Quantifier Rank Spectrum of $\mathcal{L}_{\infty,\omega}$ (PhD Thesis Defense)

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Definition 1. If L is a relational language then $\mathcal{L}_{\omega_1,\omega}(L)$ is the smallest collection of formulas such that if $\phi(\mathbf{x}) \in \mathcal{L}_{\omega_1,\omega}(L)$ then

- $L \subseteq \mathcal{L}_{\omega_1,\omega}(L)$
- $\neg\phi(\mathbf{x}) \in \mathcal{L}_{\omega_1,\omega}(L)$
- $(\forall y)\phi(\mathbf{x}) \in \mathcal{L}_{\omega_1,\omega}(L)$
- $(\exists y)\phi(\mathbf{x}) \in \mathcal{L}_{\omega_1,\omega}(L)$

and if $\{\psi_i(\mathbf{x}) : i \in \omega\} \subseteq \mathcal{L}_{\omega_1,\omega}(L)$ where $\bigcup_{i\in\omega}$Free Variables(ϕ_i) is finite then

- $\bigwedge_{i\in\omega} \psi_i(\mathbf{x}) \in \mathcal{L}_{\omega_1,\omega}(L)$
- $\bigvee_{i\in\omega} \psi_i(\mathbf{x}) \in \mathcal{L}_{\omega_1,\omega}(L)$

Definition 2. If L is a relational language and $\phi(\mathbf{x}) \in \mathcal{L}_{\omega_1,\omega}(L)$ we define the quantifier rank of $\phi(\mathbf{x})$ ($\mathrm{qr}(\phi(x))$) by induction:

- $\mathrm{qr}(R(\mathbf{x})) = 0$ if R is a relation in L.

- $\mathrm{qr}(\neg\phi(\mathbf{x})) = \mathrm{qr}(\phi(\mathbf{x}))$.

- $\mathrm{qr}(\bigwedge_{i\in\omega}\psi_i(\mathbf{x})) = \sup\{\mathrm{qr}(\psi_i) : i \in \omega\}$.

- $\mathrm{qr}((\forall y)\psi(\mathbf{x})) = \mathrm{qr}(\psi(\mathbf{x})) + 1$.

Definition 3. If M, N are models of the language L then we say M is equivalent to N up to α ($M \equiv_\alpha N$) if and only if for all $\phi \in \mathcal{L}_{\omega_1,\omega}(L)$ with $\mathrm{qr}(\phi) \leq \alpha$

$$M \models \phi \Leftrightarrow N \models \phi$$

Definition 4. We say that the quantifier rank of M ($\mathrm{qr}(M)$) is α if α is the least ordinal such that for all models N of L

$$M \equiv_\alpha N \Rightarrow (\forall \beta < \omega_1) M \equiv_\beta N$$

We know, by the following theorem of Dana Scott, that in the case of countable models this is notion is well defined and further the quantifier rank of any countable model is itself countable.

Theorem 5 (Scott)**.** *If M is a countable model of the language L then there is a formula ϕ_M of $\mathcal{L}_{\omega_1,\omega}$ such that*

- $M \models \phi_M$
- *For all models N of L*

$$N \models \phi_M \rightarrow N \cong M$$

Further, as we will only be interested in countable models for this talk we will assume all models are countable and all models have countable quantifier rank.

Definition 6. Let $\phi \in \mathcal{L}_{\omega_1,\omega}(L)$. We define the quantifier rank spectrum of ϕ (qr(ϕ)) to be

$$\{\mathrm{qr}(M) : M \models \phi \wedge |M| = \omega\}$$

In this talk we will primarily be interested in particularly well behaved formulas.

Definition 7. Let $\phi \in \mathcal{L}_{\omega_1,\omega}$. We say that ϕ is Scattered if

$$(\forall \alpha \in \mathrm{qr}(\phi))|\{M : M \models \phi \wedge |M| = \omega \\ \wedge \mathrm{qr}(M) = \alpha\}| = \omega$$

Theorem 8 (Morley). *Let $\phi \in \mathcal{L}_{\omega_1,\omega}$. Then ϕ is scattered if and only if $|\{M : M \models \phi$ and M is countable$\}| = \omega$ or ω_1 in all forcing extensions of the universe.*

The main result of Part I of my thesis and the main result of this talk is

Theorem 9. *Let $\omega * \alpha$ be a limit ordinal. Then there is a scattered sentence $\phi_{\omega*\alpha}$ such that*

- *Quantifier rank of $\phi_{\omega*\alpha} \leq \omega$*

- *Quantifier rank spectrum of $\phi_{\omega*\alpha}$ is unbounded in $\omega * \alpha$*

- *$\phi_{\omega*\alpha}$ is scattered.*

Definition 10. Let $L_P = \{P^n : P^n$ is an n-ary predicate$\}$.

