S \in J_C(A)$ such that $(\forall f \in S(B)) f^*T \in J_C^\alpha(B)$. Now $g^*S \in J_C(B)$. So $(\forall h \in g^*S(D)) g \circ h \in S(D)$ and $(g \circ h)^*T = h^*(g^*T) \in J_C^\alpha(D)$. Hence $g^*T \in J_{\alpha+1}(B)$. So (C, J_C^{ORD}) satisfies (Change of Base). □

As we will see (C, J_C^{ORD}) is not only a site, but the smallest site containing (C, J_C).

Corollary 4.18. *If (C, J_C) is a weak site then so is (C, J_C^{α}) for all $\alpha \leq ORD$.*

Definition 4.19. *Let $L_{WS} = L_{Cat} \cup \{Cov(x, y)\}$ where $Cov(x, y)$ is a binary relation, and $Th_{WS}(X)$ be the formula which says*

- *If $C = X|_{L_{Cat}}$ then $Th_{Cat}(C)$.*
- *$Cov(S, A) \to S \in \mathcal{P}(X)$ and $A \in X$.*
- *If we let $S \in J_C(A)$ be a short hand for $Cov(S, A)$ then (C, J_C) is a weak site.*

We will consider all weak sites (C, J_C) as models of the language L_{WS} using the interpretation of $Cov(S, A) \leftrightarrow S \in J_C(A)$.

Definition 4.20. *Let $L_{Site} = L_{WS}$ and let $Th_{Site}(X)$ be the formula which says*

- *$Th_{WS}(X)$.*
- *If $C = X|_{L_{Cat}}$ and $S \in J(A)$ is a short hand for $Cov(S, A)$ then (C, J_C) is a site.*

So Th_{Site} is the higher order theory of sites. Next we want to show that every site relativizes for Th_{Site}.

Proposition 4.21. *If (C, J_C) is a weak site then (C, J_C^{ORD}) is an initial object in $Exp(Th_{Site}, (C, J_C))$.*

Proof. By Proposition 4.17 $(C, J_C^{\mathrm{ORD}}) \models Th_{Site}$. However, if (D, J_D) is an object in $Exp(Th_{Site}, (C, J_C))$ then C is a subcategory of D and $J_C \subseteq J_D$. So the (unique) map from (C, J_C^{ORD}) into (D, J_D) mapping C to itself is a map of weak sites. Hence (C, J_C^{ORD}) is an initial object in $Exp(Th_{Site}, (C, J_C))$. □

Corollary 4.22. *If $V_0 \subseteq V_1$ are standard models of set theory with $V_0 \models$ "$(C, J_C) \models Th_{Site}$" then $(C, J_C^{ORD})^{V_1}$ is the relativization of (C, J_C) to V_1 for Th_{Site}*

Not surprisingly the connection between (C, J_C) and (C, J_C^{ORD}) extends to their separated presheaves and sheaves.

Proposition 4.23. *Suppose F is a presheaf on C. Then*

(a) F is separated for (C, J_C) if and only if F is separated for (C, J_C^{ORD}).

(b) F is a sheaf for (C, J_C) if and only if F is a sheaf for (C, J_C^{ORD}).

Proof. Part (a):
First it is clear that if F is separated for (C, J_C^{ORD}) then F is separated for (C, J_C) as $J_C \subseteq J_C^{\mathrm{ORD}}$.

For the other direction let's assume, to get a contradiction, that F is separated for (C, J_C) but not separated for (C, J_C^{ORD}). Then there is a $\langle (a_i, i) : i \in S\rangle$ with $S \in J_C^{\mathrm{ORD}}(A)$ and there are $\overline{a}_0, \overline{a}_1 \in F(A)$ such that $\overline{a}_0$ and $\overline{a}_1$ are covered by $\langle (a_i, i) : i \in S\rangle$. Assume that S has minimal degree of $\alpha+1$ such that the above is true (it can't have degree 0 as F is separated for $(C, J_C) = (C, J_C^0)$). Then there is a $T \in J_C(A)$ such that for all $f \in T(B)$, $f^*S \in J_C^{\alpha}(B)$. So in particular $\overline{a}_0|_f = \overline{a}_1|_f$ for all $f \in T$ as $\overline{a}_0|_f, \overline{a}_1|_f$ are both covered by $\langle (a_{f\circ i}, i) : i \in f^*S\rangle$ and f^*S has degree less than $\alpha + 1$. But we also have $\overline{a}_0, \overline{a}_1$ are covered by $\langle (a_0|_f, f) : f \in T\rangle$ (because it is a compatible collection) and hence $\overline{a}_0 = \overline{a}_1$ as T has degree 0.

Part (b):
First it is clear that if F is a sheaf for (C, J_C^{ORD}) then F is also a sheaf for (C, J_C), as $J_C \subseteq J_C^{\mathrm{ORD}}$.

For the other direction assume, to get a contradiction, that F is a sheaf for (C, J_C) but not a sheaf for (C, J_C^{ORD}). By Part (a) we know that F is separated for (C, J_C^{ORD}). So there must be a collection $\langle (a_i, i) : i \in S\rangle$ with $S \in J_C^{\mathrm{ORD}}(A)$ which is compatible but which doesn't cover any element of F. Let S have minimal degree of $\alpha + 1$ such that the above is true (it can't have degree 0 as F is a sheaf for $(C, J_C) = (C, J_C^0)$). Then there is a $T \in J_C(A)$ such that for all $f \in T(B)$, $f^*S \in J_C^{\alpha}(B)$. So in particular $\langle (a_{f\circ g}|g, g) : g \in f^*S\rangle$, with $S \in J_C^{\mathrm{ORD}}(A)$, is a compatible collection of elements for all $f \in T$ and hence must cover an element $a_f \in F(\mathrm{dom}(f))$ by the inductive hypothesis. But the collection $\langle (a_f, f) : f \in T\rangle$ is also compatible and hence must cover an element $a \in A$ also by the inductive hypothesis. And, as this a must also be covered by $\langle (a_i, i) : i \in S\rangle$ by construction, we have our contradiction. □

Corollary 4.24. *If (C, J_C) is an almost subcanonical weak site then so is (C, J_C^{ORD}).*

Corollary 4.25. *If (C, J_C) is a weak site then $\boldsymbol{Sep}(C, J_C) = \boldsymbol{Sep}(C, J_C^{\boldsymbol{ORD}})$ and $\boldsymbol{Sheaf}(C, J_C) = \boldsymbol{Sheaf}(C, J_C^{\boldsymbol{ORD}})$.*

Corollary 4.26. *There is a functor $\boldsymbol{a} : \boldsymbol{Sep}(C, J_C) \to \boldsymbol{Sheaf}(C, J_C)$ such that if $\boldsymbol{i} : \boldsymbol{Sheaf}(C, J_C) \to \boldsymbol{Sep}(C, J_C)$ is the inclusion functor then $\boldsymbol{i}$ is right adjoint to $\boldsymbol{a}$ and the unit $\iota : 1_{\boldsymbol{Sep}(C, J_C)} \Rightarrow \boldsymbol{a} \circ \boldsymbol{i}$ is such that for all $F \in obj(\boldsymbol{Sep}(C, J_C))$ and all $A \in obj(C)$, $(\iota_F)_A$ is the identity on its domain. We call $\boldsymbol{a}$ the sheafification functor.*

Obviously it is possible to define a sheafification functor on any presheaves and not just those which are separated. However the point of Corollary 4.26 is that we are defining sheafification in such a way that there is an inclusion map from the presheaf into the sheaf which is the identity on its domain. In order for this to happen though our presheaf must be separated to start with.

