1. Verify that a set $X \subseteq \mathbb{P}^n$ is a projective variety if and only if $X \cap U_i$ is an affine variety in $U_i \cong \mathbb{A}^n$ for each $i = 0, \ldots, n$. Check that the projective subvarieties of $\mathbb{P}^n$ are the closed sets of a topology on $\mathbb{P}^n$. (So we have defined the Zariski topology on $\mathbb{P}^n$ two equivalent ways.)

2. Let $\phi : \mathbb{Z}^{n+1} \to A$ be any homomorphism of abelian groups. The $\phi$-degree of a monomial $c x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ is defined to be $\phi(\alpha_0, \ldots, \alpha_n)$. Given $f \in k[x_0, \ldots, x_n]$ and $a \in A$, let $f_a$ be the sum of the terms of $f$ of $\phi$-degree $a$; call a polynomial $f \in k[x_0, \ldots, x_n]$ $\phi$-homogeneous if all its terms have the same $\phi$-degree.

Let $I \subseteq k[x_0, \ldots, x_n]$ be a nonzero ideal. Prove that following are equivalent:

(i) $I = \langle f_1, \ldots, f_s \rangle$ for $\phi$-homogeneous polynomials $f_i$;

(ii) For every polynomial $f$, $f \in I$ implies $f_a \in I$ for all $a \in A$.

(iii) Every reduced Gröbner basis for $I$ consists of $\phi$-homogeneous polynomials.

Now taking $\phi : \mathbb{Z}^{n+1} \to \mathbb{Z}$ to be the sum map, recover a theorem about homogeneous ideals. What about taking $\phi : \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ to be the identity map?

3. (a) Let $F \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $d > 0$. Prove Euler’s relation $\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = d \cdot F$.

(b) Now suppose $F \in k[x, y, z]$ is homogeneous of degree $d > 0$, with char $k$ not dividing $d$. Write $F_x, F_y,$ and $F_z$ for its first partial derivatives. The singular locus of the plane curve $C = V(F) \subset \mathbb{P}^2$ is the intersection of $C$ with $V(F_x, F_y, F_z)$. Show that a point $(u : v : 1)$ on $C$ is singular if and only if $(u, v)$ is singular as a point of the affine plane curve $V(F(x, y, 1)) \subset \mathbb{A}^2$.

(c) Similarly, the tangent line to a nonsingular point $p$ of $C$ is the line $F_x(p) \cdot x + F_y(p) \cdot y + F_z(p) \cdot z = 0$.

Check that if $p \in U_i$ then $L \cap U_i$ is the tangent line at $p$ to the affine curve $C \cap U_i$, for any $i = 0, 1, 2$.

4. (a) Assume char $k \neq 2$. A point $(a : b : c : d : e : f) \in \mathbb{P}^5$ uniquely determines a plane conic, i.e. the variety of $F(x, y, z) = ax^2 + bxy + cy^2 + dxy + eyz + fz^2$.

Write down the equation for the hypersurface in $\mathbb{P}^5$ that parametrizes singular plane conics.
(b) Let $F, G \in \mathbb{C}[x, y, z]$ be homogeneous quadratic polynomials, not scalar multiples of each other. Check that for any $(t : u) \in \mathbb{P}^1$, $tF + uG$ is again a (nonzero) quadratic polynomial. Thus $F$ and $G$ determine a pencil $\mathcal{P}$ of conics

$$\mathcal{P} = \{ V(tF + uG) \in \mathbb{P}^2 \mid (t : u) \in \mathbb{P}^1 \}. $$

Show that either

i. every conic in $\mathcal{P}$ is singular, or

ii. there are exactly three singular conics in the pencil, counted with multiplicity.

(c) Give an example of a pencil $\mathcal{P}$ of type i.

(d) Finally, show that the family of conics over $\mathbb{C}$ passing through the four points

$$(1 : 0 : 0) \quad (0 : 1 : 0) \quad (0 : 0 : 1) \quad (1 : 2 : 3)$$

is a pencil of type (ii); write down the equations for the three singular members of this family and draw pictures of the three varieties.