CATEGORIES ENRICHED IN QUANTALOIDS ASSOCIATED TO A FRAME

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Abstract. There are two quantaloids which can be constructed from a frame Γ: $\overline{\Gamma}$, the one object quantaloid with morphisms Γ, and $\mathrm{R}(\Gamma)$ the quantaloid of relations on Γ. We consider the relationship between $\mathrm{CAT}(\overline{\Gamma})$, the category of skeletal symmetric categories enriched in $\overline{\Gamma}$, and $\mathrm{CAT}(\mathrm{R}(\Gamma))$, the category of skeletal symmetric categories enriched in $\mathrm{R}(\Gamma)$. We give two full and faithful functors from $\mathrm{CAT}(\overline{\Gamma})$ to $\mathrm{CAT}(\mathrm{R}(\Gamma))$ and show that there compositions with the Cauchy completion functors are isomorphic. One of these functors comes from the inclusion of quantaloids of $\overline{\Gamma}$ in $\mathrm{R}(\Gamma)$ and the other comes from an equivalence of categories between $\mathrm{CAT}(\overline{\Gamma})$ and the category of flabby separated presheaves on Γ.

1. Introduction

Given a frame Γ there are two quantaloids associated with Γ which occur naturally when studying quantaloid enriched categories. These are $\overline{\Gamma}$, the one object quantaloid where $\overline{\Gamma}[*_{\overline{\Gamma}}, *_{\overline{\Gamma}}] = \Gamma$, and the quantaolid $\mathrm{R}(\Gamma)$ of relations on Γ (when Γ is considered as a category).

Both $\mathrm{CAT}(\overline{\Gamma})$, the category of skeletal symmetric $\overline{\Gamma}$-enriched categories, and $\mathrm{CAT}(\mathrm{R}(\Gamma))$, the category of skeletal symmetric $\mathrm{R}(\Gamma)$-enriched categories have occurred naturally in other guises. In the case of $\overline{\Gamma}$ it is easy to see that every skeletal symmetric $\overline{\Gamma}$-enriched category is also a Γ^{op}-ultrametric space (see [1] for a discussion of Γ^{op}-ultrametric spaces and the connection to categories enriched in $\overline{\Gamma}$). In the case of $\mathrm{R}(\Gamma)$ it was shown in [11] that there is an equivalence between the category of Cauchy complete skeletal symmetric $\mathrm{R}(\Gamma)$-enriched categories and the category of sheaves on Γ.

The purpose of this paper is to study the relationship between $\mathrm{CAT}(\overline{\Gamma})$ and $\mathrm{CAT}(\mathrm{R}(\Gamma))$. In Section 2 we will review the basics of categories enriched

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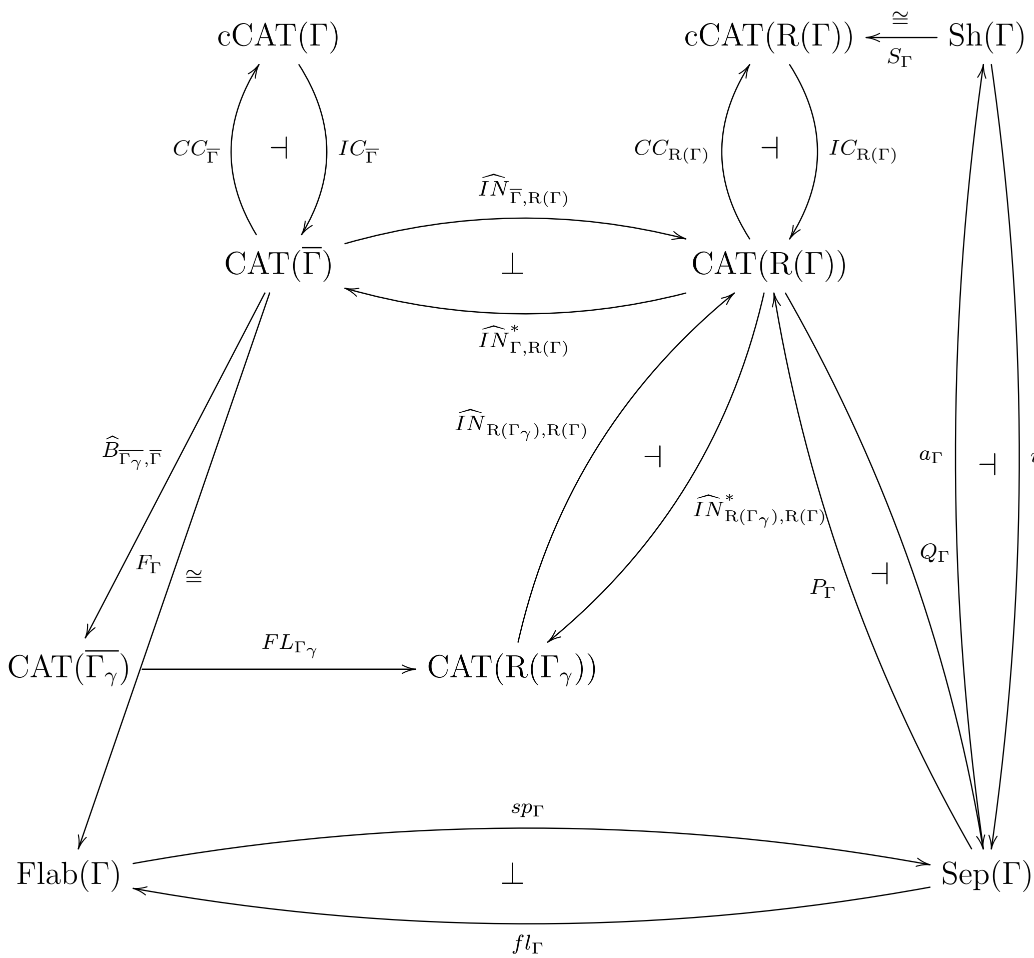
in a quantaloid. In Section 3 we will show that there are two distinct full and faithful functors from $\mathrm{CAT}(\overline{\Gamma})$ into $\mathrm{CAT}(\mathrm{R}(\Gamma))$. First there is the full and faithful functor $\widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}$ which comes from the full and faithful inclusion of quantaloids from $\overline{\Gamma}$ into $\mathrm{R}(\Gamma)$. Then we will construct an equivalence F_Γ between the category $\mathrm{CAT}(\overline{\Gamma})$ and the category $\mathrm{Flab}(\Gamma)$ of flabby separated presheaves on Γ. This, composed with the full and faithful functor $P_\Gamma \circ fl_\Gamma : \mathrm{Flab}(\Gamma) \to \mathrm{CAT}(\mathrm{R}(\Gamma))$, gives us our second full and faithful functor from $\mathrm{CAT}(\overline{\Gamma})$ into $\mathrm{CAT}(\mathrm{R}(\Gamma))$.

In Section 4 we will consider the relation of these full and faithful functors to two different notions of completeness. We will show that the composition of each of these full and faithful functors with the Cauchy completion functor are isomorphic. Then we will show that the functor $P_\Gamma \circ fl_\Gamma \circ F_\Gamma$ restricts to an equivalence between the full subcategory of injective objects in $\mathrm{CAT}(\overline{\Gamma})$ and the full subcategory of injective objects in $\mathrm{CAT}(\mathrm{R}(\Gamma))$.

1.1. **Diagram of Functors.** As we are dealing with several categories and several functors between them it will be useful to have a diagram containing this information as a reference. This is not a commutative diagram.



Below are the expanded names of each of the categories along with the places where they are defined:

- Γ_γ: The restriction of Γ to those elements less than or equal to γ (Definition 2.10)
- $\overline{\Gamma}$, $\overline{\Gamma_\gamma}$: The unitial quantal obtained from Γ and Γ_γ respectively (Definition 2.3).
- $\mathrm{R}(\Gamma)$, $\mathrm{R}(\Gamma_\gamma)$: The quantaloid of relations on Γ and Γ_γ respectively (Definition 2.5)
- $\mathrm{CAT}(\overline{\Gamma}), \mathrm{CAT}(\overline{\Gamma_\gamma}), \mathrm{CAT}(\mathrm{R}(\Gamma)), \mathrm{CAT}(\mathrm{R}(\Gamma_\gamma))$: Skeletal symmetric categories enriched in the quantaloids $\overline{\Gamma}$, $\overline{\Gamma_\gamma}$, $\mathrm{R}(\Gamma)$ and $\mathrm{R}(\Gamma_\gamma)$ respectively. (Definition 2.6)

- $\mathrm{cCAT}(\overline{\Gamma}), \mathrm{cCAT}(\mathrm{R}(\Gamma))$: Cauchy complete skeletal symmetric categories enriched in $\overline{\Gamma}$ and $\mathrm{R}(\Gamma)$ respectively. (Definition 2.14)
- $\mathrm{Flab}(\Gamma)$: Flabby separated presheaves on Γ. (Definition 3.10)
- $\mathrm{Sep}(\Gamma)$: Separated presheaves on Γ. (Definition 3.2)
- $\mathrm{Sh}(\Gamma)$: Sheaves on Γ. (Definition 3.2)

Below are the places where the functors are defined:

- $IC_{\overline{\Gamma}}, IC_{\mathrm{R}(\Gamma)}$: Definition 2.14. $CC_{\overline{\Gamma}}, CC_{\mathrm{R}(\Gamma)}$: Lemma 2.15.
- $\widehat{B}_{\overline{\Gamma},\overline{\Gamma_\gamma}}$: Definition 2.10. $\widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}$: Subsection 3.1. $\widehat{IN}_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)}$: Subsection 3.3.
- $\widehat{IN}^*_{\overline{\Gamma},\mathrm{R}(\Gamma)}$, $\widehat{IN}^*_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)}$: Lemma 2.8.
- P_Γ: Definition 3.3. Q_Γ: Lemma 3.7.
- a_Γ, i_Γ: Definition 3.2. sp_Γ, fl_Γ: Lemma 3.11.
- F_Γ: Definition 3.13. FL_Γ: Subsection 3.3. S_Γ: Lemma 3.5.

Here is a list of the functor isomorphisms which are proved in this paper along with the places where they are proved.