4.3. Subcanonical Sites

We saw in the previous section that every site relativizes. However often we will want to consider sites which are subcanoinical. As such in this section we will show that every subcanonical site relativizes as a subcanonical site (which may result in a different site than its relativization merely as a site).

Lemma 4.27. *Suppose (C, J_C) is an almost subcanonical weak site. Then $(\forall S \in J_C^{ORD}(A))$ S covers $y_C(A)$.*

Proof. Suppose $x \in y_C(A)(B)$ and $S \in J_C^{\mathrm{ORD}}(A)$. Then $x \in C[B, A]$ and x^*S is a cover of B. Hence $\langle (x \circ i, i) : i \in x^*S \rangle \subseteq S$ is a compatible collection of elements which cover x. So S is a covering subsheaf for $y_C(A)$ (as x was arbitrary). $\square$

Theorem 4.28. *Let $Th_{SubCan}(X) = Th_{Site}(X) \cup \{X$ is a subcanonical site$\}$. Then for any almost subcanonical site (C, J_C),[6] $Exp(Th_{SubCan}, (C, J_C))$ has an initial object.*

Proof. First we need to show that $(\exists X) X \in \mathrm{obj}(Exp(Th_{SubCan}, (C, J_C)))$. Define $X' = (\overline{C}, J_{\overline{C}})$ such that

[6] If we don't require our site to be almost subcanonical then there will non-equal $\equiv$-equivalence classes which are covered by the same cover. So, in any subcanonical extension those $\equiv$-equivalence classes would have to be identified. But, because both $\equiv$ and $\not\equiv$ are relations in our models we can't identify distinct $\equiv$-equivalence classes in extensions.

- $\mathrm{obj}(C) = \mathrm{obj}(\overline{C})$
- $\overline{C}[A, B] = \{\langle (b_i, i) : i \in S\rangle$ such that $S \in J_C(A)$ and $(\forall i \in S) b_i : \mathrm{dom}(i) \to B, b_{i \circ j} = b_i \circ j\}/\equiv$
- $\langle (b_i, i) : i \in S\rangle \equiv \langle (b'_i, i) : i \in S'\rangle$ if and only if $b_i = b'_i$ for all $i \in S \cap S'$.

Notice that in order to completely determine a map in $\overline{C}[A, B]$ we need to determine where it sends (in $y_{\overline{C}}(B)$) each element of a cover $S \subseteq y_{\overline{C}}(A)$. So a map in $\overline{C}[A, B]$ is an equivalence class of matching families from $y_{\overline{C}}(B)$ for a sieve $S \in J_C(A)$.

That $\equiv$ is an equivalence relation follows immediately from Lemma 4.4 and Lemma 4.9.

If $\langle (b_i, i) : i \in S\rangle \in \overline{C}[A, B]$ and $\langle (d_i, i) : i \in S'\rangle \in \overline{C}[B, D]$ we define $\langle (d_j, j) : j \in S'\rangle \circ \langle (b_i, i) : i \in S\rangle = \langle (d_{b_i}, i) : i \in S''\rangle$ where $S'' = \{i : b_i \in S'\}$. In order to show that this definition makes sense, we first need to show that $\langle (d_{b_i}, i) : i \in S''\rangle \in \overline{C}[A, D]$. Or, more specifically, that $S'' \in J_C(A)$. But we know that for all $i \in S$, $i^* S'' = \{\alpha : i \circ \alpha \in S''\} = \{\alpha : b_{i\circ\alpha} \in S'\} = \{\alpha : b_i \circ \alpha \in S'\} = b_i^* S' \in J_C(\mathrm{dom}(b_i))$. Hence $S'' \in J_C(A)$ by (Local Character).

To show that composition is well defined we need to show that composition is closed under $\equiv$. It suffices to show that if $S_0 \subseteq S$, $S'_0 \subseteq S'$ then $\langle (d_{b_i}, i) : i \in S''_0\rangle = \langle (d_j, j) : j \in S'_0\rangle \circ \langle (b_i, i) : i \in S_0\rangle \equiv \langle (d_j, j) : j \in S'\rangle \circ \langle (b_i, i) : i \in S\rangle = \langle (d_{b_i}, i) : i \in S''\rangle$. But it is clear that $S''_0 \subseteq S''$ by construction and hence $S''_0 \cap S'' = S''_0$. So $\langle (d_{b_i}, i) : i \in S''_0\rangle \equiv \langle (d_{b_i}, i) : i \in S''\rangle$.

We have shown that $\overline{C}$ is a well defined category. In general though we don't have $C \subseteq \overline{C}$. But what we do have is a injective function $f : C \to \overline{C}$ where f is the identity on objects and $f(\alpha) = \langle (\alpha \circ i, i) : i \in C[-, \mathrm{dom}(\alpha)]\rangle$. Further $f(\alpha \circ \beta) = \langle (\alpha \circ \beta \circ i, i) : i \in C[-, \mathrm{dom}(\alpha \circ \beta)]\rangle = f(\alpha) \circ f(\beta)$. Hence f is a functor. Further if $f(\alpha) \equiv f(\beta)$ then there is a covering sieve S on $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ such that $(\forall i \in S) \alpha \circ i = \beta \circ i$ and hence $f(\alpha)$ covers α and β. But then as (C, J_C) is almost subcanonical $\alpha = \beta$. So f is injective (up to $\equiv$).

If we let $X = (X' - image(f)) \cup C$ then we have an isomorphism between X and X' which is the identity on $(X' - image(f))$ and f on C. We define composition on X so as to make this an isomorphism of categories.

All that remains is to define the collection of covering sieves. To simplify notation we will work with X' and define the covering sieves on X to be those which are the images of covering sieves on X' under the isomorphism.

Definition 4.29. *For a map $\boldsymbol{f} = \langle (f_i, i) : i \in T\rangle$ we say $f \in \boldsymbol{f}$ if there is an $i \in T$ such that $f = f_i$. For a sieve S we say $S \in J_{\overline{C}}(A)$ if and only if $\widetilde{S} = \{b : (\exists \boldsymbol{b} \in S) b \in \boldsymbol{b}\} \in J_C(A)$.*

Given a sieve S on C we will often want to consider the sieve it generates in $\overline{C}$. As such we will let $S_{\overline{C}} = \{f \circ \mathbf{x} : f \in S, \mathbf{x} \in \overline{C}\}$.

