1. Show that every infinite descending chain $V_1 \supseteq V_2 \supseteq \cdots$ of subvarieties of $\mathbb{A}^n_k$ eventually stabilizes, i.e. there is some $N$ such that $V_N = V_{N+1} = \cdots$.

2. Fix $n > 1$ and let $\text{Nil}_n$ be the set of $n \times n$ complex matrices $X$ such that $X^n = 0$, regarded as a subset of $\mathbb{A}^2_{\mathbb{C}}$. Find, with proof, a minimal set of at most $n$ polynomials $f_1, \ldots, f_s$ such that $\text{Nil}_n = V(f_1, \ldots, f_s)$.

3. Consider the set $I$ of polynomials $f \in k[x, y, z]$ such that

$$f(0) = \frac{\partial f}{\partial x}(0) = \frac{\partial^2 f}{\partial x^2}(0) = \frac{\partial f}{\partial y}(0) = \frac{\partial f}{\partial z}(0) = 0.$$ 

Show that $I \subset k[x, y, z]$ is an ideal, and find a minimal generating set for it.

4. Show that $X = \{(t^2, t^3, t^4) : t \in \mathbb{C}\}$ is a subvariety of $\mathbb{A}^3_\mathbb{C}$, and find $I(X)$. (One way to do this involves using the division algorithm.)

5. Monomial orderings:

(a) Verify that $\text{grevlex}$ (defined on p. 58) is a monomial order.

(b) For any $w \in \mathbb{R}^n_{\geq 0}$ and any monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$, show that the total order $>_w$ defined by $\alpha >_w \beta$ iff

$$w \cdot \alpha > w \cdot \beta \quad \text{or} \quad w \cdot \alpha = w \cdot \beta \text{ and } \alpha > \beta$$

is a monomial order.

6. The radical of an ideal $J$ in a ring $R$ is the set

$$\sqrt{J} = \{ f \in R : f^m \in J \text{ for some } m \}.$$ 

(a) Verify that $\sqrt{J}$ is an ideal.

(b) Now suppose $R = k[x_1, \ldots, x_n]$. Verify that $\sqrt{J} \subseteq I(V(J))$.

(c) Let $>$ be any monomial order and let $J \subseteq k[x_1, \ldots, x_n]$ be an ideal. Show that $J$ is radical if $\langle \text{LT}(J) \rangle$ is radical.