- $\widehat{IN}^*_{\overline{\Gamma},\mathrm{R}(\Gamma)} \circ IC_{\mathrm{R}(\Gamma)} \circ CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)} \cong IC_\Gamma \circ CC_\Gamma$: Lemma 3.1.
- $CC_{\mathrm{R}(\Gamma)} \circ P_\Gamma \cong S_\Gamma \circ a_\Gamma$: Lemma 3.6.
- $CC_{\mathrm{R}(\Gamma)} \cong S_\Gamma \circ a_\Gamma \circ Q_\Gamma$: Corollary 3.8.
- F_Γ is an equivalence of categories: Theorem 3.12.
- $FL_{\Gamma_\gamma} \circ \widehat{B}_{\overline{\Gamma_\gamma},\overline{\Gamma}} \cong \widehat{IN}^*_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)} \circ FL_\Gamma$: Lemma 3.21.
- $CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)} \cong CC_{\mathrm{R}(\Gamma)} \circ FL_\Gamma$: Theorem 4.1.

1.2. **Background.** In this paper we will work in a fixed background model, SET, of Zermelo-Frankael set theory. We will also abuse notation and refer to the category of sets and functions in our background model as SET (when no confusion can arise). In general we will not assume the axiom of choice unless it is necessary. If a result does use the axiom of choice we will mark it by $(*)$. For any set theoretic ideas which are not explicitly mentioned the reader is referred to such standard works as [5].

All categories in this paper will be locally small. We will use the convention that when C is a category with objects A and B, $C[A, B]$ is the set of morphisms whose domain is A and whose codomain is B.[1] We also let C_o be the collection of objects of C. The reader is referred to such standard texts

[1] This will be consistent with our notation for enriched categories in Section 2.

as [7] for general category theoretic notions, to [8] for general sheaf theoretic notions and to [9] and [10] for information about quantaloids and categories enriched in a quantaloid.

2. Enriched Categories

In this section we review some of the basic theory of categories enriched in a quantaloid which we will use.

2.1. **Quantaloids.** In what follows we will be interested in quantaloid enriched categories where the quantaloids have specific properties.

Definition 2.1. *We call a quantaloid* $\mathcal{Q}$ **symmetric** *if*

- *For each pair of* **objects** $X, Y \in \mathcal{Q}$, $\mathcal{Q}[X,Y] = \mathcal{Q}[X,Y]$.
- *For each* $f \in \mathcal{Q}[X,Y]$ *and* $g \in \mathcal{Q}[Y,Z]$, $g \circ f = f \circ g$.

If there is a frame Γ *and for all objects* $X, Y \in \mathcal{Q}_o$ *an injection* $i_{X,Y} : \mathcal{Q}[X,Y] \to \Gamma$ *such that*

- $(\forall a, b \in \mathcal{Q}[X,Y])\ i_{X,Y}(a \wedge_{\mathcal{Q}} b) = i_{X,Y}(a) \wedge_\Gamma i_{X,Y}(b)$.
- $(\forall a \in \mathcal{Q}[X,Y])(\forall b \in \mathcal{Q}[Y,Z]) i_{X,Z}(a \circ_{\mathcal{Q}} b) = i_{X,Y}(a) \wedge_\Gamma i_{Y,Z}(b)$.

then we say $\mathcal{Q}$ *is* **frame like**.

All symmetric quantaloids in this paper will be small (i.e. the set of all morphisms will be a set in SET).

Definition 2.2. *A* **symmetric unitial quantale**, $\mathcal{Q}$, *is a symmetric quantaloid with only one object (which we denote* $*_{\mathcal{Q}}$*).*

Frames and frame like symmetric unitial quantales are essentially the same things.

Definition 2.3. *If* Γ *is a frame let* $\overline{\Gamma}$ *be the symmetric unitial quantale such that* $\overline{\Gamma}[*_{\overline{\Gamma}}, *_{\overline{\Gamma}}] = \Gamma$ *and* $(\forall a, b \in \Gamma)\ a \circ b = a \wedge b$, *and* $id_{*_{\overline{\Gamma}}} = \top_\Gamma$.

Lemma 2.4. *For any frame like symmetric unitial quantale* G, $\overline{G[*_G, *_G]} \cong G$ *(as quantaloids). For any frame* Γ, $\overline{\Gamma}[*_{\overline{\Gamma}}, *_{\overline{\Gamma}}] \cong \Gamma$.

Definition 2.5. *The* **quantaloid of relations** *on* Γ, $R(\Gamma)$, *is the quantaloid where:*

- $R(\Gamma)_o = \Gamma$.

- *$R(\Gamma)[a,b] = \{w \in \Gamma : w \leq a \wedge b\}$.*
- *$R(\Gamma)[a,b](c,d) = \Gamma[c,d]$ (in particular $c \leq d$ in $R(\Gamma)[a,b]$ if and only if $c \leq d$ in Γ).*
- *If $\alpha \in R(\Gamma)[a,b]$ and $\beta \in R(\Gamma)[b,c]$ then $\beta \circ \alpha = \alpha \wedge \beta$.*

It is easy to see that $\mathrm{R}(\Gamma)$ is always frame like and symmetric. From here on Γ will always be a frame and $\mathcal{Q}$ will always be a symmetric quantaloid.

2.2. **Quantaloid Enriched Categories.** Our notation in this section is very similar to that of [10]. In particular if $\mathcal{Q}$ is a symmetric quantaolid and $\mathbb{A}$ is a $\mathcal{Q}$-enriched category then $\mathbb{A}_o$ is the collection of objects of $\mathbb{A}$ and $\rho_{\mathbb{A}} : \mathbb{A}_o \to \mathcal{Q}_o$ is the the function which takes an object of $\mathbb{A}$ and returns its **type**, an object of $\mathcal{Q}$. For $q \in \mathcal{Q}_o$ we will use the shorthand $\mathbb{A}_o(q) = \rho_{\mathbb{A}}^{-1}(q)$.

Definition 2.6. *Suppose $\mathcal{Q}$ is a symmetric quantaloid and $\mathbb{A}$ is a $\mathcal{Q}$-enriched category. If $\mathbb{A}$ satisfies*

(Symmetry) $\mathbb{A}[x,y] = \mathbb{A}[y,x]$ for all $x, y \in \mathbb{A}_o$.

we say $\mathbb{A}$ is **symmetric***.*

If $\mathbb{A}$ further satisfies

- *$\mathbb{A}[x,y] \wedge \mathbb{A}[y,x] \geq id_q \leftrightarrow x = y$ (whenever $\rho_{\mathbb{A}}(x) = q = \rho_{\mathbb{A}}(y)$).*

then we say $\mathbb{A}$ is **skeletal***.*

We let $\mathrm{CAT}(\mathcal{Q})$ be the category of skeletal symmetric $\mathcal{Q}$-enriched categories and $\mathcal{Q}$-functors. In this paper by a **$\mathcal{Q}$-category** we will always mean a skeletal symmetric $\mathcal{Q}$-enriched category.

The following straight forward lemmas will be useful later on.

Lemma 2.7. *Suppose $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are frame like symmetric quantaloids and $I : \mathcal{Q}_0 \to \mathcal{Q}_1$ is a map of quantaloids, i.e. a functor of the underlying categories which preserves supreme. Then I induces a functor $\hat{I} : CAT(\mathcal{Q}_0) \to CAT(\mathcal{Q}_1)$. Further if I is injective $\hat{I}$ is injective.*

Proof. For each $\mathbb{A} \in \mathrm{CAT}(\mathcal{Q}_0)$:

- When $a, b \in \mathbb{A}_o$, let $a \sim_{\hat{I}} b$ if $I(\rho_{\mathbb{A}}(a)) = q = I(\rho_{\mathbb{A}}(b))$ and $I(\mathbb{A}[a,b]) \geq id_q$.
- $\hat{I}(\mathbb{A})_o = \mathbb{A}_o / \sim_{\hat{I}}$.

- $\rho_{\hat{I}(\mathbb{A})}(a) = I(\rho_{\mathbb{A}}(a))$ for all $a \in \mathbb{A}_o$.
- $\hat{I}(\mathbb{A})[a,b] = I(\mathbb{A}[a,b])$ for all $a, b \in \mathbb{A}_o$.

We also let $\hat{I}(f) = f$ for all $\mathcal{Q}_0$-functors f.

It is then easily checked that $\sim_{\hat{I}}$ is an equivalence relation, that $\hat{I}(\mathbb{A})[a,b] = \hat{I}(\mathbb{A})[a',b']$ whenever $a \sim_{\hat{I}} a'$ and $b \sim_{\hat{I}} b'$ and hence that $\hat{I}$ is a functor. It is also easily checked that $\hat{I}$ is injective when I is injective. □

If I is an inclusion functor (i.e. $I : \mathcal{Q}_0 \subseteq \mathcal{Q}_1$) then $\hat{I}$ is also an inclusion functor. In general though for an injective I we will only have that $\hat{I}$ is isomorphic to an inclusion functor. However in this case we will still treat $\hat{I}$ as an inclusion functor when no confusion can arise (i.e. we won't distinguish between $\mathcal{Q}_0$-categories and their image under functors of the form $\hat{I}$).

Lemma 2.8. *If I (as in Lemma 2.7) is injective, full and faithful then $\hat{I}$ has a right adjoint, $\hat{I}^*$.*

Proof. Let X be the collection of objects in the range of I. For a skeletal symmetric $\mathcal{Q}_1$-category $\mathbb{A}$ we define the $\mathcal{Q}_0$-category

- $\hat{I}^*(\mathbb{A})_o = \bigcup_{q \in X} \mathbb{A}_o(q)$.
- $\rho_{\hat{I}^*(\mathbb{A})}(a) = \rho_{\mathbb{A}}(a)$ for all $a \in \hat{I}^*(\mathbb{A})_o$.
- $\hat{I}^*(\mathbb{A})[a,b] = \mathbb{A}[a,b]$ for all $a, b \in \hat{I}^*(\mathbb{A})_o$.

For a $\mathcal{Q}$-functor $f : \mathbb{A} \to \mathbb{B}$ we let $\hat{I}^*(f)$ be the restriction of f to $\hat{I}^*(\mathbb{A})_o$.