Claim 4.30. *If $\langle (g_j, j) : j \in S\rangle \in \overline{C}[A, B]$ and $i \in S(D)$ then $\langle (g_j, j) : j \in S\rangle \circ i \equiv g_i$.*

Proof. Let $\mathbf{i} = f(i) = \langle (i \circ \alpha, i), \alpha \in C[-, D]\rangle$. We know that $\langle (g_j, j) : j \in S\rangle \circ i \equiv \langle (g_i, i) : i \in S\rangle \circ \mathbf{i} = \langle (g_{i\circ\alpha}, \alpha) : i \circ \alpha \in S\rangle = \langle (g_i \circ \alpha, \alpha) : \alpha \in i^* S\rangle \equiv g_i$ □

Corollary 4.31. *If S is a sieve on $(\overline{C}, J_{\overline{C}})$ then $\widetilde{S} = S \cap morph(C)$.*

Claim 4.32. *If $\langle (g_j, j) : j \in S\rangle \in \overline{C}[A, B]$ and $i \in C[B, D]$ then $i \circ \langle (g_j, j) : j \in S\rangle \equiv \langle (i \circ g_j, j) : j \in S\rangle$.*

Proof. Let $\mathbf{i} = f(i) = \langle (i \circ \alpha, i), \alpha \in C[-, D]\rangle$. We know that $i \circ \langle (g_j, j) : j \in S\rangle \equiv \mathbf{i} \circ \langle (g_j, j) : j \in S\rangle = \langle (i \circ g_j, j) : g_j \in C[-, D]\rangle = \langle (i \circ g_j, j) : j \in S\rangle$ □

Claim 4.33. *If $S \in J_{\overline{C}}(A)$ and $S \subseteq S'$ then $S' \in J_{\overline{C}} A$.*

Proof. We have $\widetilde{S} \subseteq \widetilde{S}'$ and $\widetilde{S} \in J_C(A)$ and so $\widetilde{S}' \in J_C A$ (because (C, J_C) is a site). □

Claim 4.34. *$(\overline{C}, J_{\overline{C}})$ is a site.*

Proof. It is clear that $J_{\overline{C}}$ satisfies (Identity).

Suppose $S \in J_{\overline{C}}(B)$ and T is a sieve on B such that $\mathbf{f}^* T \in J_{\overline{C}}(\text{dom}(\mathbf{f}))$ for all $\mathbf{f} \in S$ and hence $\widetilde{\mathbf{f}^* T} \in J_C(\text{dom}(\mathbf{f}))$. If $f \in \widetilde{S}$ then $f^* \widetilde{T} = \{g : f \circ g \in \widetilde{T}\}$. But we also have $f^* T = \{\langle g_i : i \in W\rangle : f \circ \langle g_i : i \in W\rangle \in T\} = \{\langle g_i : i \in W\rangle : \langle f \circ g_i : i \in W\rangle \in T\}$. So if $g \in \widetilde{f^* T}$ then $f \circ g \in \widetilde{T}$ and $g \in f^* \widetilde{T}$. Hence $\widetilde{f^* T} \subseteq f^* \widetilde{T}$ and so $f^* \widetilde{T} \in J_C(A)$. But then, because (C, J_C) satisfies (Local Character) we have $\widetilde{T} \in J_C(B)$ and hence $T \in J_{\overline{C}}(B)$. So $(\overline{C}, J_{\overline{C}})$ satisfies (Local Character).

Suppose $S \in J_{\overline{C}}(B)$ and let $\mathbf{f} = \langle f_i : i \in T\rangle \in \overline{C}[A, B]$. Then for all $i \in T$,

$$\begin{aligned} i^*(\widetilde{\mathbf{f}^* S}) &= \{g : i \circ g \in \widetilde{\mathbf{f}^* S}\} \\ &= \{g \in \mathrm{morph}(C) : i \circ g \in \mathbf{f}^* S\} \\ &= \{g \in \mathrm{morph}(C) : \mathbf{f} \circ i \circ g \in S\} \\ &= \{g \in \mathrm{morph}(C) : f_i \circ g \in S\} \\ &= f_i^* \widetilde{S} \end{aligned}$$

But $f_i^* \widetilde{S} \in J_C(\mathrm{dom}(f_i))$ because $S \in J_{\overline{C}}(B)$ and hence $\widetilde{S} \in J_C(B)$. So $i^*(\widetilde{\mathbf{f}^* S}) \in J_C(\mathrm{dom}(i))$ for all $i \in T$. Hence, by (Local Character) we have $\widetilde{\mathbf{f}^* S} \in J_C(\mathrm{dom}(\mathbf{f}))$ and therefore $\mathbf{f}^* S \in J_{\overline{C}}(B)$. So $(\overline{C}, J_{\overline{C}})$ satisfies (Base Change) and hence is a site. □

Claim 4.35. $(\overline{C}, J_{\overline{C}})$ *is a subcanonical site.*

Proof. First notice that if $\boldsymbol{\beta} = \langle (\beta_i, i) : i \in S'\rangle, \boldsymbol{\gamma} = \langle (\gamma_i, i) : i \in S''\rangle \in C[B, A]$ are such that $\boldsymbol{\beta}|_j = \boldsymbol{\gamma}_j$ for all $j \in S$ (with $S \in J_{\overline{C}}$) then the same holds for all $j \in \widetilde{S}$. So in particular for all $j \in \widetilde{S} \cap S' \cap S''$ $\beta_j = \gamma_j$. So, because C is almost subcanonical, we have that $\beta_j = \gamma_j$ for all $j \in S' \cap S''$ as β_j and γ_j are both covered by $\langle (\beta_j \circ i, i) : i \in j^* \widetilde{S}\rangle$. Hence $\boldsymbol{\beta} \equiv \boldsymbol{\gamma}$. So $y_{\overline{C}}(A)$ is a separated presheaf (and $(\overline{C}, J_{\overline{C}})$ is almost subcanonical).

Suppose $S \in J_{\overline{C}}(B)$ and $\langle (\alpha_i, i) : i \in S\rangle$ is a compatible collection of elements in $y_{\overline{C}}(A)$. Then $\langle (\alpha_i, i), i \in \widetilde{S}\rangle$ is a compatible collection and $\overline{S} \in J_C(B)$

If $i \in \widetilde{S}(D_i)$ then $\alpha_i \in y_{\overline{C}}(A)(D_i)$ and hence $\alpha_i : D_i \to A$. Now if $j \in C[D_k, D_i]$ and $k = i \circ j$ then $\alpha_k \equiv \alpha_i|_j = \alpha_i \circ j$. But if $\alpha_i = \langle f_{i,n} = n \in T_i\rangle$ then $\alpha_i \circ j = \langle f_{i,j \circ n} : n \in j^* T_i\rangle$. So $(\forall n \in j^* T_i \cap T_j) f_{k,n} = f_{i,j \circ n}$. Further, if $\overline{T} = \langle k \circ n : n \in T_k\rangle$ then $\overline{T}$ is a sieve and for all $i \in S$ $T_i \subseteq i^* \overline{T}$, and so $i^* \overline{T} \in J_C(D_i)$ and hence, by (Local Character), $\overline{T} \in J_C(B)$.

So, if $\alpha = \langle (f_{k,n}, k \circ n\rangle : k \circ n \in \overline{T}\rangle$ then $\alpha \in y_{\overline{C}}(A)(B)$ and $\alpha|_i = \alpha_i$. Hence $y_{\overline{C}}(A)$ is a sheaf and $(\overline{C}, J_{\overline{C}})$ is subcanonical. □

Claim 4.36. $(\overline{C}, J_{\overline{C}})$ *is an initial object of* $Exp(Th_{SubCan}, (C, J_C))$.

Proof. If $S \in J_{\overline{C}}(A)$ and $\langle (f_i, i) : i \in S'\rangle \in S(B)$ then for each $i \in S'$ $\langle (f_i, i) : i \in S'\rangle \circ i = f_i$ and hence $\widetilde{S}_{\overline{C}} \subseteq S$. So in particular site on $\overline{C}$ containing J_C must contain $J_{\overline{C}}$ because if $S \in J_{\overline{C}}(A)$ and $S \subseteq S'$ then $S' \in J_{\overline{C}}(A)$. □

□

Corollary 4.37. *If $V_0 \subseteq V_1$ are standard models of set theory and $V_0 \models$ "$(C, J_C) \models Th_{SubCan}$" then $(\overline{C}, J_{\overline{C}})$ is the relativization of (C, J_C) to V_1 for Th_{SubCan}*

Corollary 4.38. *For any almost subcanonical weak site (C, J_C) the category $Exp(Th_{SubCan}, (C, J_C))$ has an initial object.*

Proof. This is because if (C, J_C) is an almost subcanonical weak site then (C, J_C^{ORD}) is an almost subcanonical site such that for any site (D, J_D) containing (C, J_C) there is a unique map (C, J_C^{ORD}) into (D, J_D). □

Notice that this proof only works if our site is almost subcanonical to start with. Otherwise in the sheafification process we have to make different maps become the same and hence we loose preservation of "not equals". And, if we don't require preservation of not equals, there are other ways we could turn (C, J_C) into a subcanonical site (e.g. we could collapse all morphisms and turn the resulting partial order into a subcanonical site. This might be minimal (depending on J_C) because even though we have added new elements to the $\equiv$ relation we might be able to get away with adding fewer new covers than we otherwise could).