Definition 11. Let T_P be universal closure of the following L_P sentences:

$$(\forall i_1, \cdots i_n \in n) P^n(x_1, \cdots, x_n) \\ \rightarrow P^n(x_{i_1}, \cdots, x_{i_n})$$

$$P^{n+1}(x_0, \cdots, x_n) \rightarrow P^n(x_1, \cdots, x_n)$$

This theory puts a tree structure under subseteq ($\subseteq$) on the finite subsets of our model.

Definition 12. Define the color of $\overline{a} \in M$ ($\|\overline{a}\|$) as follows:

- $\neg P(\overline{a}) \leftrightarrow \|\overline{a}\| = -\infty$
- $P(\overline{a}) \leftrightarrow \|\overline{a}\| \geq 0$
- If the tree extending $\overline{a}$ is wellfounded then $\|\overline{a}\| = \sup\{\|\overline{a}b\| : b \in M\}$
- $\|\overline{a}\| = \infty$ otherwise.

Definition 13. Let $M \models T_P$. Then the Spectrum of M (Spec(M)) $= \{\|\overline{a}\| : \overline{a} \subseteq M\}$

Definition 14. Let $f : m \to \mathrm{ORD}$ such that $f(m+1)+1 = f(m)$. Then we say that f is a slow slant line.

Definition 15. Let f be a slant line. We say that two tuples $\langle a_i : i \in n\rangle$ and $\langle b_i : i \in n\rangle$ are the same up to f if for all $S \subseteq n$

- $\|\langle a_i : i \in S\rangle\| = \|\langle b_i : i \in S\|\rangle$
- $\|\langle a_i : i \in S\rangle\| > f(|S|)$ and $\|\langle b_i : i \in S\|\rangle > f(|S|)$

Color

$\omega * \alpha + 6$

f

g

$\omega * \alpha + 5$

$\omega * \alpha + 4$

$\omega * \alpha + 3$

$\omega * \alpha + 2$

$\omega * \alpha + 1$

$\omega * \alpha$

Tuples

a_1 a_2 a_3 a_1a_2 a_2a_3 a_1a_3 $a_1a_2a_3$

Definition 16. Let $L_R = L_P \cup \{R_{\leq}^{i,j} : R_{\leq}^{i,j}$ is an $i+j$-ary predicate$\}$.

We will abuse notation and consider $R_{\leq}^{i,j}$ as a predicate of two arguments, one of arity i and one of arity j.

Definition 17. Let T_R be universal closure of the following L_R sentences:

$$\begin{aligned} & T_P \\ & R_{\leq}(\mathbf{x}, \mathbf{y}) \leftrightarrow \quad [[\neg P(\mathbf{x})] \vee [P(\mathbf{x}) \rightarrow P(\mathbf{y}) \\ & \qquad\qquad \wedge (\forall a)(\exists b) R_{\leq}(\mathbf{x}a, \mathbf{y}b)]] \end{aligned}$$

This expanded theory T_R will be useful because we have the following theorem

Theorem 18. *If* $M \models T_R$ *and has no tuples of color* ∞ *then* $M \models (\forall \overline{a}, \overline{b}) R_{\leq}(\overline{a}, \overline{b}) \leftrightarrow \|\overline{a}\| \leq \|\overline{b}\|$

The "nice" scattered sentences will have what we call a Collection of Archetypes. The collection of archetypes for a sentence T will consist of four pieces of information

- A set AT(T) of archetypes

- A partial order $\langle 2-\mathrm{AT}(T), \leq\rangle$ of on certain pairs of archetypes (called consistent pairs of archetypes)

- A collection BP(T) of base predicates (along with consistent pairs of base predicates $\langle 2 - BP(T), \leq\rangle$)

- An "Extra Information" function $EI_T : \mathrm{AT}(T) \cup \{M : M \models T\} \to X \times \mathrm{ORD}$

