Derived Algebraic Geometry XIV: Representability Theorems

March 14, 2012

Contents

Introduction

Let R be a commutative ring and let X be an R-scheme. Suppose that we want to present X explicitly. We might do this by choosing a covering of X by open subschemes $\{U_\alpha\}_{\alpha \in I}$, where each U_α is an affine scheme given by the spectrum of a commutative R-algebra A_α. Let us suppose for simplicity that each intersection $U_\alpha \cap U_\beta$ is itself an affine scheme, given as the spectrum of $A_\alpha[x_{\alpha,\beta}^{-1}]$ for some element $x_{\alpha,\beta} \in A_\alpha$. To describe X, we need to specify the following data:

(a) For each $\alpha \in I$, a commutative R-algebra A_α. Such an algebra might be given by generators and relations as a quotient

$$R[x_1, \ldots, x_n]/(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_m))$$

for some collection of polynomials f_i.

(b) For every pair of indices $\alpha, \beta \in I$, a pair of elements $x_{\alpha,\beta} \in A_\alpha$ and $x_{\beta,\alpha} \in A_\beta$, together with an R-algebra isomorphism $\phi_{\alpha,\beta} : A_\alpha[x_{\alpha,\beta}^{-1}] \simeq A_\beta[x_{\beta,\alpha}^{-1}]$.

Moreover, the isomorphisms $\phi_{\alpha,\beta}$ should be the identity when $\alpha = \beta$, and satisfy the following cocycle condition:

(c) Given $\alpha, \beta, \gamma \in I$, the commutative ring $A_\gamma[x_{\gamma,\beta}^{-1}, \phi_{\beta,\gamma}(x_{\beta,\alpha})^{-1}]$ should be a localization of $A_\gamma[x_{\gamma,\alpha}^{-1}]$. Moreover, the composite map

$$A_\alpha \to A_\alpha[x_{\alpha,\gamma}^{-1}] \xrightarrow{\phi_{\alpha,\gamma}} A_\gamma[x_{\gamma,\alpha}^{-1}] \to A_\gamma[x_{\gamma,\beta}^{-1}, \phi_{\beta,\gamma}(x_{\beta,\alpha})^{-1}]$$

should be obtained by composing (localizations of) $\phi_{\alpha,\beta}$ and $\phi_{\beta,\gamma}$.

In [52], we introduced the notion of a *spectral scheme*. The definition of a spectral scheme is entirely analogous the classical notion of a scheme. However, the analogues of (a), (b), and (c) are much more complicated in the spectral setting. For example, giving an affine spectral scheme over a commutative ring R is equivalent to giving an $\mathbb{E}_\infty$-algebra over R. These are generally quite difficult to describe using generators and relations. For example, the polynomial ring $R[x]$ generally does not have a finite presentation as an $\mathbb{E}_\infty$-algebra over R, unless we assume that R has characteristic zero. These complications are amplified when we pass to the non-affine situation. In the spectral setting, (b) requires us to construct equivalences between $\mathbb{E}_\infty$-algebras, which are often difficult to describe. Moreover, since $\mathbb{E}_\infty$-rings form an ∞-category rather than an ordinary category, the analogue of the cocycle condition described in (c) is not a condition but an additional datum (namely, a homotopy between two $\mathbb{E}_\infty$-algebra maps $A_\alpha \to A_\gamma[x_{\gamma,\beta}^{-1}, \phi_{\beta,\gamma}(x_{\beta,\alpha})^{-1}]$ for every triple $\alpha, \beta, \gamma \in I$), which must be supplemented by additional coherence data involving four-fold intersections and beyond. For these reasons, it is generally very difficult to provide "hands-on" constructions in the setting of spectral algebraic geometry.

Fortunately, there is another approach to describing a scheme X. Rather than trying to explicitly construct the commutative rings associated to some affine open covering of X, one can instead consider the functor $\mathcal{F}_X$ represented by X, given by the formula $\mathcal{F}_X(R) = \mathrm{Hom}(\mathrm{Spec}\, R, X)$. The scheme X is determined by the functor $\mathcal{F}_X$ up to canonical isomorphism. The situation for spectral schemes is entirely analogous: every spectral scheme $\mathfrak{X}$ determines a functor $\mathcal{F}_{\mathfrak{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and the construction $\mathfrak{X} \mapsto \mathcal{F}_{\mathfrak{X}}$ determines a fully faithful embedding from the ∞-category of spectral schemes to the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ (Theorem V.2.4.1). Generally speaking, it is much easier to describe a spectral scheme (or spectral Deligne-Mumford stack) $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ by specifying the functor $\mathcal{F}_{\mathcal{X}}$ than it is to specify the structure sheaf $\mathcal{O}_{\mathcal{X}}$ explicitly. This motivates the following general question:

Question 0.0.1. Given a functor $\mathcal{F} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, under what circumstances is $\mathcal{F}$ representable by a spectral Deligne-Mumford stack?