Definition 4.39. *Suppose (C, J_C) and (D, J_D) are weak sites. We say that $F : C \to D$ is an equivalence of sites for of (C, J_C) and (D, J_D) if*

- *F is an equivalence of categories.*
- *For all $A \in obj(C)$ and for all sieves S on A, $S \in J_C(A) \leftrightarrow \overline{F[S]} \in J_D(F(A))$ where $\overline{F[S]} = \{F(f) \circ g : f \in S, g \in morph(D)\}$ is the sieve generated by $F[S] = \{F(f) : f \in S\}$.*

Lemma 4.40. *If $E : C \to D$ is an equivalence of sites for (C, J_C) and (D, J_D) and $E' : D \to C$ is a functor such that $E \circ E' \simeq 1_D$ and $E' \circ E \simeq 1_C$ then E' is an equivalence of sites for (D, J_D) and (C, J_C).*

Proof. Immediate. □

Proposition 4.41. *Suppose (C, J_C) and (D, J_D) are almost subcanonical weak sites with $(\overline{C}, J_{\overline{C}})$ and $(\overline{D}, J_{\overline{D}})$ the corresponding initial objects of the categories $Exp(Th_{SubCan}, (C, J_C))$ and $Exp(Th_{SubCan}, (D, J_D))$ respectively. If $E : C \to D$ is an equivalence of sites then E extends to an equivalence of sites $\overline{E} : \overline{C} \to \overline{D}$. Where $\overline{E}(\langle(\alpha_i, i) : i \in S\rangle) = \langle(E(\alpha_i) \circ g, E(i) \circ g) : i \in S, g \in D\rangle$.*

Proof. This follows immediately from the definition of $(\overline{C}, J_{\overline{C}})$ and $(\overline{D}, J_{\overline{D}})$. $\square$

Definition 4.42. *Suppose (C, J_C) is a almost subcanonical weak site. Let $\overline{(C, J_C)}$ be the full subcategory of **Sheaf**(C, J_C) whose objects are $\{\mathbf{a}(y_C(A)) : A \in obj(C)\}$.*

Lemma 4.43. *Suppose (C, J_C) is an almost subcanonical site with $(\overline{C}, J_{\overline{C}})$ the corresponding initial element of $Exp(Th_{SubCan}, (C, J_C))$. Then $\overline{(C, J_C)}$ is isomorphic to $(\overline{C}, J_{\overline{C}})$. Further, under this isomorphism epimorphic families are exactly those families which come from covers in $J_{\overline{C}}$.*

Proof. We now define a functor $E : \overline{(C, J_C)} \to (\overline{C}, J_{\overline{C}})$ as follows. For $\mathbf{a}(y_C(A)) \in \text{obj}(\overline{(C, J_C)})$ let $E(\mathbf{a}(y_C(A))) = A$. If $f : \mathbf{a}(y_C(A)) \to \mathbf{a}(y_C(B))$ then, because $y_C(A)$ and $y_C(B)$ are separated

$$(\exists S \in J_C(A))(\forall i \in S(D)) f(i) \in y_C(B)(D).$$

Let $E(f) = [\langle (f(i), i) : i \in S \rangle]$.

Notice that if $S \in J_C(A)$ and $\overline{b} = [\langle (b_i, i) : i \in S \rangle] \in \overline{C}[A, B]$ then, because S covers $\mathbf{a}(y_C(A))$, there is a unique $\hat{b} : \mathbf{a}(y_C(A)) \to \mathbf{a}(y_C(B))$ such that $\hat{b} = b_i$ for all $i \in S$. So, by construction, we have $E(\hat{b}) = \overline{b}$ and E is surjective.

Next suppose $f, g : \mathbf{a}(y_C(A)) \to \mathbf{a}(y_C(B))$ and $E(f) = E(g)$. If $E(f) = [\langle (f(i), i) : i \in S \rangle]$ and $E(g) = [\langle (g(i), i) : i \in S' \rangle]$ then we have $(\forall i \in S \cap S') f(i) = g(i)$ and so $E(f) = E(g)$. Hence E is injective, and therefore an isomorphism of categories.

Further it is clear that $S \in J_{\overline{C}}(A)$ if and only if $\widetilde{S} \in J_C(A)$ (where $\widetilde{S}$ is as in Definition 4.29). Now by Lemma 4.27 any $\widetilde{S} \in J_C(A)$ covers $y_C(A)$ and hence covers $\mathbf{a}(y_C(A))$ (and therefore is an epimorphic family). Similarly, given any epimorphic family S' covering $\mathbf{a}(y_C(A))$, $S' \cap y_C(A) \in J_C(A)$. So we have $S \in J_{\overline{C}}(A)$ if and only if $\widetilde{S} \in J_C(A)$ if and only if $E^{-1}[S]$ is an epimorphic family in $\overline{(C, J_C)}$. $\square$

Proposition 4.44. *Suppose (C, J_C) is an almost subcanonical weak sites with $(\overline{C}, J_{\overline{C}})$ the initial element of $Exp(Th_{SubCan}, (C, J_C))$. Then **Sheaf**$(C, J_C)$ is isomorphic to **Sheaf**$(\overline{C}, J_{\overline{C}})$.*

Proof. If $X \in \text{obj}(\mathbf{Sheaf}(\overline{C}, J_{\overline{C}}))$ let $E(X)(A) = X(A)$ for all $A \in \text{obj}(\overline{C}) = \text{obj}(C)$ and if $f : B \to A$ let $E(X)(f) : E(X)(A) \to E(X)(B)$ be the function

$f : X(A) \to X(B)$ for all $f \in C[B, A]$. Further if $\alpha \in \mathbf{Sheaf}(\overline{C}, J_{\overline{C}})[X, Y]$ let $E(\alpha)_A = \alpha_A$ for all $A \in \mathrm{obj}(C)$. We then have $E : \mathbf{Sheaf}(\overline{C}, J_{\overline{C}}) \to \mathbf{Sheaf}(C, J_C)$ is a functor.

If $X \in \mathrm{obj}(\mathbf{Sheaf}(C, J_C))$ let $F(X)(A) = X(A)$ for all $A \in \mathrm{obj}(C) = \mathrm{obj}(\overline{C})$ and if $f : B \to A$ let $F(X)(\langle (f_i, i) : i \in T \rangle) : F(X)(A) \to F(X)(B)$ be the function such that for all $b \in F(X)(B)$, $F(X)(\langle (f_i, i) : i \in T \rangle)(a)$ is the unique element covered by $\langle X(f_i)(a) : i \in T \rangle$. Further if $\alpha \in \mathbf{Sheaf}(C, J_C)[X, Y]$ let $F(\alpha)_A = \alpha_A$ for all $A \in \mathrm{obj}(\overline{C})$. We then have $F : \mathbf{Sheaf}(C, J_C) \to \mathbf{Sheaf}(\overline{C}, J_{\overline{C}})$ is a functor.