Definition 19. Let T be our theory with nice properties in a language L. Further let $\mathcal{M} \models T_P$. Then define

$$L(\mathcal{M}) = L^1 \cup L^2 \cup \{Q, R^2_{\leq}\} \cup \{c_i : i \in \mathcal{M}\}$$

Definition 20. Let $T(\mathcal{M})$ be universal closure of the following $L(\mathcal{M})$ sentences:
<u>Q:</u>

- $Q(x) \leftrightarrow \bigvee_{a \in \mathcal{M}} x = c_a$
- $Q \models \phi(c_{a_1}, \cdots c_{a_n})$ in L^2 iff $\mathcal{M} \models \phi(a_1, \cdots a_n)$
- $Q(\mathbf{x}) \wedge \neg Q(\mathbf{y}) \rightarrow \neg U(\mathbf{x}, \mathbf{y})$ where U is any predicate other than $R^2_{\leq}$ and $|\mathbf{x}| > 0$

$\underline{L^2:}$

- $(\forall x)(\exists c)Q(c)R^2_{\leq}(x,c)$
- $(\forall c)(\exists x)\neg Q(x) \wedge R^2_{\leq}(x,c)$

Other Axioms:

- $\neg Q \models T^2$
- $\neg Q \models T^1$
- $\neg Q \models P^1(\mathbf{x}) \rightarrow P^2(\mathbf{x})$

- <u>Homogeneity:</u>
  For all $(A, A_*), (B, B_*)$ consistent pairs of base predicates such that $(A, A_*) \leq (B, B_*)$, and all $m \in \omega$

$$\neg Q \models [(\forall \mathbf{x})[A^1(\mathbf{x}) \wedge A^2_*(\mathbf{x})] \rightarrow (\exists \mathbf{y}_1, \cdots \mathbf{y}_m)(B^1(\mathbf{x}, \mathbf{y}) \wedge B^2_*(\mathbf{x}, \mathbf{y}))]$$

- <u>Completeness:</u>

$$(\forall \mathbf{x})(\exists \mathbf{y}) \bigvee_{(A, A') \in 2-BP(T)} (A, A')(\mathbf{xy})$$

We are going to want our theory T to have properties which allow us to prove the following

Theorem 21. *If*

- $M, N \models T(\mathcal{M})$
- $M|L^1 \equiv_{\omega*\alpha} N|L^1$
- $M|L^2 \cong N|L^2$

then $M \equiv_{\omega*\alpha} N$

Theorem 22. *If Spec*$(M) \subseteq$ *Spec*$(\mathcal{M})$ *then there is a model* $M' \models T(\mathcal{M})$ *such that* $M'|L^1 \cong M$*.*

Theorem 23. *If* $\omega * \alpha <$ *Spec*$(\mathcal{M})$ *which is a limit ordinal then there are* M, N *such that* $M \equiv_{\omega*\alpha} N$ *and Spec*$(M) \cup$*Spec*$(N) \subseteq$ *Spec*$(\mathcal{M})$*.*

We are now ready to give our definition of a collection of archetypes

(Truth on Atomic Formulas for Archetypes)
If $M \models \phi(\mathbf{x})$ and $N \models \phi(\mathbf{y})$ where ϕ is an archetype then for every atomic formula ψ, $M \models \psi(\mathbf{x})$ iff $N \models \psi(\mathbf{y})$.

(Truth on Color)
If $\phi \in \mathrm{AT}(T)$ and $\phi(x_1, \cdots, x_n), \phi(y_1, \cdots, y_n)$ then $\|\{x_i : i \in S\| = \|\{y_i : i \in S\|$ for all $S \subseteq n$.

(Truth on Atomic Formulas for Base Predicates)
If $M \models A(\mathbf{x})$ and $N \models B(\mathbf{y})$ where B is a base predicatethen for every atomic formula ψ, $M \models \psi(\mathbf{x})$ iff $N \models \psi(\mathbf{y})$.

(Restriction of Arity for Archetypes)

If ϕ is an archetype on a tuple **x** and **y** is a subset of **x** then we can restrict ϕ to **y** and get an archetype.

(Completeness for Archetypes)

If ϕ is an archetype which describes a tuple **x** and $\mathbf{x}^\wedge\mathbf{y}$ extends **x** then there is some archetype which describes $\mathbf{x}^\wedge\mathbf{y}$.

