1. Show that a regular map of quasiprojective varieties is continuous with respect to the Zariski topology.

2. This problem expands the notion of irreducibility that we have developed so far. We say that a nonempty topological space $X$ is irreducible if whenever $X = X_1 \cup X_2$ for closed sets $X_1$ and $X_2$, we have $X_1 = X$ or $X_2 = X$. We say that a subset $U$ of $X$ is irreducible if it is irreducible with respect to the induced topology.

   (a) Let $U \subseteq X$ be a subset. Show that $U$ is irreducible if and only if its closure $\overline{U}$ is.

   (b) Show that a continuous map $X \to Y$ sends irreducible sets of $X$ to irreducible sets of $Y$.

3. (Shafarevich, exercise 1.4.10) Let $d,n \geq 1$ and let $N = \binom{d+n}{n} - 1$. Show that the image of $\mathbb{P}_\mathbb{C}^n \to \mathbb{P}_\mathbb{C}^N$ under the $d$-fold Veronese embedding is not contained in any proper linear subspace of $\mathbb{P}^N$.

4. (Reid Undergraduate algebraic geometry exercise 5.13) Now let’s study in greater detail the Veronese surface $S$, defined as the image of $\phi : \mathbb{P}_\mathbb{C}^2 \to \mathbb{P}_\mathbb{C}^5$ given by

   $$(x : y : z) \mapsto (x^2 : xy : xz : y^2 : yz : z^2).$$

Show that $\phi$ is an isomorphism $\mathbb{P}^2 \to S$ by writing down equations of the inverse map. Prove that $\phi$ sends the lines of $\mathbb{P}^2$ to conics sitting inside 2-planes in $\mathbb{P}^5$.

Now for any line $\ell \subset \mathbb{P}^2$, write $\pi(\ell) \subset \mathbb{P}^5$ for the 2-plane spanned by the conic $\phi(\ell)$. Prove that the union of $\pi(\ell)$ taken over all $\ell \subset \mathbb{P}^2$ is a cubic hypersurface $\Sigma \subset \mathbb{P}^5$. In fact, identifying $S$ with the projectivization of the rank 1 locus of a symmetric $3 \times 3$ matrix $M$ in the 6 coordinates of $\mathbb{P}^5$, show that $\Sigma = V(\det M)$.