Further it is immediate that $E \circ F = id_{\mathbf{Sheaf}(C, J_C)}$ and $F \circ E = id_{\mathbf{Sheaf}(\overline{C}, J_{\overline{C}})}$. □

Proposition 4.45. *Suppose (C, J_C) and (D, J_D) are equivalent weak sites. Then $\mathbf{Sheaf}(C, J_C)$ and $\mathbf{Sheaf}(D, J_D)$ are equivalent categories.*

Proof. Immediate. □

Corollary 4.46. *If (C, J_C) is an almost subcanonical weak site let*

- *(C_0, J_{C_0}) = initial element in $(Exp(Th_{SubCan}, (C, J_C)))^{V_0}$.*
- *$(C_{\overline{0}}, J_{C_{\overline{0}}})$ = initial element in $(Exp(Th_{SubCan}, (C_0, J_{C_0})))^{V_1}$.*
- *(C_1, J_{C_1}) = initial element in $(Exp(Th_{SubCan}, (C, J_C)))^{V_1}$*

then $V_1 \models \mathbf{Sheaf}(C_0, J_{C_0}) \simeq \mathbf{Sheaf}(C_{\overline{0}}, J_{C_{\overline{0}}}) \simeq \mathbf{Sheaf}(C_1, J_{C_1})$.

5. Sheaves on a Site

5.1. $Sh(x, \langle C, J_C \rangle)$

In this section we want to explicitly construct a category equivalent to the category of sheaves on a weak site $\langle C, J_C \rangle$ and show that the category relativizes. Ideally we would like to consider the (class sized) model which is the category of all sheaves on a site. Unfortunately though the notion of being a sheaf is not absolute and so this model would not relativize. Instead we consider a category equivalent to the category of sheaves on a weak site but whose objects are actually separated presheaves (a notion which is absolute).

Definition 5.1. *Let $Sh(x, \langle C, J_C \rangle)$ be the formula which says*

$$x = \langle Obj, Morph, Dom, Codom, Id, \equiv, \not\equiv \rangle$$

and

(a) $\langle C, J_C\rangle$ *is a weak site (here* $\langle C, J_C\rangle$ *is treated as a parameter).*

(b) $Th_{Cat}(x|_{L_{Cat}})$.

(c) $Obj = \{F : C^{op} \to$ ***SET*** *such that* F *is a separated presheaf for* $\langle C, J_C\rangle\}$.

(d i) $Morph = \{\langle D, d, R, r, F\rangle : D, d, R, r \in Sep, F : d \Rightarrow r,\ F$ *is a natural transformation,* d, r *cover* D, R *respectively in* $(C, J_C^{ORD})\}$.

(d ii) $dom(\langle D, d, R, r, F\rangle) = D$ *and* $codom(\langle D, d, R, r, F\rangle) = R$. *We will often refer to* $\langle D,\ d,\ R,r,\ F\rangle$ *simply as* F.

(d iii) $\langle X, d, R, r, F\rangle \circ \langle D, d', X, r', G\rangle = \langle D, G^{-1}[r] \cap d', R, r, F \circ G\rangle$

(e) *If* $X \in Obj$ *then* $Id(X) = \langle X, X, X, X, id_X\rangle$.

(f i) $(\forall F, G \in Obj)F \equiv G$ *if and only if* $F = G$.

(f ii) *For all* $\langle D_F, d_F, R_F, r_F, F\rangle, \langle D_G, d_G, R_G, r_G, G\rangle \in Morph$,

$$\langle D_F, d_F, R_F, r_F, F\rangle \equiv \langle D_G, d_G, R_G, r_G, G\rangle$$

if and only if

- $D_F = D_G$ *and* $R_F = R_G$
- $(\forall x \in d_F \cap d_G)F(x) = G(x)$

There are a few points in Definition 5.1 worth highlighting. First lets fix a model $Sh(C, J_C)$ of $Sh(x, \langle C, J_C\rangle)$. First, despite having objects which are only separated presheaves we will see that $Sh(C, J_C)$ is equivalent to **Sheaf**(C, J_C). If $A \in Sh(C, J_C)$ then we want to think of A as a stand in for $\mathbf{a}(A)$ and if $f : A \to B$ is a map in $Sh(C, J_C)$ then we want to think of f as a stand in for $\mathbf{a}(f) : \mathbf{a}(A) \to \mathbf{a}(B)$. To see why this is the case notice that any map $g : \mathbf{a}(A) \to \mathbf{a}(B)$ is determined by where it sends A. However, it is not necessarily the case that such a g will send all elements of A to B. All we know is that there is a subset $a \subseteq A$ which generates A and which is sent, by a, to a subset of B. It is this idea which motivates the definition of a map in $Sh(C, J_C)$.

Proposition 5.2. *Set* $\models (\exists x)Sh(x, \langle C, J_C\rangle)$

Proof. Obj and $Morph$ exist as classes in Set and are uniquely defined. As such it suffices to show that the $\equiv$ is an equivalence relation and satisfies conditions of Definition 3.10. First notice that if $\langle D, d_F, R, r_F, F\rangle \equiv \langle D, d_G, R, r_G, G\rangle$ then $\langle \mathbf{a}(D), d_F, \mathbf{a}(R), r_F, F\rangle \equiv \langle D, d_F, R_F, r_F, F\rangle$. Further the unique map from $\mathbf{a}(D)$ to $\mathbf{a}(R)$ induced by F is the same as the map induced by G, so $\langle D_F, d_F, R_F, r_F, F\rangle \equiv \langle D_G, d_G, R_G, r_G, G\rangle$ if and only if the natural transformations they induce from $\mathbf{a}(D)$ to $\mathbf{a}(R)$ are identical.

In particular this means $\equiv$ is an equivalence relation. It is then easy to check the other conditions of Definition 3.10. $\square$

Lemma 5.3. *$Set \models (\forall x, y) Sh(x, \langle C, J_C\rangle) \wedge Sh(y, \langle C, J_C\rangle) \rightarrow x = y$.*

Proof. Because all relations are definable by formulas of set theory which do not mention each other. $\square$

Definition 5.4. *If $Set \models Sh(x, \langle C, J_C\rangle)$ then we use $Sh(C, J_C)$ as a shorthand for x.*

Proposition 5.5. *$Sh(C, J_C)$ is equivalent to* $\textbf{\textit{Sheaf}}(C, J_C^{\textbf{\textit{ORD}}})$

Proof. Let $E(A) = \mathbf{a}(A)$ if A is a separated presheaf. Now if $f = (D, d, R, r, F)$ then there is a unique map $\mathfrak{f} : \mathbf{a}(D) \rightarrow \mathbf{a}(R)$ such that $\mathfrak{f}$ restricted to d is F (this follows from the fact that d covers $\mathbf{a}(D)$ and r covers $\mathbf{a}(R)$). If we let $E(f) = \mathfrak{f}$ then $E : Sh(C, J_C) \rightarrow \mathbf{Sheaf}(C, J_C^{\mathbf{ORD}})$ is a functor. Let $\mathbf{i}$ be the map $\mathbf{Sheaf}(C, J_C^{\mathbf{ORD}}) \rightarrow Sh(C, J_C)$ given by $\mathbf{i}(A) = A$ if $A \in \mathrm{obj}(\mathbf{Sheaf}(C, J_C^{\mathbf{ORD}}))$ and $\mathbf{i}(\alpha) = (\mathrm{dom}(\alpha), \mathrm{dom}(\alpha), \mathrm{codom}(\alpha), \mathrm{codom}(\alpha), \alpha)$ if $\alpha \in \mathrm{morph}(\mathbf{Sheaf}(C, J_C^{\mathbf{ORD}}))$.