It is then immediate that $\hat{I}^*$ is right adjoint to $\hat{I}$ where the counit is the inclusion $\varepsilon_{\mathbb{A}} : (\hat{I} \circ \hat{I}^*(\mathbb{A}))_o \subseteq (\mathbb{A})_o$ and the unit $\eta_{\mathbb{A}} = id_{\mathbb{A}}$. □

2.2.1. *Symmetric Unitial Quantale Enriched Categories.* It is easy to see that if Γ is a frame then $\mathrm{CAT}(\overline{\Gamma})$ is equivalent to the category of generalized ultrametric spaces and non-expanding maps where the distance function in the generalized ultrametric spaces takes values in Γ instead of $\mathbb{R}^{\geq 0}$ (for more on this isomorphism see [1]). Motivated by this equivalence we introduce the closed ball functor.

Definition 2.9. *Suppose $\mathcal{Q}$ is a symmetric unitial quantale and $\mathbb{A}$ is a $\mathcal{Q}$-category. For $a \in \mathbb{A}_o$ and $\gamma \in \mathcal{Q}[*_{\mathcal{Q}}, *_{\mathcal{Q}}]$ we define the* **closed ball around** *a* **of radius** *γ to be the set $B^{\mathbb{A}}(a, \gamma) = \{r \in \mathbb{A}_o : \mathbb{A}[a,r] \geq \gamma\}$.*

We will omit superscripts on closed balls when it is clear which enriched category the balls are in.

Definition 2.10. *For $\gamma \in \Gamma$ let $\Gamma_\gamma = \{\zeta \in \Gamma : \zeta \leq \gamma\}$ and let $B_{\overline{\Gamma_\gamma},\overline{\Gamma}} : \overline{\Gamma} \to \overline{\Gamma_\gamma}$ be the map given by $B_{\overline{\Gamma_\gamma},\overline{\Gamma}}(\zeta) = \zeta \wedge \gamma$.*

Note that $B_{\overline{\Gamma_\gamma},\overline{\Gamma}}$ is a map of unitial quantals as $B_{\overline{\Gamma_\gamma},\overline{\Gamma}}(\bigvee_{a\in A} a) = \gamma \wedge \bigvee_{a\in A} a = \bigvee_{a\in A}(a \wedge \gamma) = \bigvee_{a\in A} B_{\overline{\Gamma_\gamma},\overline{\Gamma}}(a)$.

Definition 2.11. *If $\mathbb{A}$ is a $\overline{\Gamma}$-category and $\gamma \in \Gamma$ we let $\mathbb{A}_\gamma$ be the $\overline{\Gamma_\gamma}$-category such that*

- $(\mathbb{A}_\gamma)_o = \{B^{\mathbb{A}}(x, \gamma) : x \in \mathbb{A}_o\}$.
- $\mathbb{A}_\gamma[B^{\mathbb{A}}(x, \gamma), B^{\mathbb{A}}(y, \gamma)] = \mathbb{A}[x, y] \wedge \gamma$.

If $f : \mathbb{A} \to \mathbb{B}$ is a map of $\overline{\Gamma}$-categories, let $f_\gamma : \mathbb{A}_\gamma \to \mathbb{B}_\gamma$ be such that

$$f(B^{\mathbb{A}}(x, \gamma)) = B^{\mathbb{B}}(f(x), \gamma).$$

It is easy to check that in this situation $\mathbb{A}_\gamma$ is an $\overline{\Gamma_\gamma}$-category and f is a $\overline{\Gamma_\gamma}$-functor. Further the map $(\cdot)_\gamma$ is isomorphic to $\widehat{B}_{\overline{\Gamma_\gamma},\overline{\Gamma}}$. We end with a lemma on closed balls.

Lemma 2.12. *Let*

- $\{x_i : i \in I\} \subseteq \mathbb{A}_o$ *and* $\{\gamma_i : i \in I\} \subseteq \Upsilon$.
- $\alpha = \bigvee\{\gamma_i : i \in I\}$ *and* $\beta = \bigwedge\{\gamma_i : i \in I\}$.
- $B = \bigcap_{i\in I} B(x_i, \gamma_i)$.

If $B \neq \emptyset$ then

(1) $(\forall i, j \in I) B(x_i, \beta) = B(x_j, \beta)$.
(2) $(\forall x \in B) B = B(x, \alpha)$.

Proof. Let $x \in B$. To see (1) observe that $x \in B(x_i, \gamma_i)$ and $x \in B(x_j, \gamma_j)$ so $B(x_i, \beta) = B(x, \beta) = B(x_j, \beta)$.

To see (2) let $\zeta_B = \bigwedge\{\mathbb{A}[x, y] : x, y \in B\}$. As $(\forall i \in I) x \in B(x_i, \gamma_i)$ we have $B(x, \gamma_i) = B(x_i, \gamma_i)$. Hence $(\forall i \in I) B \subseteq B(x, \gamma_i)$ and $\zeta_B \leq \gamma_i$. Therefore $\zeta_B \leq \alpha$ and in particular $B \subseteq B(x, \alpha)$. But we also have $(\forall i \in I) B(x, \alpha) \subseteq B(x, \gamma_i)$ and so $B(x, \alpha) \subseteq B$. Therefore (2) holds. □

2.3. Cauchy Completeness in Quantaloid Enriched Categories.

2.3.1. *Definitions.*

Definition 2.13. *Suppose $\mathcal{Q}$ is a symmetric quantaloid and $\mathbb{A}$ is a symmetric $\mathcal{Q}$-enriched category. For $q \in \mathcal{Q}$ we define a* **virtual q-element** *to be a function φ with domain $\mathbb{A}_o$ satisfying:*

(a) $(\forall x \in \mathbb{A}_o)\varphi(x) \in \mathcal{Q}[q, \rho_{\mathbb{A}}(x)]$.
(b) $(\forall x, y \in \mathbb{A}_o)\mathbb{A}[x, y] \circ \varphi(x) \leq \varphi(y)$.
(c) $(\forall x, y \in \mathbb{A}_o)\varphi(x) \circ \varphi(y) \leq \mathbb{A}[y, x]$.
(d) $\bigvee_{x \in \mathbb{A}_o} \varphi(x) \geq id_q$.

If φ is a virtual q-element, we say φ is **realized** *if there is an element $\phi \in \mathbb{A}_o(q)$ such that $(\forall x \in \mathbb{A}_o)\varphi(x) = \mathbb{A}[\phi, x]$.*

Definition 2.14. *If $\mathcal{Q}$ is a frame like symmetric quantaloid and $\mathbb{A}$ is a $\mathcal{Q}$-category then $\mathbb{A}$ is* **Cauchy complete** *if every virtual element is realized. We let $cCAT(\mathcal{Q})$ be the full subcategory of $CAT(\mathcal{Q})$ consisting of Cauchy complete $\mathcal{Q}$-categories with inclusion functor $IC_{\mathcal{Q}} : CAT(\mathcal{Q}) \to cCAT(\mathcal{Q})$.*

The notion of a Cauchy complete category was first introduced by Lawvere in [6] and has been extended to categories enriched in quantaloids (see [10]). The general definition is that a $\mathcal{Q}$-enriched category $\mathbb{A}$ is Cauchy complete if, for every $\mathcal{Q}$-enriched category $\mathbb{B}$, every adjoint pair of distributors from $\mathbb{B}$ to $\mathbb{A}$ comes from an actual $\mathcal{Q}$-functor from $\mathbb{B}$ to $\mathbb{A}$. As such it is worth taking a moment to discuss why the general definition reduces to Definition 2.14 when $\mathcal{Q}$ is frame like and symmetric and $\mathbb{A}$ is a $\mathcal{Q}$-category.

First observe that when considering Cauchy completeness it suffices to restrict our attention to the case when $\mathbb{B}$ is of the form $\hat{q}$, where $\hat{q}$ is the $\mathcal{Q}$-category with a unique element $*$ such that $\rho_{\hat{q}}(*) = q$ and $\hat{q}[*, *] = id_q$.[2] For a proof see Proposition 7.1 of [10].

Now suppose (φ_l, φ_r) is a pair of adjoint distributors from $\hat{q}$ to $\mathbb{A}$. We know that $\mathbb{A}$ has a Cauchy completion which we can call $\mathbb{A}_{cc}$ and that there is an element $\phi \in \mathbb{A}_{cc}(q)$ such that $(\forall x \in (\mathbb{A}_{cc})_o)$ $\varphi_l(x) = \mathbb{A}_{cc}[\phi, x]$ and $\varphi_r(x) = \mathbb{A}_{cc}[x, \phi]$.

Because $\mathcal{Q}$ is frame like and symmetric it is immediate that $\mathcal{Q}$ satisfies the **modular law**. In other words for all $u, v, w \in \mathcal{Q}_o$,

$$(\forall \alpha \in \mathcal{Q}[u, v])(\forall \beta \in \mathcal{Q}[v, w])(\forall \tau \in \mathcal{Q}[u, w])\tau \wedge (\beta \circ \alpha) \leq \beta \circ ((\beta \circ \tau) \wedge \alpha)$$

[2] These are called **Cauchy presheaves** in [10].

But then by [3] (the First Theorem of Section 3) we have that $\mathbb{A}_{cc}$ is symmetric and hence $(\forall x \in \mathbb{A}_o)\varphi_l(x) = \varphi_r(x)$.

We then have the following

- φ is a virtual q-element of $\mathbb{A}$ if and only if (φ, φ) is an adjoint pair of distributors from $\hat{q}$ to $\mathbb{A}$.
- Every adjoint pair of distributors $(\varphi_l, \varphi_r) : \hat{q} \rightleftarrows \mathbb{A}$ is of the form (φ, φ) for a virtual q-element of A.
- A virtual element φ is realized if and only if the adjoint pair of distributors (φ, φ) comes from a functor.

This implies that when $\mathcal{Q}$ is a frame like symmetric quantaloid and $\mathbb{A}$ is a $\mathcal{Q}$-category $\mathbb{A}$ is Cauchy complete in the sense of [10] if and only if it is Cauchy complete in the sense of Definition 2.14. The following lemma therefore follows from know facts about Cauchy complete categories.

