1. Draw pictures of the following varieties over \( \mathbb{R} \):

(a) \( V(xz, yz, z^2 - z, (x^2 - y^2)(z - 1)) \)

(b) \( V(y - xz) \)

2. Let \( k \) be algebraically closed and let \( I, J \subseteq k[x_1, \ldots, x_n] \) be ideals.

(a) Prove that \( V(I : J^\infty) = V(I) \setminus V(J) \).

Does equality necessarily hold if \( k \) is not algebraically closed?

(b) Given a list of generators for \( I \) and \( J \), describe an algorithm for computing a generating set for \( (I : J^\infty) \).

You might find it helpful to look at the very related exercises CLO 4-§4-8 and 9. In case you’re interested, saturation is implemented in Macaulay2 as \texttt{saturate(I,J)}.

3. Let \( V \subseteq \mathbb{A}_k^n \) be a variety. The affine \( N^\text{th} \) secant variety \( \text{Sec}^N(V) \) is the Zariski closure of the set

\[
\{ t_1v_1 + \cdots + t_Nv_N: v_i \in V, t_i \in k, t_1 + \cdots + t_N = 1 \} \subseteq \mathbb{A}_k^n.
\]

(For example, if \( N = 2 \), this is the closure of the union of secant lines spanned by \( V \).)

Let \( C \) be the affine rational normal curve \( \{(u, u^2, u^3, u^4): u \in \mathbb{C}\} \).

(a) Compute \( \text{Sec}^2(C) \), i.e. write down equations for it.

(b) Compute the variety \( \text{Tan}(V) \), defined as the closure of the union of all tangent lines to \( V \).

(c) Show that \( \text{Tan}(C) \subseteq \text{Sec}^2(C) \).

We haven’t officially defined the tangent line to a space curve, but you can guess how to parametrize such a line from your knowledge of calculus. Alternatively, see CLO pp. 19-20 for a very similar example.

4. Fix \( m, n, r \geq 1 \) and let \( D_r \) be the set of \( m \times n \) complex matrices of rank at most \( r \).

(a) Check that \( D_r \) is a variety in \( \mathbb{A}_k^{mn} \) by showing that it is the zero locus of the \( (r+1) \times (r+1) \) minors of an \( m \times n \) matrix. (A minor is a determinant of a square submatrix.)

(b) Show that \( D_r \) is irreducible and \( D_r = \text{Sec}^r(D_1) \) for all \( r \).
5. (a) Let $S \subseteq \mathbb{A}_C^2$ be the set of pairs $(a, b)$ such that the plane curve

$$V_{a,b} = V(y^2 = x^3 + ax + b)$$

is singular. Show that $S$ is itself a singular plane curve, give an equation for it, and draw a picture of its real points.

(b) For each $(a, b) \in S$, determine which point(s) of $V_{a,b}$ are singular, and describe the type of singularity exhibited at those points.

6. Email me with a couple of possible dates for your in-class presentation. (Earlier is better, especially if you anticipate a busy end of the semester.) If you have a topic in mind, let me know; alternatively, let me know if you would like some suggestions.

On this and all subsequent homeworks, you may always use Macaulay2 to compute your answers (unless stated otherwise.)

Please see the course website for a clarification on the course collaboration policy. In particular, it is never OK to copy or post solutions from/to the Internet or to represent the work of others as your own.