If we let $\eta_A : \mathbf{i} \circ E(A) \rightarrow A$ be $(\mathbf{a}(A), A, \mathbf{a}(A), A, id_A)$ then η_A is an isomorphism and η is a natural transformation. Similarly if we let $\varepsilon_A : E \circ \mathbf{i}(A) \rightarrow x$ be id_A then ε_A is an isomorphism and ε is a natural transformation. In particular we have E and $\mathbf{i}$ are equivalences of categories (up to $\equiv$). $\square$

Corollary 5.6. *Suppose $A, B \in obj(Sh(C, J_C))$. Then $(\exists \alpha \in Sh(C, J_C))[A, B]$ α is an isomorphism if and only if $(\exists \alpha' \in \textbf{\textit{Sheaf}}(C, J_C^{\textbf{\textit{ORD}}})[\textbf{\textit{a}}(A), \textbf{\textit{a}}(B)])$ α' is an isomorphism.*

For the rest of this paper let $V_0, V_1 \models ST$ be standard models with $V_0 \subseteq V_1$ and let (C, J_C) be an almost subcanonical weak site in V_0.

Lemma 5.7. *Suppose $(\overline{C}, J_{\overline{C}}) = (C, J_C^{ORD})^{V_0}$. Then $V_1 \models (C, J_C^{ORD}) = (\overline{C}, J_{\overline{C}}^{ORD})$.*

Proof. Because $J_C \subseteq J_{\overline{C}}$, $J_C^{\mathrm{ORD}} \subseteq J_{\overline{C}}^{\mathrm{ORD}} \subseteq (J_C^{\mathrm{ORD}})^{\mathrm{ORD}} = J_C^{\mathrm{ORD}}$. □

Lemma 5.8. *If F is a separated presheaf for (C, J_C) in V_0 then F is a separated presheaf for (C, J_C) in V_1.*

Proof. F is a separated presheaf for (C, J_C) if and only if F is a separated presheaf for $(C, J_C^{\mathrm{ORD}})^{V_0}$ if and only if F is a separated presheaf for $(C, J_C^{\mathrm{ORD}})^{V_1}$. □

Proposition 5.9. *If $V_0 \models Sh(x, \langle C, J_C\rangle)$ then $V_1 \models (\exists y) Sh(y, \langle C, J_C\rangle) \wedge x \subseteq y$*

Proof. We see that $Obj^x \subseteq Obj^y$ by Lemma 5.8. Also if F is a separated presheaf then $d \in V_0$ is a covering set for F in V_0 if and only if d is a covering set for F in V_1. Hence $Morph^x \subseteq Morph^y$, $\equiv^x \subseteq \equiv^y$, and $\not\equiv^x \subseteq \not\equiv^y$. So $x \subseteq y$ □

Corollary 5.10. *If $V_0 \models Sh(x, \langle C, J_C\rangle)$ then $V_1 \models Exp(Sh(x, \langle C, J_C\rangle), x)$ has an initial object.*

5.2. Limits and Colimits

In this section we will show that colimits and finite limits of $Sh(C, J_C)$ relativize.

Proposition 5.11. *Suppose K is a diagram in $Sh(C, J_C)^{V_0}$.*

(a) If $CoLim \in Sh(C, J_C)^{V_0}$ is a colimit cocone of K in V_0 then $CoLim$ is a colimit cone of K in $Sh(C, J_C)^{V_1}$.

(b) If K is finite and Lim is a limit cone of K in $Sh(C, J_C)^{V_0}$ then Lim is a limit cone of K in $Sh(C, J_C)^{V_1}$.

Proof. Part (a):
$CoLim$ is a colimit of K in $Sh(C, J_C)^{V_0}$ if and only if $V_0 \models \mathbf{a}(CoLim)$ is a colimit cone of $\mathbf{a}"[K]$ [7] (by Proposition 5.5 and the fact that $\mathbf{a}$ preserves colimits). And similarly $CoLim$ is a colimit of K in $Sh(C, J_C)^{V_1}$ if and only if $V_1 \models \mathbf{a}(CoLim)$ is a limit cone of $\mathbf{a}"[K]$. The result then follows from the fact that $V_1 \models \mathbf{a}(\mathbf{a}(CoLim)^{V_0})$ is isomorphic to $\mathbf{a}(CoLim)$.
Part (b):
This is done in an identical way. □

[7] $\mathbf{a}"[K]$ is the cone obtained by applying $\mathbf{a}$ to K pointwise.

Of course this doesn't mean that whenever $CoLim \in Sh(C, J_C)^{V_0}$ and $V_1 \models CoLim$ is a colimit of K that we also have $V_0 \models CoLim$ is a colimit of K.

Corollary 5.12. *Suppose $(G \subset obj(Sh(C, J_C)))^{V_0}$. If $V_0 \models$ "G is a generating set for $Sh(C, J_C)$" then $V_1 \models$ "G is a generating set for $Sh(C, J_C)$".*

Proof. We have $y_C(A)$ is a colimit, in V_0, of elements of G. Hence $y_C(A)$ is a colimit of elements of G in V_1. So, as in V_1 all objects of $Sh(C, J_C)$ are colimits of elements of $\{y_C(A) : A \in \text{obj}(C)\}$, in V_1 all objects are colimits of elements of G.

And, because $Sh(C, J_C)$ is a Grothendieck Topos, this implies that G is a generating set for $Sh(C, J_C)$ (in V_1). □

What this corollary shows is that being a generating set is upwards absolute. However, it is not in general downwards absolute.

5.3. Generating Sets

In this section we will show one of the key results of the paper. We will show that if two almost subcanonical weak sites have equivalent categories of sheaves in one standard model of set theory then they have equivalent categories of sheaves in all larger models of set theory. This will be crucial when we define our theory of Grothendieck topoi.

Definition 5.13. *Let $G \subseteq obj(Sh(C, J_C))$. Define $(\widehat{G}, J_{\widehat{G}})$ to be the site where*

- *$\widehat{G}$ is the full subcategory of $Sh(C, J_C)$ with objects G.*
- *$S \in J_{\widehat{G}}(A)$ if and only if S is an epimorphic family in $Sh(C, J_C)$.*

Proposition 5.14. *Let $V_0 \models$ "(C, J_C) is an almost subcanonical weak site" and let $\mathcal{C} = \{y_C(A) : A \in obj(C)\}$. If $V_0 \models \mathcal{C} \subseteq D \subseteq obj(Sh(C, J_C))$ then $V_1 \models Sh((\widehat{\mathcal{C}}, J_{\widehat{\mathcal{C}}})^{V_0})$ is equivalent to $Sh((\widehat{D}, J_{\widehat{D}})^{V_0})$*

Proof. First notice by Corollary 4.46 $V_1 \models Sh(C, J_C) \simeq Sh((\widehat{C}, J_{\widehat{C}})^{V_0})$. To simplify notation we will use (C, J_C) instead of $(\widehat{\mathcal{C}}, J_{\widehat{\mathcal{C}}})^{V_0}$. Also notice that $V_1 \models (C, J_C)$ and $(\widehat{D}, J_{\widehat{D}})^{V_0}$ are almost subcanonical by Proposition 4.15. Working in V_1 let $G = Sh(C, J_C)$ and let $H = Sh((\widehat{D}, J_{\widehat{D}})^{V_0})$.