Lemma 2.15. *The inclusion functor $IC_{\mathcal{Q}}$ has a left adjoint $CC_{\mathcal{Q}}$.*

In particular, if $\mathbb{A}$ is a $\mathcal{Q}$-category and $\mathbb{A}_{cc} = IC_{\mathcal{Q}} \circ CC_{\mathcal{Q}}(\mathbb{A})$ then

- If $q \in \mathcal{Q}_o$ then $\mathbb{A}_{cc}(q) = \{\varphi : \varphi \text{ is a virtual } q\text{-element}\}$.
- If $a \in \mathbb{A}_{cc}(q)$ and $b \in \mathbb{A}_{cc}(p)$ then $\mathbb{A}_{cc}[a, b] = \bigvee\{a(x) \circ b(x) : x \in \mathbb{A}_o\}$.

(see [10] Proposition 7.7 for a proof).

3. $\overline{\Gamma}$-Categories and R(Γ)-Categories

3.1. **Inclusion of $\overline{\Gamma}$ in R(Γ).** Notice that there is an inclusion of quantaloids $IN_{\overline{\Gamma},\mathrm{R}(\Gamma)} : \overline{\Gamma} \to \mathrm{R}(\Gamma)$ where $IN_{\overline{\Gamma},\mathrm{R}(\Gamma)}(*_{\overline{\Gamma}}) = \top_\Gamma$ and $IN_{\overline{\Gamma},\mathrm{R}(\Gamma)}$ is the identity on $\overline{\Gamma}[*_{\overline{\Gamma}}, *_{\overline{\Gamma}}]$.

Lemma 3.1. *The two functors $\widehat{IN}^*_{\overline{\Gamma},R(\Gamma)} \circ IC_{R(\Gamma)} \circ CC_{R(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},R(\Gamma)}$ and $IC_\Gamma \circ CC_\Gamma$ (from $CAT(\overline{\Gamma})$ to $CAT(\overline{\Gamma})$) are isomorphic.*

Proof. Suppose $\mathbb{A} \in \mathrm{CAT}(\overline{\Gamma})$ and let $\mathbb{A}^* = \widehat{IN}^*_{\overline{\Gamma},\mathrm{R}(\Gamma)} \circ IC_{\mathrm{R}(\Gamma)} \circ CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(\mathbb{A})$. Then $\mathbb{A}^*_o = \{\varphi : \varphi \text{ is a virtual } \top_\Gamma\text{-element of } \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(\mathbb{A})\}$. Further for $\varphi, \psi \in \mathbb{A}^*_o$ we also have $\mathbb{A}^*[\varphi, \psi] = \bigvee\{\varphi(x) \wedge \psi(x) : x \in \mathbb{A}_o\}$.

However φ is a virtual $\top_\Gamma$-element of $\widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(\mathbb{A})$ if and only if φ is a virtual $*_{\overline{\Gamma}}$-element of $\mathbb{A}$. Hence $\mathbb{A}^*_o = CC_\Gamma(\mathbb{A})_o$ and the identity is an isomorphism of $\overline{\Gamma}$-categories between $\mathbb{A}^*$ and $CC_\Gamma(\mathbb{A})$.

Lastly, notice that for any Γ-functor $f : \mathbb{A} \to \mathbb{B}$ and any $a \in \mathbb{A}_o$,

$f(a) = \widehat{IN}^*_{\overline{\Gamma},\mathrm{R}(\Gamma)} \circ IC_{\mathrm{R}(\Gamma)} \circ CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(f)(a) = IC_\Gamma \circ CC_\Gamma(f)(a)$. Hence $\widehat{IN}^*_{\overline{\Gamma},\mathrm{R}(\Gamma)} \circ IC_{\mathrm{R}(\Gamma)} \circ CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(f) = IC_\Gamma \circ CC_\Gamma(f)$ as $\overline{\Gamma}$-functors (because the domain is the Cauchy completion of $\mathbb{A}$).

We therefore have $\widehat{IN}^*_{\overline{\Gamma},\mathrm{R}(\Gamma)} \circ IC_{\mathrm{R}(\Gamma)} \circ CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)} \cong IC_\Gamma \circ CC_\Gamma$ as functors. □

Lemma 3.1 tells us that if we start with a $\overline{\Gamma}$-category, then transform it into a $\mathrm{R}(\Gamma)$-category, take the Cauchy completion (in $\mathrm{CAT}(\mathrm{R}(\Gamma))$) and then return to $\mathrm{CAT}(\overline{\Gamma})$ we get the same result as just taking the Cauchy completion in $\mathrm{CAT}(\overline{\Gamma})$.

3.2. **Separated Presheaves and Sheaves.**

Definition 3.2. *Let $Sh(\Gamma)$ be the category of sheaves on Γ, let $Sep(\Gamma)$ be the full subcategory of separated presheaves, let $i_\Gamma : Sep(\Gamma) \to Sh(\Gamma)$ be the inclusion functor and let $a_\Gamma : Sh(\Gamma) \to Sep(\Gamma)$ be the sheafification functor.*

Definition 3.3. *Suppose X is a separated presheaf on Γ. We let $P_\Gamma(X)$ be the $R(\Gamma)$-category where*

- $P_\Gamma(X)_o = \bigcup_{\gamma\in\Gamma}\{\langle x,\gamma\rangle : x \in X(\gamma)\}$.
- $P_\Gamma(X)[\langle a,\gamma_a\rangle,\langle b,\gamma_b\rangle] = \bigvee\{\gamma \in \Gamma : a|_\gamma = b|_\gamma\}^3$ *for all $a, b \in P_\Gamma(X)$.*

Further if $f : X \to Y$ is a map in $Sep(\Gamma)$ we let $P_\Gamma(f)(\langle a,\gamma\rangle) = \langle f_\gamma(a),\gamma\rangle$.[4]

The following are then immediate from the literature (specifically [2] and [11])

Lemma 3.4. *P_Γ is a full and faithful functor from $Sep(\Gamma)$ into $CAT(R(\Gamma))$.*

Lemma 3.5. *$S_\Gamma = CC_{R(\Gamma)} \circ P_\Gamma \circ i_\Gamma$ is an equivalence of categories between $Sh(\Gamma)$ and $cCAT(R(\Gamma))$.*

Lemma 3.6. *$CC_{R(\Gamma)} \circ P_\Gamma \cong S_\Gamma \circ a_\Gamma$.*

In particular, a separated presheaf on Γ is a sheaf if and only if it's image under P_Γ is Cauchy complete.

Lemma 3.7. *P_Γ has a left adjoint Q_Γ with $Q_\Gamma \circ P_\Gamma \cong id_{Sep(\Gamma)}$.*

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[3]If X is a presheaf on Γ, $\gamma,\gamma' \in \Gamma$ with $\gamma \le \gamma'$, and $x \in X(\gamma')$ then $x|_\gamma = X(i_{\gamma,\gamma'})(x) \in X(\gamma)$, i.e. the restriction of x to γ.

[4]If $f : X \to Y$ is a map of presheaves on Γ then $f_\gamma : X(\gamma) \to Y(\gamma)$ is the γ component.

Proof. For each $\mathbb{A} \in \mathrm{CAT}(\mathrm{R}(\Gamma))$ and $\gamma \in \Gamma$ let $X_\gamma = \{\langle x, \gamma\rangle : x \in \mathbb{A}_o(\zeta), \zeta \geq \gamma\}$ and let $Q_\Gamma(\mathbb{A})(\gamma) = X_\gamma / \sim_\gamma$ where $\langle x, \gamma\rangle \sim_\gamma \langle y, \gamma\rangle$ if and only if $\mathbb{A}[x, y] \geq \gamma$[5]. If $\langle x, \gamma\rangle \in Q_\Gamma(\mathbb{A})$ and $\gamma' \leq \gamma$ we let $\langle x, \gamma\rangle|_{\gamma'} = \langle x, \gamma'\rangle$. It is then easily checked that $Q_\Gamma(\mathbb{A})$ is a presheaf.

Now if $\langle x, \gamma\rangle$ and $\langle y, \gamma\rangle$ are both covered by a compatible collection of elements $\{\langle\langle z_\eta, \eta\rangle, \eta\rangle : \eta \in I \subseteq \Gamma\}$ then $\bigvee I = \gamma$. But this means that $\langle x, \gamma\rangle|_\eta = \langle z_\eta, \eta\rangle = \langle y, \gamma\rangle|_\eta$ for all $\eta \in I$. Hence $\mathbb{A}[x, y] \geq \eta$ for all $\eta \in I$ and $\mathbb{A}[x, y] \geq \bigvee I = \gamma$. But this implies that $\langle x, \gamma\rangle \sim_\gamma \langle y, \gamma\rangle$ and hence they belong to the same equivalence class of $Q_\Gamma(\mathbb{A})$. In particular this implies that $Q_\Gamma(\mathbb{A})$ is separated.

Next for $f : \mathbb{A} \to \mathbb{C}$ we define $Q_\Gamma(\mathbb{A})(f)(\langle x, \gamma\rangle) = (\langle f(x), \gamma\rangle)$. It is immediate that $Q_\Gamma(\mathbb{A})(f) : Q_\Gamma(\mathbb{A}) \to Q_\Gamma(\mathbb{C})$ is a map of separated presheaves. Hence $Q_\Gamma : \mathrm{CAT}(\mathrm{R}(\Gamma)) \to \mathrm{Sep}(\Gamma)$ is a functor.

Now for the counit and unit. For $X \in \mathrm{Sep}(\Gamma)$ we let $\varepsilon_X : Q_\Gamma \circ P_\Gamma(X) \to X$ be the map where $(\forall x \in X(\gamma))(\varepsilon_X)_\gamma(\langle\langle x, \gamma\rangle, \gamma\rangle) = x$. For a $\mathrm{R}(\Gamma)$-category $\mathbb{A}$ we let $\eta_\mathbb{A} : \mathbb{A} \to P_\Gamma \circ Q_\Gamma(\mathbb{A})$ be the map where $(\forall x \in \mathbb{A}_o)\eta_\mathbb{A}(x) = \langle\langle x, \gamma\rangle, \gamma\rangle$.