(Amalgamation for Archetypes)

If ϕ and ψ are archetypes which agree on the what they force to be true on their common domain then there is a consistent extension of A and B which forces all "new" colors to be $-\infty$.

(Amalgamation for Base Predicates)

If A and B are base predicates which agree on the what they force to be true on their common domain then there is a consistent extension of A and B which forces all "new" colors to be $-\infty$.

(Homogeneity for Base Predicates)

If B is a base predicate which forces another base predicate A to hold and $M \models A(\overline{a})$ then there are infinitely many extensions $\{b_i : i \in \omega\}$ of $\overline{a}$ such that $M \models A(\overline{a}^\wedge \overline{b})$

(Uniqueness of Base Predicate)

This says that each tuple realizes at most one base predicate

(Completeness of Extra Information)

This says that the extra information predicate for a model is just the union of the extra information from each archetype which is realized.

Now we come to two of the most important properties of a collection of archetypes.

(Prediction)

If σ, τ are archetypes such that $\tau(\mathbf{x}, \mathbf{y})$ forces $\sigma(\mathbf{x})$ then there is an archetype $\eta_\tau(\overline{a})$ and a base predicate A_τ such that

- $M \models (\exists \mathbf{x}, \mathbf{y})\tau(\mathbf{x}, \mathbf{y})$ if and only if $M \models (\exists \overline{a})\eta_\tau(\overline{a})$
- $(\forall M \models T)\ M \models [\eta_\tau(\overline{a}) \wedge \sigma(\mathbf{x}) \wedge A_\tau(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a})] \rightarrow \tau(\mathbf{x}, \mathbf{y})$.

If $\eta(\overline{a})$ $A(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a})$ $\sigma(\mathbf{x})$ then $\eta(\overline{a})$ $A(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a})$ $\tau(\mathbf{x}, \mathbf{y})$ $\sigma(\mathbf{x})$

(Prediction up to a Slant Line)

If σ, σ', τ are archetypes such that

- $\tau(\mathbf{x}, \mathbf{y})$ forces $\sigma(\mathbf{x})$
- σ and σ' force the colors on their domains to be the same up to a slant line sl
- $sl(1) = \omega * \lambda + |\mathbf{xy}| + n$

then there is an archetype $\eta_{\tau|sl}(\overline{a})$ and a base predicate $A_{\tau|sl}$ such that

- If $M \models (\exists \overline{a})\sigma'(\overline{a})$ then $M \models (\exists \overline{b})(\eta_{\tau|sl}(\overline{b})$
- For all $M \models T$ if $M \models [\eta_{\tau|sl}(\overline{a}) \wedge \sigma'(\mathbf{x}) \wedge A_{\tau|sl}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a})] \wedge \tau'(\mathbf{x}, \mathbf{y})$ then τ and τ' force the colors of the tuples they describe to be the same up to slant line sl

(Consistency of Color)

If (ϕ, ϕ') is a consistent pair archetypes then any color ϕ' forces must be at least as large as the color ϕ forces on the same tuple.

(Consistency of $\leq$)

$\leq$ on consistent archetype pairs is transitive and if $(\phi_0, \phi_1) \leq (\psi_0, \psi_1)$ then ψ_i is the restriction of ϕ_i to its domain.

(Restriction of Arity for 2-Seq. of Archetypes)

If $(\phi_0, \phi_1)(\mathbf{x}, \mathbf{y}) \leq (\psi_0, \psi_1)(\mathbf{x})$ and (ζ_0, ζ_1) is a restriction of $(\phi_0, \phi_1)(\mathbf{x}, \mathbf{y})$ to $\mathbf{x}, \mathbf{z}$ with $\mathbf{z} \subseteq \mathbf{y}$ then $(\zeta_0, \zeta_1) \leq (\psi_0, \psi_1)$

(Amalgamation for 2-Sequences of Archetypes)

If (ϕ_0, ϕ_1) and (ψ_0, ψ_1) are consistent pairs of archetypes which each force the same information on their common domain then the amalgamations which give all "new" tuples color $-\infty$ is also a consistent archetype pair and $\leq$ (ϕ_0, ϕ_1) and (ψ_0, ψ_1).