In this proof we will have the same separated presheaves occurring in several different categories. As such it is worth fixing some notation to deal with this. If $D \subseteq \text{obj}(Sh(C, J_C)^{V_0})$ then $D \subseteq \text{obj}(Sh(C, J_C)^{V_1}) = \text{obj}(G)$ as well (because as sets $\text{obj}(Sh(C, J_C)^{V_0}) \subseteq \text{obj}(Sh(C, J_C)^{V_1})$). When we wish to refer to D as a subset of $\text{obj}(Sh(C, J_C)^{V_1})$ we will call it D_G.

In addition, there is a Yoneda embedding of every object of D into $Sh((\overline{D}, J_D)^{V_0})^{V_1} = \text{obj}(H)$. So it will be useful to let $D_H = \{y_{\overline{D}}(d) : d \in D\} \subseteq \text{obj}(H)$.

Now by Proposition 5.5 $V_1 \models Sh(C, J_C)$ is equivalent to $Sh(\widehat{\mathcal{C}}, J_{\widehat{\mathcal{C}}})$ which is equivalent to $Sh(\widehat{D_G}, J_{\widehat{D_G}})$ (because $\mathcal{C}$ is a generating set for $Sh(\overline{C}, J_{\overline{C}})$ so is D_G). Also by Corollary 5.12 then $V_1 \models H$ is equivalent to $Sh(\widehat{D_H}, J_{\widehat{D_H}})$. So in particular to show that $V_1 \models Sh(\widehat{D_H}, J_{\widehat{D_H}})$ is equivalent to $Sh(\widehat{D_G}, J_{\widehat{D_G}})$ it suffices to show H is equivalent to G. Or it suffices to show $V_1 \models (\exists E) E : (\widehat{D_G}, J_{\widehat{D_G}}) \to (\widehat{D_H}, J_{\widehat{D_H}})$ where E is an isomorphism of sites.

We will construct E explicitly. First notice that there are injective maps $F_G : (\widehat{D}, J_{\widehat{D}})^{V_0} \to (\widehat{D_G}, J_{\widehat{D_G}})$ and $F_H : (\widehat{D}, J_{\widehat{D}})^{V_0} \to (\widehat{D_H}, J_{\widehat{D_H}})$ which are isomorphisms on objects. The map F_H coming from the fact that, by Corollary 4.46 $(\widehat{D_H}, J_{\widehat{D_H}})^{V_1}$ is an initial object in $Exp(Th_{SubCan}, (\widehat{D}, J_{\widehat{D}})^{V_0})$. The map F_G coming from the fact that $Sh(C, J_C)^{V_0} \subseteq Sh(C, J_C)^{V_1}$ (by Proposition 5.9). Lastly notice that these maps are maps of sites and not just categories as any cover in $(\widehat{D}, J_{\widehat{D}})^{V_0}$ is also a cover in the other categories, $(\widehat{D_G}, J_{\widehat{D_G}})$ and $(\widehat{D_H}, J_{\widehat{D_H}})$, under these maps

If $X \in \text{obj}(\widehat{D_G})$ let $E(X) = F_H(F_G^{-1}(X))$ and if $Y \in \text{obj}(\widehat{D_H})$ let $E'(Y) = F_G(F_H^{-1}(Y))$. Then E, E' are inverses functions on the objects of $\widehat{D_G}, \widehat{D_H}$ respectively because F_G, F_H are bijections on objects.

For each $A \in \text{obj}(\widehat{D})$ let $Cov_G(A) = \{F_G(f) : f \in C[X, A], X \in \text{obj}(C)\}$ and let $Cov_H(A) = \{F_H(f) : f \in C[X, A], X \in \text{obj}(C)\}$. Then $Cov_G(A)$ is a covering set for $F_G(A)$ because $\{f : f \in C[X, A], X \in obj(C)\}$ is a covering set of A and $F_G(A)$ is a sheafification of A. Similarly $Cov_H(A)$ is a covering set of $F_H(A)$ because $\{y_{\widehat{D}}(f) : f \in C[X, A], X \in \text{obj}(C)\}$ is a covering set of $y_{\widehat{D}}(A)$ and $F_H(A)$ is a sheafification of $y_{\widehat{D}}(A)$. Further $(\forall A \in \text{obj}(\widehat{D}))(\exists$ isomorphism $i : Cov_G(A) \simeq Cov_H(A)$ such that $i(f) = y_{\widehat{D}}(f)$.

If $\alpha \in \widehat{D_G}[F_G(A), F_G(B)]$ let $E(\alpha) \in \widehat{D_H}[E(F_G(A)), E(F_G(B))]$ be the unique map such that $E(\alpha)(i(f)) = i(\alpha(f))$ for all $f \in Cov_G(A)$. Similarly if $\alpha' \in \widehat{D_H}[F_H(A), F_H(B)]$ let $E'(\alpha') \in \widehat{D_H}[E'(F_H(A)), E'(F_H(B))]$ be the unique map such that $E'(\alpha')(i^{-1}(f)) = i^{-1}(\alpha(f))$ for all $f \in Cov_H(A)$. It is then clear that E, E' are functors and inverses of each other. Hence E is an isomorphism.

All that is left is to show that $J_{\widehat{D_H}} = J_{\widehat{D_G}}$. However we know that $S \in J_{D_G}(A)$ if and only if the map $\phi_S : \coprod\{y_{D_G}(\mathrm{dom}(f)) : f \in S\} \to A$ is an epimorphism if and only if there are covering sets $X_i \subseteq Cov(y_{D_G}(\mathrm{dom}(f)))$ and $X_A \subseteq Cov(A)$ such that if X is the coproduct of the the X_i's then X_A is in the image of X. But this holds if and only if the same holds once E is applied. □

Theorem 5.15. *Suppose (C, J_C) and (D, J_D) are almost subcanonical weak sites such that $V_0 \models Sh(C, J_C)$ is equivalent to $Sh(D, J_D)$. Then $V_1 \models Sh(C, J_C)$ is equivalent to $Sh(D, J_D)$*

Proof. First notice by Corollary 4.46 we can assume without loss of generality that (C, J_C) and (D, J_D) are subcanonical sites in V_0 (and hence almost subcanonical in V_1).

By assumption there are maps, such that $V_0 \models E_C : Sh(C, J_C) \to Sh(D, J_D)$ and $E_D : Sh(D, J_D) \to Sh(C, J_C)$ are equivalences of categories. Let $(\mathcal{C}, J_\mathcal{C})$ be the full subcategory of $Sh(C, J_C)^{V_0}$ whose objects are $\{y_C(A) : A \in \mathrm{obj}(C)\} \cup \{E_D(y_C(B)) : B \in \mathrm{obj}(D)\}$ and whose covering sieves are epimorphic families. Similarly let $(\mathcal{D}, J_\mathcal{D})$ be the full subcategory of $Sh(D, J_D)^{V_0}$ whose objects are $\{y_D(A) : A \in \mathrm{obj}(D)\} \cup \{E_C(y_D(B)) : B \in \mathrm{obj}(C)\}$ and whose covering sieves are epimorphic families. Because E_C, E_D are equivalences of categories, they restrict an equivalences of sites between $(\mathcal{C}, J_\mathcal{C})$ and $(\mathcal{D}, J_\mathcal{D})$.