It is then easily checked that ε and η are the counit and unit of an adjunction $Q_\Gamma \dashv P_\Gamma$ and that ε is a natural isomorphism witnessing $Q_\Gamma \circ P_\Gamma \cong id_{\mathrm{Sep}(\Gamma)}$. □

Corollary 3.8. $CC_{R(\Gamma)} \cong S_\Gamma \circ a_\Gamma \circ Q_\Gamma$.

We end this section with a lemma which will be important later on.

Lemma 3.9. *Suppose X is a separated Γ presheaf, $\mathbb{A} = P_\Gamma(X)$ and $\gamma \in \Gamma$. Then*

(a) $(\forall x \in \mathbb{A}_o)(\exists x_\gamma \in \mathbb{A}_o(\rho_\mathbb{A}(x) \wedge \gamma))\mathbb{A}[x, x_\gamma] = \rho_\mathbb{A}(x_\gamma) = \rho_\mathbb{A}(x) \wedge \gamma$. *Further x_γ is unique.*

(b) *If X is flabby then $(\forall x \in \mathbb{A}_o)(\exists y \in \mathbb{A}_o(\top))\mathbb{A}[x, y] = \rho_\mathbb{A}(y)$ (y may not be unique).*

(c) *For all $x, x' \in \mathbb{A}_o$, if φ is a virtual γ-element in $\mathbb{A}$ then $\mathbb{A}[x, x'] \wedge \varphi(x') = \mathbb{A}[x, x'] \wedge \varphi(x)$.*

(d) *For all $x \in \mathbb{A}_o$, if φ is a virtual γ-element of $\mathbb{A}$ and x_γ is such that $\mathbb{A}[x, x_\gamma] = \rho_\mathbb{A}(x_\gamma) = \rho_\mathbb{A}(x) \wedge \gamma$ then $\varphi(x) = \varphi(x_\gamma)$.*

[5]Note it is immediate from (Transitivity) that $\sim_\gamma$ is an equivalence relation.

Proof. Parts (a) and (b) follow immediately from the properties of presheaves and flabby presheaves. (c) follows immediately from properties of virtual elements. (d) holds because $\varphi(x_\gamma) = \varphi(x) \wedge \rho_{\mathbb{A}}(x_\gamma) = \varphi(x) \wedge \rho_{\mathbb{A}}(x) \wedge \gamma$. But by construction $\varphi(x) \leq \rho_{\mathbb{A}}(x) \wedge \gamma$ so $\varphi(x_\gamma) = \varphi(x)$. □

3.3. **Flabby Presheaves.**

Definition 3.10. *We let Flab*(Γ) *be the full subcategory Sep*(Γ) *whose objects are the flabby separated presheaves. We let FlabSh*(Γ) *to be the full subcategory of Sh*(Γ) *whose objects are flabby sheaves.*

Flab(Γ) is a reflexive subcategory of Sep(Γ).

Lemma 3.11. *The inclusion map* $sp_\Gamma : \mathit{Flab}(\Gamma) \to \mathit{Sep}(\Gamma)$ *has a right adjoint* $fl_\Gamma : \mathit{Sep}(\Gamma) \to \mathit{Flab}(\Gamma)$

Proof. For any separated presheaf X let $fl_\Gamma(X)(\gamma) = \{a \in X(\gamma) : (\exists b \in X(\top))b|_\gamma = a\}$. For any map $\alpha : X \Rightarrow Y$ and any $x \in X(\gamma)$ let $fl_\Gamma(\alpha)(x) = \alpha(x)$ (i.e. $fl_\Gamma(\alpha)$ is the restriction of α to $fl_\Gamma(X)$). It is then easily checked that fl_Γ is a right adjoint to the inclusion map sp_Γ. □

It is worth pointing out that fl_Γ does not restrict to a functor between Sh(Γ) and FlabSh(Γ). The reason is that it is possible to have a flabby sheaf X and a compatible collection $\langle (a_i, \gamma_i) : i \in I\rangle$ of elements of X where $\bigvee^\Gamma\{\gamma_i : i \in I\} < \top$ and for each a_i there is an $a_i' \in X(\top)$ such that $a_i'|_{\gamma_i} = a_i$, but where there is no $a \in X(\top)$ such that $a|_{\gamma_i} = a_i$ for all $i \in I$.

Theorem 3.12. *There is an equivalence of categories between Flab*(Γ) *and CAT*$(\overline{\Gamma})$.

Proof. We define maps $G_\Gamma : \mathrm{Flab}(\Gamma) \to \mathrm{CAT}(\overline{\Gamma})$ and $F_\Gamma : \mathrm{CAT}(\overline{\Gamma}) \to \mathrm{Flab}(\Gamma)$ which form an equivalence of categories.

We begin by defining the functor F_Γ.

Definition 3.13. *If* $\mathbb{A} \in \mathit{CAT}(\overline{\Gamma})$ *then we define* $F_\Gamma(\mathbb{A})(\gamma) = \mathbb{A}_\gamma$ *and* $B(a,\gamma)|_{\gamma^*} = B(a,\gamma^*)$ *for all* $\gamma^* \leq \gamma$.

If $\mathbb{A}, \mathbb{C} \in \mathit{CAT}(\overline{\Gamma})$ *and* $f : \mathbb{A} \to \mathbb{C}$ *is a* $\overline{\Gamma}$*-functor then for all* $B(a,\gamma) \in F_\Gamma(\mathbb{A})(\gamma)$ *let* $F_\Gamma(f)_\gamma(B(a,\gamma)) = B(f(a),\gamma)$.

Claim 3.14. *If* $\mathbb{A} \in \mathit{CAT}(\overline{\Gamma})$ *then* $F_\Gamma(\mathbb{A}) \in \mathit{Flab}(\Gamma)$.

Proof. First, in order to show $F_\Gamma(\mathbb{A})$ is a presheaf, we need to check that restriction is well defined. Suppose $B(a,\gamma) = B(b,\gamma) \in F_\Gamma(\mathbb{A})(\gamma)$, $\gamma^* \le \gamma$ and $x \in B(a,\gamma^*)$ (i.e. $\mathbb{A}[x,a] \ge \gamma^*$). $\mathbb{A}[a,b] \ge \gamma \ge \gamma^*$, so $\mathbb{A}[x,b] \ge \mathbb{A}[x,a] \wedge \mathbb{A}[a,b] \ge \gamma^* \wedge \gamma = \gamma^*$ and therefore $x \in B(b,\gamma^*)$. Hence $B(a,\gamma^*) = B(b,\gamma^*)$ and restriction is well defined.

To see that $F_\Gamma(\mathbb{A})$ is flabby notice that if $B(a,\gamma) \in F_\Gamma(\mathbb{A})(\gamma)$ then $B(a,\top) = \{a\} \in F_\Gamma(\mathbb{A})(\top)$ and $\{a\}|_\gamma = B(a,\gamma)$.

Finally, to see $F_\Gamma(\mathbb{A})$ is separated, suppose $\gamma = \bigvee_{i\in I} \lambda_i$ and $\boldsymbol{B} = \{B(x_i,\lambda_i) \in F_\Gamma(\mathbb{A})(\lambda_i) : i \in I\}$ is such that $B(x_i,\lambda_i)|_{\lambda_i\wedge\lambda_j} = B(x_i,\lambda_i\wedge\lambda_j) = B(x_j,\lambda_i\wedge\lambda_j) = B(x_j,\lambda_j)|_{\lambda_i\wedge\lambda_j}$. Further suppose $B(x,\gamma), B(y,\gamma) \in F_\Gamma(\mathbb{A})(\gamma)$ are both compatible with $\boldsymbol{B}$ (to show they must be equal). Then $B(x,\gamma)|_{\lambda_i} = B(x,\lambda_i) = B(x_i,\lambda_i)$. Hence $x \in B(x_i,\lambda_i)$ for all i and $x \in \bigcap_{i\in I} B(x_i,\lambda_i)$. But by Lemma 2.12 $B(x,\gamma) = \bigcap_{i\in I} B(x_i,\lambda_i)$. By a similar argument we also have $B(y,\gamma) = \bigcap_{i\in I} B(x_i,\lambda_i)$ and so $B(x,\gamma) = B(y,\gamma)$. Hence $F_\Gamma(A)$ is separated. □

Claim 3.15. *If $f : \mathbb{A} \to \mathbb{C}$ is a $\overline{\Gamma}$-functor then $F_\Gamma(f) : F_\Gamma(\mathbb{A}) \to F_\Gamma(\mathbb{C})$ is a map of flabby separated presheaves.*

Proof. First we need to show that $F_\Gamma(f)_\gamma(B^{\mathbb{A}}(a,\gamma))$ doesn't depend on our choice of a. Suppose $B^{\mathbb{A}}(a,\gamma) = B^{\mathbb{A}}(b,\gamma)$ and $x \in B^{\mathbb{C}}(f(a),\gamma)$. Then $\mathbb{C}[f(a),f(b)] \ge \mathbb{A}[a,b] \ge \gamma$ and $\mathbb{C}[x,f(a)] \ge \gamma$. So $\mathbb{C}[x,f(b)] \ge \mathbb{C}[x,f(a)] \wedge \mathbb{C}[f(a),f(b)] \ge \gamma$ and $x \in B^{\mathbb{C}}(f(b),\gamma)$. Hence we have $F_\Gamma(f)_\gamma(B^{\mathbb{A}}(a,\gamma)) = F_\Gamma(f)_\gamma(B^{\mathbb{A}}(b,\gamma))$.

Next we need to show $F_\Gamma(f)$ is a natural transformation, i.e. if $\lambda \le \gamma$ then $F_\Gamma(f)_\lambda(B^{\mathbb{A}}(a,\gamma)|_\lambda) = (F_\Gamma(f)_\gamma(B^{\mathbb{A}}(a,\gamma)))|_\lambda$. But we have $F_\Gamma(f)_\lambda(B^{\mathbb{A}}(a,\gamma)|_\lambda) = F_\Gamma(f)_\lambda(B^{\mathbb{A}}(a,\lambda)) = B^{\mathbb{C}}(f(a),\lambda) = B^{\mathbb{C}}(f(a),\gamma)|_\lambda = (F_\Gamma(f)_\gamma(B^{\mathbb{A}}(a,\gamma)))|_\lambda$. So $F_\Gamma(f)$ is a natural transformation from $F_\Gamma(\mathbb{A})$ to $F_\Gamma(\mathbb{C})$. □

Now we define our functor G_Γ.

