MODEL THEORETIC PROOF OF A RESULT OF HJORTH

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In this note we will give a model theoretic proof of Hjorth's result in [2] that if there is a counterexample to Vaught's conjecture then there is a counterexample with no models of size $\aleph_2$.

The proof will be broken into two parts. First, in Section 1 we state Proposition 1.2, which we will use to prove the main theorem. Proposition 1.2 intuitively says that it is possible to have a theory which partitions its models in to two parts in such a way that on one side we have a theory with a unique countable model up to isomorphism but no model of size $\aleph_2$, on the other side our structure is arbitrary, and yet the only relationship between two sides is that they must have the same size.

In Section 2 we will prove Proposition 1.2. This proof follows closely Section 3 of [1] and T_0 is a (very) minor variation of the example given there.

In particularall of the ideas in this note are taken from Hjorth's papers [1] and [2].

1. Main Result

In what follows we will not distinguish between a unary formula and the set it represents.

Definition 1.1. *Given a unary formula $\varphi(x)$ and a relational language L we say that T proves L is* **trivial** *on $\varphi(x)$ if for each $R \in L$ we have $T \vdash (\forall x_1, \dots, x_n)R(x_1, \dots, x_n) \rightarrow \bigwedge_{i \leq n} \neg\varphi(x_i)$.*

In other words L is trivial on $\varphi(x)$ if no relation from L holds of any tuple of elements containing a member that satisfies $\varphi(x)$.

Proposition 1.2. *Let Q be a unary predicate and let P be a binary predicate. There is a (relational) language L_0 and a sentence $T_0 \in \mathcal{L}_{\omega_1,\omega}(L_0 \cup \{P, Q\})$ such that:*

(1) *Index Axioms:*
   (a) $T_0 \vdash (\forall x, y)P(x, y) \rightarrow \neg Q(x) \wedge Q(y)$.
   (b) $T_0 \vdash (\forall x)\neg Q(x) \rightarrow (\exists! y)P(x, y)$.
   (c) $T_0 \vdash (\forall y)Q(y) \rightarrow (\exists^\infty x)P(x, y)$.

(2) *Characterizing $\aleph_1$:*
   (a) *L_0 is trivial on Q.*
   (b) *T_0 has a unique countable model $\mathcal{M}_0$.*
   (c) *T_0 has models of size $\aleph_1$ but no models of size $\aleph_2$.*

(3) *Expanding A Model. Suppose:*

(a) L_1 *is any countable (relational) language disjoint from* $L_0 \cup \{Q, P\}$.
(b) $\mathcal{M}_1$ *is a countable* $L_1 \cup \{Q\}$ *structure which is trivial for* $\neg Q$.
then there is a unique (up to isomorphism) countable $L_0 \cup L_1 \cup \{Q, P\}$ *structure* $\mathcal{N}$ *such that:*
(c) $\mathcal{N}|_{L_0} \cong \mathcal{M}_0$.
(d) $\mathcal{N}|_{L_1} \cong \mathcal{M}_1$

Proposition 1.2 is (roughly) the model theoretic analog of the statement that there exists a structure $\mathcal{M}$ such that $Aut(\mathcal{M})$ *divides* S_∞, i.e. such that there is a closed subgroup $H \subseteq Aut(\mathcal{M})$ and a continuous surjection $H \to S_\infty$.

Corollary 1.3. *If* $\varphi_{\mathcal{M}_1}$ *is a Scott sentence for* $\mathcal{M}_1$ *from Proposition 1.2 then* $\varphi_{\mathcal{M}_1} \cup T_0$ *is a Scott sentence for* $\mathcal{N}$ *from Proposition 1.2.*

Proof. This is because $\mathcal{N} \models \varphi_{\mathcal{M}_1} \cup T_0$ and by Proposition 1.2 (3) any other countable structure which satisfies $\varphi_{\mathcal{M}_1} \cup T_0$ is isomorphic to $\mathcal{N}$. □

In what follows if $\sigma \in \mathcal{L}_{\omega_1,\omega}(L_1)$ we define σ^Q to be the relativization of σ to Q, i.e. the formula obtained by relativizing all quantifiers to Q and which says L_1 is trivial on $\neg Q(x)$.

Theorem 1.4. *If* $\sigma \in \mathcal{L}_{\omega_1,\omega}(L_1)$ *is a counterexample to Vaught's conjecture then* $T_0 \cup \sigma^Q$ *is a counterexample to Vaught's conjecture which has no model of size* $\aleph_2$.

Proof. If $\mathcal{M}$ is an L_1-structure let $\mathcal{M}^Q$ be the resulting relativization to Q (i.e. the $L_1 \cup \{Q\}$ structure which satisfies $\varphi^Q_{\mathcal{M}}$ where $\varphi_{\mathcal{M}}$ is a Scott sentence of $\mathcal{M}$). It is then easily checked that for any sentence $\varphi \in \mathcal{L}_{\omega_1,\omega}(L_1)$ the map $\mathcal{M} \to \mathcal{M}^Q$ is a bijection between countable models of φ and models of φ^Q.