Hence by Proposition 4.45 we have $V_1 \models Sh(\mathcal{C}, J_\mathcal{C}) \simeq Sh(\mathcal{D}, J_\mathcal{D})$. But we also have by Proposition 5.14 that $V_1 \models Sh(C, J_C) \simeq Sh(\mathcal{C}, J_\mathcal{C})$ and $Sh(D, J_D) \simeq Sh(\mathcal{D}, J_\mathcal{D})$. □

6. Grothendieck Toposes

In this section we will give a theory whose models are exactly (a definable expansion of) the Grothendieck topoi. Further every model of this theory has a relativization to every standard model of set theory (satisfying the Axiom of Choice).

Definition 6.1. *Let G be a Grothendieck Topos. We say that (C, J_C) is a generating site for G if*

- *$C \subseteq G$ and C is a set.*

- J_C *consists of epimorphic families in* C.

- G *is equivalent to* $Sh(C, J_C)$.

Definition 6.2. *Let* $L_{Topoi} = L_{Cat} \cup \{GS\}$ *where* GS *is unary relations on the power set of the model. Let* $GT(x)$ *be the formula which says*

- $x|_{L_{Cat}}$ *has a set of generators.*

- *Sums in* $x|_{L_{Cat}}$ *are disjoint.*

- *All equivalence relations in* $x|_{L_{Cat}}$ *are effective.*

- $GS(g)$ *if and only if* $g \subseteq x$ *is a generating site for* x.

The first three of these conditions are *Giraud's axioms for a Grothendieck Topos* ([9]), and the last is there to tell us which categories of sheaves our topos should be equivalent to.

Proposition 6.3. *Suppose* $V_0 \models GT(G)$. *Then* $(obj(Exp(GT, G)))^{V_1}$ *is non-empty.*

Proof. We know that $V_0 \models (\exists(C, J_C))E_C : G \simeq Sh(C, J_C)$. Working in V_1 let $I : \text{obj}(G) \cup \text{obj}(Sh(C, J_C)) \to \text{obj}(Sh(C, J_C))$ where $I|_{Sh(C,J_C)}$ is the identity and $I|_G$ is E_C. We then let

- $\text{obj}(G') = \text{obj}(G) \cup \text{obj}(Sh(C, J_C))$

- $G'[X, Y] = \{(x, y, f) : f \in Sh(C, J_C), f : I(x) \to I(y)\}$ where $(x, y, f) \circ (y, z, g) = (x, z, g \circ f)$.

G' is then clearly equivalent to G and there is a full and faithful injection $I' : G \rightarrowtail G'$ such that $I'|_{\text{obj}(G)} = id_G$ and if $f \in G[X, Y]$ then $I'(f) = (X, Y, f)$.

Now G' isn't an extension of G because we don't have G as a subset of G'. But if we define $H = (G' - image(I')) \cup G$ with the obvious composition of morphisms then H is an extension of G (with respect to L_{Cat}) and there is an isomorphism between H and G'.

So all that is left is to show that $(H \models GS(g, J_g))^{V_1}$ whenever $(G \models GS(g, J_g))^{V_0}$. Now $(G \models GS(g, J_g))^{V_0}$ if and only if $(G \simeq Sh(g, J_g))^{V_0}$. But by Theorem 5.15 we have $(H \simeq Sh(C, J_C) \simeq Sh(g, J_g))^{V_1}$ because $V_0 \models Sh(C, J_C) \simeq Sh(g, J_g)$. Hence $(H \models GS(g, J_g))^{V_1}$. So $(\text{obj}(Exp(GT, G)))^{V_1}$ is non-empty. $\square$

Theorem 6.4 (*)**.** *If* $V_1 \models$ *Axiom of Choice then* $(Exp(GT, G))^{V_1}$ *has a object which has no non-trivial automorphisms and which maps into every other object*[8] *(and hence* G *has a relativization to* V_1 *for* GT*).*

Proof. Working in V_1 let $Skel(C, J_C)$ be a skeletal subcategory of $Sh(C, J_C)$ (which we know exists as V_1 satisfies the axiom of choice). Let H' be the full subcategory of H (from Proposition 6.3 whose objects are either from G or are from $Skel(C, J_C)$ and not isomorphic to any objects in G. H' is then equivalent to H which is equivalent to $Sh(C, J_C)$.

As H' is equivalent to H, any generating site in H also is a generating set in H' (under the equivalence) and so the inclusion map from G into H' preserves generating sites. So, $(H' \in Exp(GT, G))^{V_1}$.

By Theorem 5.15 if $(T \in Exp(GT, G))^{V_1}$ then T is equivalent to $Sh(C, J_C)$ (in V_1). But if T is any such category containing G as a subcategory, then there must be a unique injection from H' into T taking G to itself. Hence H' is an initial element of $(Exp(GT, G)^{V_1})$. □

We will end with a conjecture:

Conjecture 6.5. *If* $GT^*(x)$ *is the* L_{Cat} *formula which says,* x *satisfies Giraud's axioms for a Grothendieck Topos then all models of* GT^* *have a relativization to all standard models of set theory with the Axiom of Choice. Further the relativization of a model of* GT *is isomorphic (in* L_{Cat}*) to its relativization as a model of* GT^**.*

In other words if we remove the condition that our Grothendieck Topos preserve generating sites then we still have that all Grothendieck Topoi relativize and they all relativize to the same categories (i.e. if $G \simeq Sh(C, J_C)$ in V_0 then $G' \simeq Sh(C, J_C)$ in V_1 (where G' is the relativization)).

References

[1] Ackermann, Wilhelm, *Zur Axiomatik der Mengenlehre*, Mathematische Annalen, 1956, Volume 131, pp. 336–345.

[8] Notice that $(Exp(GT, G))^{V_1}$ will not in general have an initial object. This is because, while the model constructed in Theorem 6.4 does have a map into every other Grothendieck topos in $(Exp(GT, G))^{V_1}$, the map will only be unique if it is an isomorphism.

[2] Awodey, S., Butz, C., Simpson, A., and Streicher, T., *Relating First-Order Set Theories and Elementary Toposes*, The Bulletin of Symbolic Logic Volume 13, Number 3, September. 2007, pp. 340–358.

[3] Awodey, S., Gambino, N., Lumsdaine, P. L., and Warren, M. A., *Lawvere-Tierney Sheaves in Algebraic Set Theory*, The Journal of Symbolic Logic Volume 75, Number 3, September. 2009, pp. 862–890.

[4] Barwise, Jon, *Admissible sets and structures: An approach to definability theory*, Perspectives in Mathematical Logic, Springer-Verlang (1975).

[5] Feferman, Solomon, *Set-Theoretical Foundations of Category Theory*, Reports of the Midwest Category Seminar. III, Springer (1969), pp. 201–247.

[6] Jech, Thomas, *Set Theory: The Third Millennium Edition*, Springer Monogrphas in Mathematics, Springer-Verlag, 2000.

[7] Joyal, André and Moerdijk, Ieke, *Algebraic set theory*, London Mathematical Society Lecture Note Series, 220. Cambridge University Press, Cambridge, 1995.

[8] Mac Lane, Saunders, *Categories for the Working Mathmatician*, Graduate Texts in Mathematics Volume 5, Springer-Verlag, 1998.

[9] Mac Lane, Saunders and Moerdijk, Ieke, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Universitext, Springer-Verlag, 1992.

[10] Zermelo, Ernst, *Untersuchungen ber die Grundlagen der Mengenlehre I*, Mathematische Annalen, 1908 Volume 65, pp. 261–281.