Definition 3.16. *If $A \in Flab(\Gamma)$ let $G_\Gamma(A)_o = A(\top)$ and*

$$G_\Gamma(A)[a,b] = \bigvee\{\gamma : a|_\gamma = b|_\gamma\}.$$

If $A, C \in Flab(\Gamma)$ and $f : A \Rightarrow C$ is a map of presheaves then $G_\Gamma(f) = f_\top : A(\top) \to C(\top)$.

Claim 3.17. *If A is a flabby separated presheaf on Γ then $G_\Gamma(A)$ is a $\overline{\Gamma}$-category.*

Proof. To see $G_\Gamma(A)$ is a symmetric $\overline{\Gamma}$-enriched category let $a,b,c \in A(\top)$ with $G_\Gamma(A)[a,b] \geq \gamma$ and $G_\Gamma(A)[b,c] \geq \gamma$. Then $a|_\gamma = b|_\gamma = c|_\gamma$ and hence $G_\Gamma(A)[a,c] \geq \gamma$. In particular this means $G_\Gamma(A)[a,c] \geq G_\Gamma(A)[a,b] \wedge G_\Gamma(A)[b,c]$ and $G_\Gamma(A)$ satisfies transitivity. Symmetry and reflexivity are immediate hence $G_\Gamma(A)$ is a $\overline{\Gamma}$-enriched category.

Next notice $a|_{G_\Gamma(A)[a,b]} = b|_{G_\Gamma(A)[a,b]}$ because $\{(a|_\gamma, \gamma) : \gamma \leq G_\Gamma(A)[a,b]\}$ is a compatible collection of elements covering both $a|_{G_\Gamma(A)[a,b]}$ and $b|_{G_\Gamma(A)[a,b]}$. In particular this means if $G_\Gamma(A)[a,b] = \top$ then $a = a|_\top = b|_\top = b$. So $G_\Gamma(A)$ is skeletal and hence a $\overline{\Gamma}$-category.

□

Claim 3.18. *If $A, C \in Flab(\Gamma)$ and $f : A \Rightarrow C$ is a map of presheaves then $G_\Gamma(f) : G_\Gamma(A) \to G_\Gamma(C)$ is a $\overline{\Gamma}$-functor.*

Proof. Fix $a,b \in A(\top)$ and let $\zeta = G_\Gamma(A)[a,b]$. Then $a|_\zeta = b|_\zeta$ and so $f_\top(a)|_\zeta = f_\zeta(a|_\zeta) = f_\zeta(b|_\zeta) = f_\top(b)|_\zeta$ and hence $\zeta = G_\Gamma(A)[a,b] \leq G_\Gamma(C)[f(a), f(b)]$. □

We now show that F_Γ and G_Γ form an equivalence of categories.

Claim 3.19. *There is a natural isomorphism $\eta : F \circ G \Rightarrow id_{Flab(\Gamma)}$ such that $(\forall x \in A(\top))\eta_A(\{x\}) = x$ when $A \in Flab(\Gamma)$.*

Proof. For all $a,b \in A(\top)$, $a|_\gamma = b|_\gamma$ if and only if $G_\Gamma(A)[a,b] \geq \gamma$ if and only if $B^{G_\Gamma(A)}(a,\gamma) = B^{G_\Gamma(A)}(b,\gamma)$. So the maps $\eta_A(B^{G_\Gamma(A)}(a,\gamma)) = a|_\gamma$ is a well defined and injective natural transformation. Further, because A is flabby, η_A is also surjective and hence an isomorphism for all γ.

To show that η_A is a natural isomorphism we need to show for any map $f \in \mathrm{Flab}(\Gamma)[A,C]$ we have $\eta_C \circ F_\Gamma(G_\Gamma(f)) = f \circ \eta_A$. Now $(\forall x \in A(\top))\eta_A(\{x\}) = x$, and $(\forall x \in C(\top))\eta_C(\{x\}) = x$. Further we have $(\forall x \in A(\top))$ $F_\Gamma(G_\Gamma(f))_\top(\{x\}) = \{f_\top(x)\}$. So $(\eta_C \circ F_\Gamma(G_\Gamma(f)))_\top = (f \circ \eta_A)_\top$. But, because A is a flabby presheaf this implies that $\eta_C \circ F_\Gamma(G_\Gamma(f)) = f \circ \eta_A$.

Hence $\eta : F_\Gamma \circ G_\Gamma \Rightarrow id_{\mathrm{Flab}(\Gamma)}$ is a natural isomorphism. □

Claim 3.20. *There is a natural isomorphism $\varepsilon : id_{CAT(\overline{\Gamma})} \Rightarrow G \circ F$.*

Proof. If $\mathbb{A}' = G(F(\mathbb{A}))$ then $\mathbb{A}'_o = \{\{a\} : a \in \mathbb{A}_o\}$ and for all $a,b \in \mathbb{A}_o$, $A'[\{a\},\{b\}] = \bigvee\{\gamma : B^{\mathbb{A}}(a,\gamma) = B^{\mathbb{A}}(b,\gamma)\} = \mathbb{A}[a,b]$. Hence the map $\varepsilon_{\mathbb{A}}(a) =$

$\{a\}$ is an isomorphism of $\overline{\Gamma}$-categories. It is also follows immediately that $\varepsilon : id_{\mathrm{CAT}(\overline{\Gamma})} \Rightarrow F \circ G$ is a natural isomorphism. □

Hence F_Γ and G_Γ are equivalences of categories. □

In what follows it will be useful to use the short hand $FL_\Gamma = P_\Gamma \circ sp_\Gamma \circ F_\Gamma$. In other words FL_Γ is the map which takes a $\overline{\Gamma}$-category, turns it into a separated presheaf, and then turns that separated presheaf into a $\mathrm{R}(\Gamma)$-category.

We also let $IN_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)} : \mathrm{R}(\Gamma_\gamma) \to \mathrm{R}(\Gamma)$ be the inclusion of quantaloids.

Lemma 3.21. *The two functors $FL_{\Gamma_\gamma} \circ \widehat{B}_{\overline{\Gamma_\gamma},\overline{\Gamma}}$ and $\widehat{IN}^*_{R(\Gamma_\gamma),R(\Gamma)} \circ FL_\Gamma$ (from $CAT(\overline{\Gamma})$ to $CAT(R(\Gamma_\gamma))$) are isomorphic.*

Proof. For any $\mathbb{A} \in \mathrm{CAT}(\overline{\Gamma})$ let $K^{\mathbb{A}} = FL_{\Gamma_\gamma}(\mathbb{A}_\gamma) \in \mathrm{CAT}(\mathrm{R}(\Gamma_\gamma))$ and let $H^{\mathbb{A}} = \widehat{IN}^*_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)} \circ FL_\Gamma(\mathbb{A})$. Then $K^{\mathbb{A}}$ is the $\mathrm{R}(\Gamma_\gamma)$-category where

$$(\forall \lambda \leq \gamma)(K^{\mathbb{A}})_o(\lambda) = \{B(a', \lambda) : a' = B(a, \gamma), a \in \mathbb{A}_o\}.$$

and $(\forall B(x', \iota_x), B(y', \iota_y) \in (K^{\mathbb{A}})_o) K^{\mathbb{A}}[B(x', \iota_x), B(y', \iota_y)] = \bigvee\{\zeta : B(B(x', \iota_x), \zeta) = B(B(y', \iota_y), \zeta)\} = \bigvee^{\Gamma}\{\zeta : B(x, \zeta) = B(y, \zeta)\}$.

We also have $H^{\mathbb{A}}$ is the $\mathrm{R}(\Gamma_\gamma)$-category where

$$(\forall \lambda \leq \gamma)(H^{\mathbb{A}})_o(\lambda) = \{B^{\Gamma}(x, \lambda) : x \in \mathbb{A}_o\}$$

and $(\forall x, y \in (H^{\mathbb{A}})_o) H^{\mathbb{A}}[x, y] = \bigvee\{\zeta : B(x, \zeta) = B(y, \zeta)\}$.

Hence the $\mathrm{R}(\Gamma_\gamma)$-functor $\alpha_{\mathbb{A}} : H^{\mathbb{A}} \to K^{\mathbb{A}}$ given by $\alpha(B(x, u)) = B(B(x, \gamma), u)$ is an isomorphism. It is also immediate that α is in fact a natural isomorphism. □

We can think of this result as saying that the functor FL_Γ, which maps $\overline{\Gamma}$-categories to $\mathrm{R}(\Gamma)$-categories and factors through the equivalence with flabby presheaves on Γ, commutes with the restriction maps of enriched categories that is induced by the surjection of Γ onto Γ_γ.

Lemma 3.22 (*)**.** *If $\mathbb{A}' \in Flab(\Gamma)$ and $\mathbb{A} = P_\Gamma \circ sp_\Gamma(\mathbb{A}')$ then the following are equivalent*

1. *Every virtual γ-element of $\mathbb{A}$ is realized.*
2. *Every virtual $\top_{\Gamma_\gamma}$-element of $\widehat{IN}^*_{R(\Gamma_\gamma),R(\Gamma)}(\mathbb{A})$ is realized.*

Proof. <u>(1) implies (2):</u>

Suppose φ is a virtual $\top_{\Gamma_\gamma}$-element of $\widehat{IN}^*_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)}(\mathbb{A})$. For $x \in \mathbb{A}_o$ let $\varphi^*(x) =$

$\varphi(x')$ where x' is the unique element of $\mathbb{A}_o(\gamma \wedge \rho_{\mathbb{A}}(x))$ such that $\mathbb{A}[x, x'] = \gamma \wedge \rho_{\mathbb{A}}(x)$ (we know one exists as $\mathbb{A}$ comes from a presheaf). It is then easily checked that φ^* is a virtual γ-element of $\mathbb{A}$ and hence realized (by assumption) by an element $a \in \mathbb{A}_o(\gamma)$. But as $\mathbb{A}_o(\gamma) = \widehat{IN}^*_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)}(\mathbb{A})_o(\top_{\Gamma_\gamma})$, a also realizes φ.