In particular this implies σ^Q is a counterexample to Vaught's conjecture. However by Proposition 1.2 we also have that every countable model of σ^Q has a unique expansion to an $L_0 \cup L_1 \cup \{P, Q\}$ structure that satisfies T_0. Hence $\sigma^Q \cup T_0$ is also a counterexample to Vaught's conjecture.

However, by Proposition 1.2 (1) we have that in any model $\mathcal{M} \models T_0$ we have $|\{x : \mathcal{M} \models Q(x)\}| \leq |\{x : \mathcal{M} \models \neg Q(x)\}|$ and so by Proposition 1.2 (2) we have that there is no model of T_0 of size $\aleph_2$. But this then implies there is also no model of $\sigma^Q \cup T_0$ of size $\aleph_2$.

□

2. Proof of Proposition 1.2

In this section we will prove Proposition 1.2. This proof closely follows [1] and when the proofs are identical we will simply reference the corresponding proof in [1].

Definition 2.1 (Definition 3.2 of [1])**.** *Let* $L_0 = \{S_n : n \in \omega\} \cup \{R_k : k \in \omega\}$ *where each* S_n *is a binary relation and each* R_k *is a* $k+2$*-ary relation. We define the following sentences:*

(*1) *The conjunction of:*
- (*1a) $(\forall a, b) \bigvee_{n\in\omega} S_m(a, b)$.
- (*1b) $(\forall a, b) \bigwedge_{n\neq m}(S_n(a, b) \Rightarrow \neg S_m(a, b))$.

(*2) *The conjunction of:*
- (*2a) *For each k and $i < j < k$:*

$$(\forall a_0, a_1, b_0, \dots, b_{k-1})R_k(a_0, a_1, b_0, \dots, b_{k-1}) \Rightarrow (b_i \neq b_j \wedge a_0 \neq a_1)$$

- (*2b) *For each k and permutation π of k:*

$$(\forall a_0, a_1, b_0, \dots, b_{k-1})R_k(a_0, a_1, b_0, \dots, b_{k-1}) \Leftrightarrow R_k(a_0, a_1, b_{\pi(0)}, \dots, b_{\pi(k-1)}).$$

- (*2c) $(\forall a_0, a_1, b_0, \dots) \bigwedge_{n\in\omega} \bigwedge_{i<k} R_k(a_0, a_1, b_0, \dots) \Rightarrow (S_n(a_0, b_i) \Rightarrow S_n(a_1, b_i))$.
- (*2d) $(\forall a_0, a_1, b_0, \dots, c) \bigwedge_{n\in\omega} \bigwedge_{i<k}[R_k(a_0, a_1, b_0, \dots) \wedge_{i<k} c \neq b_i] \Rightarrow (S_n(a_0, c) \Rightarrow \neg S_n(a_1, c))$.

Definition 2.2 (Definition 3.3 of [1])**.** *An L_0-structure F is L_0-**neat** if*
1. *F is finite.*
2. $F \models (*1), (*2)$.
3. $(\forall a_0, a_1 \in F)a_0 \neq a_1 \Rightarrow (\exists k \exists \bar{b})R_k(a_0, a_1, \bar{b})$.

Definition 2.3 (Definition 3.4 of [1])**.** *An L_0-structure $\mathcal{M}$ is L_0-**full** if*
1. *for all finite $A \subseteq \mathcal{M}$ there is an L^*-neat $F \supset A$, $F \subset \mathcal{M}$.*
2. *for all L_0-neat $F \subseteq \mathcal{M}$ and L_0-neat $H_0 \subset F$ there is an L_0-neat $H_1 \subseteq \mathcal{M}$ with $F \subseteq H_1$ and an isomorphism $i : H_0 \cong H_1$ with $i|_F = id$.*

Lemma 2.4 (Lemma 3.2 of [1])**.** *Up to isomorphism there is a unique L_0-full model $\mathcal{M}_{full}$.*

Definition 2.5. *Let $T_0^* \in \mathcal{L}_{\omega_1,\omega}(L_0)$ be the Scott sentence of $\mathcal{M}_{full}$.*

Lemma 2.6 (Lemma 3.3 of [1])**.** *T_0^* has no model of size $\aleph_2$.*

Definition 2.7 (Definition 3.6 of [1])**.** *Let $L_0^+ = L_0 \cup \{Q, P\} \cup \{c_n : n \in \omega\}$ where Q is unary, P is binary and each c_n is a constant. We define the following sentence:*

(*3) *The conjunction of:*
- (*3a) $(\forall a) \bigvee_{n\in\omega} P(a, c_n)$.
- (*3b) $(\forall a) \bigwedge_{n\neq m}(P(a, c_n) \Rightarrow \neg P(a, c_m))$.