(Homogeneity of 2-Sequences of Archetypes)

Suppose

- $(\sigma, \sigma'), (\tau, \tau'), (\eta, \eta')$ are consistent pairs of archetypes
- $(\eta, \eta')(\mathbf{x}, \mathbf{y}) \leq (\sigma, \sigma')(\mathbf{x})$
- $(\eta, \eta')(\mathbf{x}, \mathbf{y})$ forces $(B, B')(\mathbf{x}, \mathbf{y})$
- $(\tau, \tau')(\mathbf{x}), (\sigma, \sigma')(\mathbf{x})$ both force $(A, A')(\mathbf{x})$

(where A, A', B, B' are base predicates). Then there is a consistent pair of archetypes (ζ, ζ') such that

- $(\zeta, \zeta')(\mathbf{x}, \mathbf{y}) \leq (\tau, \tau')(\mathbf{x})$
- $(\zeta, \zeta')(\mathbf{x}, \mathbf{y})$ forces $(B, B')(\mathbf{x}, \mathbf{y})$

(Completeness of 2-Sequences of Base Predicate)

If (τ, τ') is a consistent sequence of archetypes such that (τ, τ') forces (A, A') and σ, σ' are archetypes such that

- $\sigma(\mathbf{x})$ forces $A(\mathbf{x})$
- $\sigma'(\mathbf{x})$ forces $A'(\mathbf{x})$
- Every color which σ' forces is at least as great as the color σ forces on the same tuple

Then (σ, σ') is a consistent pair of archetypes.

(Extension of 0-Colors)

Suppose (σ, σ') is a consistent pair of archetypes. Further assume that $\tau'(\mathbf{x}, \mathbf{y})$ forces $\sigma'(\mathbf{x})$. Then, if $\tau(\mathbf{x}, \mathbf{y})$ forces $\sigma(\mathbf{x})$ and forces all "new" tuples to have color $-\infty$, (τ, τ') is a consistent pair of archetypes and $(\tau, \tau') \leq (\sigma, \sigma')$

(Extension of 1-Colors)

Suppose (σ, σ') is a consistent pair of archetypes, $\tau'(\mathbf{x}, \mathbf{y})$ forces $\sigma'(\mathbf{x})$ and there is some model which realizes both τ and σ'. Then there is an archetype τ' such that (τ, τ') is a consistent pair of archetypes and $(\tau, \tau') \leq (\sigma, \sigma')$

Color

σ' τ'

τ σ

Tuples

Color

η' σ' τ'

τ σ η

Tuples

Theorem 24. *Let* $N \models T(\mathcal{M})$*. If*

- (σ_0, σ_1), $(\tau_0, \tau_1) \in 2 - AT(T)$
- $(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y}) \leq (\sigma_0, \sigma_1)(\mathbf{x})$
- τ_i *is realized in* $N|L^i$

then $N \models (\forall \mathbf{x})(\sigma_0, \sigma_1)(\mathbf{x}) \rightarrow (\exists \mathbf{y})(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y})$.