(2) implies (1):

Suppose φ is a virtual γ-element of $\mathbb{A}$ and φ' is its restriction to $\widehat{IN}^*_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)}(\mathbb{A})$. It is then easily checked that φ' satisfies conditions $(a), (b)$ and (c) of Definition 2.13.

As $\mathbb{A}$ is the image of a flabby presheaf, for every $x \in \mathbb{A}_o$ there is an $x^* \in \mathbb{A}_o(\top_\Gamma)$ such that $\mathbb{A}[x, x^*] = \rho_{\mathbb{A}}(x)$. Further, for every $y \in \mathbb{A}_o(\top_\Gamma)$ there is an $y^+ \in \mathbb{A}_o(\gamma)$ such that $\mathbb{A}[y, y^+] = \gamma$. Hence

$$\begin{aligned}
\gamma &\leq \gamma \wedge \bigvee\{\varphi(x) : x \in \mathbb{A}_o\} = \bigvee\{\varphi(x) \wedge \gamma : x \in \mathbb{A}_o\} \\
&= \bigvee\{\varphi(x') \wedge \gamma : x' \in \mathbb{A}_o(\top_\Gamma)\} = \bigvee\{\varphi(x') \wedge \mathbb{A}[x', (x')^+] : x' \in \mathbb{A}_o(\top_\Gamma)\} \\
&\leq \bigvee\{\varphi((x^*)^+) : x' \in \mathbb{A}_o(\top_\Gamma)\} \leq \bigvee\{\varphi(y) : y \in \mathbb{A}_o(\gamma)\} \\
&= \bigvee\{\varphi(y) : y \in (\widehat{IN}^*_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)}(\mathbb{A}))_o(\top_{\Gamma_\gamma})\}.
\end{aligned}$$

with the second equality following from Lemma 3.9.

So φ' satisfies (d) of Definition 2.13 and hence is a virtual $\top_{\Gamma_\gamma}$-element of $\widehat{IN}^*_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)}(\mathbb{A})$. In particular, because we assume (2) holds, φ' is realized by an element $a \in (\widehat{IN}^*_{\mathrm{R}(\Gamma_\gamma),\mathrm{R}(\Gamma)}(\mathbb{A}))_o(\top_{\Gamma_\gamma}) = \mathbb{A}_o(\gamma)$.

All that is left is to show that a realizes φ as well. Suppose $x \in \mathbb{A}_o$ and let x_γ be the unique element of $\mathbb{A}_o(\rho_{\mathbb{A}}(x) \wedge \gamma)$ such that $\mathbb{A}[x, x_\gamma] = \rho_{\mathbb{A}}(x) \wedge \gamma = \rho_{\mathbb{A}}(x_\gamma)$ (we know one exists by Lemma 3.9). By Lemma 3.9 (d) $\varphi(x_\gamma) = \varphi(x)$. But $\mathbb{A}[x_\gamma, a] \geq \mathbb{A}[x_\gamma, x] \wedge \mathbb{A}[x, a] = \mathbb{A}[x, a] \wedge \rho_{\mathbb{A}}(x) \wedge \gamma = \mathbb{A}[x, a]$ (as $\mathbb{A}[x, a] \leq \rho_{\mathbb{A}}(x) \wedge \rho_{\mathbb{A}}(a) = \rho_{\mathbb{A}}(x) \wedge \gamma$). Further $\mathbb{A}[x, a] \geq \mathbb{A}[x_\gamma, a] \wedge \mathbb{A}[x_\gamma, x] = \mathbb{A}[x_\gamma, a] \wedge \rho_{\mathbb{A}}(x_\gamma) = \mathbb{A}[x_\gamma, a]$. Hence $\mathbb{A}[x, a] = \mathbb{A}[x_\gamma, a] = \varphi(x_\gamma) = \varphi(x)$.

In particular a realizes φ and we are done. □

Corollary 3.23 (*). *If $\mathbb{A} \in CAT(\overline{\Gamma})$ then the following are equivalent*

(1) *Every virtual γ-element of $FL_\Gamma(\mathbb{A})$ is realized.*
(2) *Every virtual $\top_{\Gamma_\gamma}$-element of $FL_\gamma((\mathbb{A})_\gamma)$ is realized.*

Proof. This follows immediately from Lemma 3.21 and Lemma 3.22. □

4. Completeness

In this section we consider two different notions of completeness in enriched categories, Cauchy completeness and Injectivity.

4.1. **Cauchy Completeness.** We have defined two functors, $CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}$ and $CC_{\mathrm{R}(\Gamma)} \circ FL_{\Gamma}$, which take $\overline{\Gamma}$-categories and return Cauchy complete $\mathrm{R}(\Gamma)$-categories. It turns out that these functors are isomorphic.

Theorem 4.1 (*). *The two functors $CC_{R(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},R(\Gamma)}$ and $CC_{R(\Gamma)} \circ FL_{\Gamma}$ (from $CAT(\overline{\Gamma})$ to $cCAT(R(\Gamma))$) are isomorphic.*

Proof. Suppose $\mathbb{A} \in \mathrm{CAT}(\overline{\Gamma})$ and let $\mathbb{A}^{\circ} = IN_{\overline{\Gamma},\mathrm{R}(\Gamma)}(\mathbb{A})$ and $\mathbb{A}^{*} = FL_{\Gamma}(\mathbb{A})$. We want to show that $CC_{\mathrm{R}(\Gamma)}(\mathbb{A}^{\circ}) \cong CC_{\mathrm{R}(\Gamma)}(\mathbb{A}^{*})$.

Suppose φ is a virtual λ-element of $\mathbb{A}^{\circ}$ (for $\lambda \in \Gamma$). We define a virtual λ-element $\varphi^{*}(x)$ of $\mathbb{A}^{*}$ as follows. For any $x \in \mathbb{A}^{*}_{o}$ let $x' \in \mathbb{A}^{*}_{o}(\top_{\Gamma})$ be such that $\mathbb{A}^{*}[x, x'] = \rho_{\mathbb{A}^{*}}(x)$ (we know such exists as $\mathbb{A}^{*}$ is the image of a flabby sheaf under P_{Γ}). Then $\varphi^{*}(x) = \varphi(x') \wedge \rho_{\mathbb{A}^{*}}(x)$.

Notice that this is well defined because if $x'' \in \mathbb{A}^{*}_{o}(\top_{\Gamma})$ with $\mathbb{A}^{*}[x'', x] = \rho_{\mathbb{A}^{*}}(x)$ then $\mathbb{A}[x', x''] \geq \rho_{\mathbb{A}^{*}}(x)$ and by Lemma 3.9 (c) $\varphi(x') \wedge \rho_{\mathbb{A}^{*}}(x) = \varphi(x'') \wedge \rho_{\mathbb{A}^{*}}(x)$. It is then immediate that φ^{*} is a virtual λ-element of $\mathbb{A}^{*}$.

Next, for $\lambda \in \Gamma$, suppose ψ is a virtual λ-element of $\mathbb{A}^{*}$. Let $\psi^{\circ}(x) = \psi(x)$ for all $x \in \mathbb{A}^{\circ}_{o} = \mathbb{A}^{*}_{o}(\top_{\Gamma})$. It is then immediate that ψ° is a virtual λ-element of $\mathbb{A}^{\circ}$.

Claim 4.2. (i) *$(\cdot)^{\circ}$ and $(\cdot)^{*}$ are inverse operations.*
(ii) *If $a, b \in CC_{R(\Gamma)}(\mathbb{A}^{\circ})_{o}$ then $CC_{R(\Gamma)}(\mathbb{A}^{\circ})[a, b] = CC_{R(\Gamma)}(\mathbb{A}^{*})[a^{*}, b^{*}]$.*
(iii) *If $a, b \in CC_{R(\Gamma)}(\mathbb{A}^{*})_{o}$ then $CC_{R(\Gamma)}(\mathbb{A}^{*})[a, b] = CC_{R(\Gamma)}(\mathbb{A}^{\circ})[a^{\circ}, b^{\circ}]$.*

Proof. (i): That $((\cdot)^{*})^{\circ}$ is the identity follows from the fact that $\mathbb{A}^{*}_{o}(\top_{\Gamma}) = \mathbb{A}^{\circ}_{o}$. That $((\cdot)^{\circ})^{*}$ is the identity follows from the fact that for all $x \in \mathbb{A}^{*}_{o}$ there is an $x' \in (\mathbb{A}^{*})_{o}(\top_{\Gamma})$ such that $\mathbb{A}^{*}[x, x'] = \rho_{\mathbb{A}^{*}}(x)$, and hence $\varphi(x') \wedge \rho_{\mathbb{A}^{*}}(x) = \varphi(x)$ (by Lemma 3.9 (c)). But as $x' \in \mathbb{A}^{*}_{o}(\top_{\Gamma})$ we have $(\varphi^{\circ})^{*}(x') = \varphi(x')$ and hence $(\varphi^{\circ})^{*}(x) = (\varphi^{\circ})^{*}(x') \wedge \rho_{\mathbb{A}}(x) = \varphi(x') \wedge \rho_{\mathbb{A}}(x) = \varphi(x)$.

<u>*(ii)*</u>: Because $(\mathbb{A}^*)_o(\top_\Gamma) = (\mathbb{A}^\circ)_o$ we have

$$\begin{aligned} CC_{\mathrm{R}(\Gamma)}(\mathbb{A}^*)[a^*, b^*] &= \bigvee\{a^*(x) \wedge b^*(x) : x \in \mathbb{A}^*_o\} \\ &= \bigvee\{a^*(x') \wedge b^*(x') : x' \in \mathbb{A}^*_o(\top_\Gamma)\} \\ &= \bigvee\{a(x') \wedge b(x') : x \in \mathbb{A}^\circ_o\} \\ &= CC_{\mathrm{R}(\Gamma)}(\mathbb{A}^\circ)[a, b]. \end{aligned}$$

with the second equality following from Lemma 3.9.