(*4) *The conjunction of:*
- (*4a) $\bigwedge_{n\neq m} c_m \neq c_n$.
- (*4b) $(\forall x)Q(x) \to \bigvee_{n\in\omega} x = c_n$.
- (*4c) *For all $U \in L_0$,* $(\forall x_1, \dots, x_n)U(x_1, \dots, x_n) \to \bigwedge_{i\leq n} \neg Q(x)$.

This definition differs from Definition 3.6 of [1] in one important way. In [1], instead of having a single binary relation P whose second argument is among $\{c_n : n \in \omega\}$, there are ω many distinct unary relations $\{P_n : n \in \omega\}$. The reason why we have chosen to interpret the ω many unary relations as a single binary relation with an index set all of whose elements are constants, is that we will later take the reduct to the language where the index set is not composed of constants. Then if we place an arbitrary structure $\mathcal{M}_1$ on the index set, for any definable subset $A \subseteq \mathcal{M}_1$ the minimal quantifier rank of a formula defining A in L_1 is the same as in $L_0 \cup L_1 \cup \{P, Q\}$.

Definition 2.8 (Definition 3.7, Definition 3.8 of [1]). *An L_0^+-structure F is* **L_0^+-neat** *if it satisfies (*3) and (*4) and $\neg Q^F$ is L_0-neat where $\neg Q^F$ is the restriction of F to those elements satisfying $\neg Q$.*

We say likewise define **L_0^+-full**.

Lemma 2.9 (Lemma 3.5 of [1]). *There is a unique countable L_0^+-full model $\mathcal{M}^+_{full}$.*

Definition 2.10. *Let $T_0^+ \in \mathcal{L}_{\omega_1,\omega}(L_0)$ be the Scott sentence of $\mathcal{M}^+_{full}$.*

The next result has a simple proof which we give here as it is the key result which makes Proposition 1.2 possible.

Lemma 2.11 (Lemma 3.6 of [1]). *Let $\tau : Q^{\mathcal{M}^+_{full}} \to Q^{\mathcal{M}^+_{full}}$ be any automorphism (of the sets). Then there is an automorphism $\tau_1 : \mathcal{M}^+_{full} \to \mathcal{M}^+_{full}$ which restricts to τ on $Q^{\mathcal{M}^+_{full}}$.*

Proof. Let $\mathcal{M}^*$ be structure such that

- $\mathcal{M}^*|_{L_0 \cup \{Q\} \cup \{c_n : n \in \omega\}} = \mathcal{M}^+_{full}|_{L_0 \cup \{Q\} \cup \{c_n : n \in \omega\}}$
- $\mathcal{M}^* \models P(a, b)$ if and only if $\mathcal{M}^+_{full} \models P(a, \tau(b))$.

It is then easily checked that $\mathcal{M}^* \models T_0^+$ and so there is an isomorphism $i : \mathcal{M}^+_{full} \to \mathcal{M}^*$. But, it is also clear that any such isomorphism must be the identity on $Q^{\mathcal{M}^+_{full}} = Q^{\mathcal{M}^*}$. Hence $\mathcal{M}^+_{full} \models P(i(a), \tau(b))$ if and only if $\mathcal{M}^* \models P(i(a), b)$ if and only if $\mathcal{M}^+_{full} \models P(a, b)$. Hence $i|_{\neg Q}$ is the desired extension. □

Lemma 2.11, essentially, says that $Aut(\mathcal{M}^+_{full})$ divides S_∞. We now define our theory.

Definition 2.12. *Let $\mathcal{M}_0$ be the reduct of $\mathcal{M}^+_{fulll}$ to $L_0 \cup \{Q, P\}$ and let $T_0 \in \mathcal{L}_{\omega_1,\omega}(L_0 \cup \{Q, P\})$ be the Scott sentence of $\mathcal{M}_0$.*

Corollary 2.13 ((Essentially) Corollary 3.2 (ii) of [1]). *T_0 has a model of size $\aleph_1$.*

We now finally give the proof of Proposition 1.2.

Proposition 1.2. That T_0 satisfies (1) and (2a) is immediate from the definition. That T_0 satisfies (2b) follows from Lemma 2.9 and (2c) follows from Lemma 2.6 and Corollary 2.13.

Finally (3) follows easily from Lemma 2.11.

□

References

[1] Greg Hjorth. Knight's model, its automorphism group, and characterizing the uncountable cardinals. *J. Math. Log.*, 2(1):113–144, 2002.

[2] Greg Hjorth. A note on counterexamples to the Vaught conjecture. *Notre Dame J. Formal Logic*, 48(1):49–51 (electronic), 2007.