$\sigma(\mathbf{x})'$

We start with

$\sigma(\mathbf{x})$

$\sigma(\mathbf{x})'$
$B'(\mathbf{xy})'$

then we have

$\sigma(\mathbf{x})$
$B(\mathbf{xy})$

$\sigma(\mathbf{x})'$ $\quad$ $\eta_{\tau'}(\bar{b})$

$B'(\mathbf{xy})'$

We know there exists

$\eta_\tau(\bar{a})$ $\quad$ $\sigma(\mathbf{x})$

$B(\mathbf{xy})$

$\sigma(\mathbf{x})'$ $\quad$ $\eta_{\tau'}(\bar{b})$

$B'(\mathbf{xy})'$

$A'(\bar{b}\mathbf{xyz}')$

and we have by Prediction

$\eta_\tau(\bar{a})$ $\quad$ $\sigma(\mathbf{x})$

$B(\mathbf{xy})$

$A(\bar{a}\mathbf{xyz})$

$$\sigma(\mathbf{x})' \qquad \eta_{\tau'}(\bar{b})$$

$$B'(\mathbf{xy})'$$

$$E'(\bar{a}\bar{b}\mathbf{xyzz}') \qquad A'(\bar{b}\mathbf{xyz}')$$

So we have

$$\eta_{\tau}(\bar{a}) \qquad \sigma(\mathbf{x})$$

$$B(\mathbf{xy})$$

$$A(\bar{a}\mathbf{xyz})$$

$$E(\bar{a}\bar{b}\mathbf{xyzz}')$$

$$\sigma(\mathbf{x})' \qquad \eta_{\tau'}(\bar{b})$$

$$\tau'(\mathbf{xy})'$$

$$E'(\bar{a}\bar{b}\mathbf{xyzz}') \qquad A'(\bar{b}\mathbf{xyz}')$$

and this implies

$$\eta_{\tau}(\bar{a}) \qquad \sigma(\mathbf{x})$$

$$\tau(\mathbf{xy})$$

$$A(\bar{a}\mathbf{xyz})$$

$$E(\bar{a}\bar{b}\mathbf{xyzz}')$$

Theorem 25. *If*

- (σ_0, σ_1),(σ'_0, σ'_1), $(\tau_0, \tau_1) \in 2 - AT(T)$

- $(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y}) \leq (\sigma_0, \sigma_1)(\mathbf{x})$

- $\sigma_0|sl = \sigma'_0|sl$

then there is a τ'_0 *such that* $(\tau'_0, \tau_1) \leq (\sigma'_0, \sigma'_1)$ *and* $\tau'_0|sl = \tau_0|sl$

Lemma 26. *If*

- $M, N \models T$
- $M \equiv_{\omega * \alpha} N$
- $I_{\eta * \omega + n} = \{f : M \to N$ *s.t.* f *is a bijection,* $|$*dom*$(f)| < \omega$*, there exists a slant line* $sl < (\eta + 1) * \omega$ *such that if* $M \models \sigma_f($*dom*$(f))$ *and* $N \models \tau_f($*range*$(f))$ *then* $\sigma_f | sl = \tau_f | sl$ *and where* $sl(|$*dom*$(f)| + n) \geq \eta * \omega\}$

Then $\langle I_\eta : \eta < \omega * \alpha \rangle$ *is a sequence of partial isomorphisms which witness that* $M \equiv_{\omega * \alpha} N$*.*

Theorem 27. *If*

- $M, N \models T(\mathcal{M})$
- $M|L^1 \equiv_{\omega*\alpha} N|L^1$
- $M|L^2 \cong N|L^2$

then $M \equiv_{\omega*\alpha} N$

Proof. Let $I_{\omega*\eta+n} = \{f :$

- $|\text{dom}(f)| < \omega$,
- There exists a slant line $sl < (\eta+1)*\omega$ such that if $M \models (\sigma_0, \sigma_1)(\text{dom}(f))$ and $N \models (\tau_0, \tau_1)(\text{range}(f))$ then $\sigma_0|sl = \tau_0|sl$, $\tau_1 = \sigma_1$ and where $sl(|\text{dom}(f)| + n) \geq \eta * \omega\}$

We know that I_η is non-empty for all $\eta < \omega * \alpha$ by the previous lemma, and by the previous theorems we know (with out to much work) that in fact $\langle I_\eta : \eta \in \omega * \alpha \rangle$ has the back and forth property and hence witnesses that $M \equiv_{\omega * \alpha} N$ $\square$

Theorem 28. *If Θ is as in "The Vaught's Conjecture: A Counter Example" then Θ has a collection of archetypes and Θ is scattered.*

Theorem 29. *If $M, N \models \Theta$, have no tuples of color ∞ and Spec$(M) \cap$ORD, Spec$(N) \cap$ORD $\geq \omega * \alpha$ then $M \equiv_{\omega * \alpha} N$*

Theorem 30. *If $N, \mathcal{M}$ models Θ and if Spec$(N) \subseteq$ Spec$(\mathcal{M})$ then there is a model $N' \models \Theta(\mathcal{M})$ such that $N'|L^1 \cong N$ and $N'|L^2 \cong \mathcal{M}$*

Theorem 31. *If* $\mathcal{M} \models \Theta$ *and Spec*$(M) = \{-\infty\} \cup \omega * \alpha$ *then*

- $\Theta(\mathcal{M})$ *has quantifier rank* ω
- *Quantifier Rank Spectrum*$(\Theta(\mathcal{M}))$ *is unbounded in* $\omega * \alpha$
- $\Theta(\mathcal{M})$ *is Scattered*