<u>*(iii)*</u>: Notice

$$\begin{aligned} CC_{\mathrm{R}(\Gamma)}(\mathbb{A}^\circ)[a^\circ, b^\circ] &= \bigvee\{a^\circ(x) \wedge b^\circ(x) : x \in \mathbb{A}^\circ_o\} \\ &= \bigvee\{a(x) \wedge b(x) : x \in \mathbb{A}^*_o(\top_\Gamma)\} \\ &= \bigvee\{a(x) \wedge b(x) : x \in \mathbb{A}^*_o\} \\ &= CC_{\mathrm{R}(\Gamma)}(\mathbb{A}^*)[a, b]. \end{aligned}$$

with the third equality following from the fact that for any $x \in \mathbb{A}^*_o$, if $x' \in \mathbb{A}^*_o(\top_\Gamma)$ with $\mathbb{A}[x, x'] = \rho_{\mathbb{A}}(x)$ then $a(x) \wedge b(x) = a(x') \wedge b(x') \wedge \rho_{\mathbb{A}}(x) \leq a(x) \wedge b(x)$. □

This gives us an isomorphism between $\alpha_{\mathbb{A}} : CC_{\mathrm{R}(\Gamma)}(\mathbb{A}^\circ) \to CC_{\mathrm{R}(\Gamma)}(\mathbb{A}^*)$ which is constant on $\mathbb{A}_o$.

If $h : \mathbb{A} \to \mathbb{B}$ is a Γ-functor then $\alpha_{\mathbb{B}} \circ [CC_{\mathrm{R}(\Gamma)} \circ FL_\Gamma(h)]$ and $[CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(h)] \circ \alpha_{\mathbb{A}}$ are two maps in $\mathrm{cCAT}(\mathrm{R}(\Gamma))$ from $CC_{\mathrm{R}(\Gamma)} \circ FL_\Gamma(\mathbb{A})$ to $CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(\mathbb{B})$ which agree on $\widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(\mathbb{A})$. Hence as $CC_{\mathrm{R}(\Gamma)} \circ FL_\Gamma(\mathbb{A})$ is a Cauchy completion of $\widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(\mathbb{A})$ (by Claim 4.2) we must have $\alpha_{\mathbb{B}} \circ CC_{\mathrm{R}(\Gamma)} \circ FL_\Gamma(h) = CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}(h) \circ \alpha_{\mathbb{A}}$. But this is exactly the condition that $(\alpha_{\mathbb{A}})_{\mathbb{A} \in \mathrm{CAT}(\overline{\Gamma})}$ is a natural transformation. So $(\alpha_{\mathbb{A}})_{\mathbb{A} \in \mathrm{CAT}(\overline{\Gamma})}$ is a natural isomorphism between the functors $CC_{\mathrm{R}(\Gamma)} \circ \widehat{IN}_{\overline{\Gamma},\mathrm{R}(\Gamma)}$ and $CC_{\mathrm{R}(\Gamma)} \circ FL_\Gamma$. □

In particular we now have the following characterization of when a $\overline{\Gamma}$-category is Cauchy complete.

Lemma 4.3 (*)**.** *If $\mathbb{A} \in CAT(\overline{\Gamma})$ then the following are equivalent:*

(1) $\mathbb{A}$ *is Cauchy complete.*

(2) *Every virtual* $\top_\Gamma$*-element of* $\widehat{IN}_{\Gamma,R(\Gamma)}(\mathbb{A})$ *is realized.*
(3) *Every virtual* $\top_\Gamma$*-element of* $FL_\Gamma(\mathbb{A})$ *is realized.*

Proof. (1) is equivalent to (2):
It is easily checked that φ is a virtual $\top_\Gamma$ element of $\widehat{IN}_{\Gamma,\mathrm{R}(\Gamma)}(\mathbb{A})$ if and only if it is also a virtual $*_{\overline{\Gamma}}$ element of $\mathbb{A}$.

(2) is equivalent to (3):
Using the notation of Proposition 4.1 a virtual $\top_\Gamma$-element φ of $\widehat{IN}_{\Gamma,\mathrm{R}(\Gamma)}(\mathbb{A})$ is realized if and only if φ^* is realized and a virtual $\top_\Gamma$-element ψ of $FL_\Gamma(\mathbb{A})$ is realized if and only if ψ° is realized. Hence as $(\cdot)^*$ and $(\cdot)^\circ$ are inverses (by Claim 4.2) (2) is equivalent to (3). □

4.2. **Injectivity.** An injective object in a category is one where every partial map into the injective object can be extended to a total map. Injectivity is a type of completeness which has applications in areas raging from sheaf cohomology to Banach space theory to the theory of modules over a ring (to name just a few). For a category C we let Inj(C) be the full subcategory of C containing the injective objects.

Lemma 4.4. *The injective objects of Sep*(Γ) *are exactly the flabby sheaves.*

Proof. See [4]. □

Because $sp_\Gamma : \mathrm{Flab}(\Gamma) \to \mathrm{Sep}(\Gamma)$ is full and faithful the following lemma is not difficult (for a complete proof see, for example, [1]).

Lemma 4.5. sp_Γ, fl_Γ *restrict to equivalences of categories between Inj*(*Sep*(Γ)) *and Inj*(*Flab*(Γ))*.*

Finally, we have

Lemma 4.6. P_Γ *and* Q_Γ *restrict to equivalences of categories between Inj*(*Sep*(Γ)) *and Inj*($CAT(R(\Gamma))$)*.*

Proof. Let Q^i_Γ be the restriction of Q_Γ to Inj(CAT(R(Γ))) and let P^i_Γ be the restriction of P_Γ to Inj(Sep(Γ)).

Claim 4.7. $P^i_\Gamma \circ Q^i_\Gamma \cong id_{InjCAT(R(\Gamma))}$.

Proof. Assume $\mathbb{A} \in \mathrm{Inj}(\mathrm{CAT}(\mathrm{R}(\Gamma)))$. Then $\eta_{\mathbb{A}} : \mathbb{A} \to P_\Gamma \circ Q_\Gamma(\mathbb{A})$ is a monic and so the identity map $id_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ must extend (along $\eta_{\mathbb{A}}$) to a map $\alpha : P_\Gamma \circ Q_\Gamma(\mathbb{A}) \to \mathbb{A}$. But, because of how Q_Γ is defined, α must be an isomorphism.

Now suppose $f : X \to Q_\Gamma(\mathbb{A})$ and $\beta : X \to Y$ is a monic (with $X, Y \in \mathrm{Sep}(\Gamma)$). Then $P_\Gamma(f) : P_\Gamma(X) \to P_\Gamma \circ Q_\Gamma(\mathbb{A})$ and $P_\Gamma(\beta) : P_\Gamma(X) \to P_\Gamma(Y)$ is a monic. But $P_\Gamma \circ Q_\Gamma(\mathbb{A}) \cong \mathbb{A}$ and hence is injective. So there is a map $f^* : P_\Gamma(Y) \to P_\Gamma \circ Q_\Gamma(\mathbb{A})$ such that $f^* \circ P_\Gamma(\beta) = P_\Gamma(f)$. But $Q_\Gamma \circ P_\Gamma \cong id_{\mathrm{Sep}(\Gamma)}$ and so by applying Q_Γ to both sides we see there is a $g : Y \to Q_\Gamma(\mathbb{A})$ such that $g \circ \beta = f$. Hence $Q_\Gamma(\mathbb{A})$ is an injective object in Sep(Γ)).

In particular, because $P_\Gamma \circ Q_\Gamma(\mathbb{A}) \cong \mathbb{A}$ for any injective $\mathbb{A}$ we have $P^i_\Gamma \circ Q^i_\Gamma \cong id_{\mathrm{Inj}(\mathrm{CAT}(\mathrm{R}(\Gamma)))}$ (as $P^i_\Gamma \circ Q^i_\Gamma$ are well defined). ☐

Claim 4.8. $Q^i_\Gamma \circ P^i_\Gamma \cong id_{Inj(Sep(\Gamma))}$.

Proof. Suppose $X \in \mathrm{Sep}(\Gamma)$ is an injective object and $f : \mathbb{A} \to P_\Gamma(X)$ and $\beta : \mathbb{A} \to \mathbb{B}$ is a monic (with $\mathbb{A}, \mathbb{B} \in \mathrm{CAT}(\mathrm{R}(\Gamma))$)). Then $Q_\Gamma(f) : Q_\Gamma(\mathbb{A}) \to Q_\Gamma \circ P_\Gamma(X)$ and $Q_\Gamma(\beta) : Q_\Gamma(\mathbb{A}) \to Q_\Gamma(\mathbb{B})$ is a monic. But $Q_\Gamma \circ P_\Gamma(X) \cong X$ and hence is injective. So there is a map $f^* : Q_\Gamma(\mathbb{B}) \to Q_\Gamma \circ P_\Gamma(X)$ such that $f^* \circ Q_\Gamma(\beta) = Q_\Gamma(f)$.

Let $\eta : id_{\mathrm{CAT}(\mathrm{R}(\Gamma))} \to P_\Gamma \circ Q_\Gamma$ be the unit of the adjunction $P_\Gamma \dashv Q_\Gamma$. So η is a natural monomorphism. But by applying P_Γ to both sides above we get $\eta_{\mathbb{B}} \circ f = [P_\Gamma \circ Q_\Gamma(f)] \circ \eta_{\mathbb{A}} = P_\Gamma(f^*) \circ [P_\Gamma \circ Q_\Gamma(\beta)] \circ \eta_{\mathbb{A}} = P_\Gamma(f^*) \circ \eta_{\mathbb{B}} \circ \beta$. But as $\eta_{\mathbb{B}}$ is a monic, it is an isometry and so as functions $f = P_\Gamma(f^*) \circ \eta_{\mathbb{B}} \circ \beta$. Hence $P_\Gamma(f^*) \circ \eta_{\mathbb{B}}$ witness the injectivity of $P_\Gamma(X)$.

In particular, because $Q_\Gamma \circ P_\Gamma \cong id_{\mathrm{Sep}(\Gamma)}$ this implies that $Q^i_\Gamma \circ P^i_\Gamma \cong id_{\mathrm{Inj}(\mathrm{Sep}(\Gamma))}$ (as $Q^i_\Gamma \circ P^i_\Gamma$ are well defined). ☐

☐

Corollary 4.9. *FL_Γ restricts to an equivalence of categories between Inj($CAT(\overline{\Gamma})$) and Inj($CAT(R(\Gamma))$).*

Proof. This follows immediately from Lemma 4.5 and Lemma 4.6. ☐

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