In the setting of classical algebraic geometry, the analogous question is addressed by the following theorem of Artin:

Theorem 1 (Artin Representability Theorem). *Let R be a Grothendieck ring (see Definition 0.0.4) and let $\mathcal{F}$ be functor of commutative R-algebras to the category of sets. Then $\mathcal{F}$ is representable by an algebraic space which is locally of finite presentation over R if the following conditions are satisfied:*

(1) *The functor $\mathcal{F}$ commutes with filtered colimits.*

(2) *The functor $\mathcal{F}$ is a sheaf for the étale topology.*

(3) *If B is a complete local Noetherian R-algebra with maximal ideal $\mathfrak{m}$, then the natural map $\mathcal{F}(B) \to \varprojlim \mathcal{F}(B/\mathfrak{m}^n)$ is bijective.*

(4) *The functor $\mathcal{F}$ admits an obstruction theory and a deformation theory, and satisfies Schlessinger's criteria for formal representability.*

(5) *The diagonal map $\mathcal{F} \to \mathcal{F} \times_{\operatorname{Spec} R} \mathcal{F}$ is representable by algebraic spaces (which must be quasi-compact schemes, if we wish to require that $\mathcal{F}$ is quasi-separated).*

This result is of both philosophical and practical interest. Since conditions (1) through (5) are reasonable expectations for any functor $\mathcal{F}$ of a reasonably geometric nature, Theorem 1 provides evidence that the theory of algebraic spaces is natural and robust (in other words, that it exactly captures some intuitive notion of "geometricity"). On the other hand, if we are given a functor $\mathcal{F}$, it is usually reasonably easy to check whether or not Artin's criteria are satisfied. Consequently, Theorem 1 can be used to construct a great number of moduli spaces.

Remark 0.0.2. We refer the reader to [1] for the original proof of Theorem 1. Note that in its original formulation, condition (3) was replaced by the weaker requirement that the map $\mathcal{F}(B) \to \varprojlim \mathcal{F}(B/\mathfrak{m}^n)$ has dense image (with respect to the inverse limit topology). Moreover, Artin's theorem proof required a stronger assumption on the commutative ring R. For a careful discussion of the removal of this hypothesis, we refer the reader to [8].

Our goal in this paper is to prove an analogue of Theorem 1 in the setting of spectral algebraic geometry. Let R be an Noetherian $\mathbb{E}_\infty$-ring such that $\pi_0 R$ is a Grothendieck ring, and suppose we are given a functor $\mathcal{F} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$. Our main result (Theorem 2) supplies necessary and sufficient conditions for $\mathcal{F}$ to be representable by a spectral Deligne-Mumford n-stack which is locally almost of finite presentation over R. For the most part, these conditions are natural analogues of the hypotheses of Theorem 1. The main difference is in the formulation of condition (4). In the setting of Artin's original theorem, a deformation and obstruction theory are auxiliary constructs which are not uniquely determined by the functor $\mathcal{F}$. The meaning of these conditions are clarified by working in the spectral setting: they are related to the problem of extending the functor $\mathcal{F}$ to $\mathbb{E}_\infty$-rings which are nondiscrete. In this setting, the analogue of condition (4) is that the functor $\mathcal{F}$ should be *infinitesimally cohesive* (Definition 2.1.9) and admits a *cotangent complex* (Definition 1.3.13). This assumption is more conceptually satisfying: in the setting of spectral algebraic geometry, the cotangent complex of a functor $\mathcal{F}$ *is* uniquely determined by $\mathcal{F}$.

Let us now outline the contents of the this paper. We begin in §1 with a general discussion of the cotangent complex formalism. Recall that if A is an $\mathbb{E}_\infty$-ring, then the cotangent complex L_A is an A-module spectrum which is *universal* among those A-modules for which the projection map $A \oplus L_A \to A$ admits a section (which we can think of as a derivation of A into L_A; see §A.8.3.4). This definition can be globalized: if $\mathfrak{X}$ is an arbitrary spectral Deligne-Mumford stack, then there is a quasi-coherent sheaf $L_{\mathfrak{X}}$ with the following property: for every étale map $u : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $u^* L_{\mathfrak{X}}$ is the quasi-coherent sheaf on $\operatorname{Spec} A$ associated to the A-module L_A. Moreover, we will explain how to describe the quasi-coherent sheaf $L_{\mathfrak{X}}$ directly in terms of the functor represented by $\mathfrak{X}$, and use this description to define the cotangent complex for a large class of functors $\mathcal{F} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which admits a cotangent complex $L_{\mathcal{F}}$. By definition, $L_{\mathcal{F}}$ controls the deformation theory of $\mathcal{F}$ along trivial square-zero extensions. That is, if A is a connective $\mathbb{E}_\infty$-ring and M is a connective A-module, then the space $\mathcal{F}(A \oplus M)$ is determined by the space $\mathcal{F}(A)$ and cotangent complex $L_{\mathcal{F}}$. However, for many applications of deformation theory, this is not enough: we would like also to describe the spaces $\mathcal{F}(\widetilde{A})$, where $\widetilde{A}$ is a *nontrivial* square-zero extension of A by M. For this, we need to make some additional assumptions on the functor $\mathcal{F}$. In §2, we will study several conditions on a functor $\mathcal{F} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, which are generally satisfied by functors of a reasonably "geometric" nature (for example, functors which are representable by spectral Deligne-Mumford stacks).

In §3, we will apply the ideas of §1 and 2 to formulate and prove the following analogue of Theorem 1:

Theorem 2 (Spectral Artin Representability Theorem)**.** *Let $\mathcal{F} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor, and suppose we are given a natural transformation $\mathcal{F} \to \mathrm{Spec}^{\mathrm{f}} R$, where R is a Noetherian $\mathbb{E}_\infty$-ring and $\pi_0 R$ is a Grothendieck ring. Let $n \geq 0$. Then $\mathcal{F}$ is representable by a spectral Deligne-Mumford n-stack which is locally almost of finite presentation over R if and only if the following conditions are satisfied:*

(1) *For every discrete commutative ring A, the space $\mathcal{F}(A)$ is n-truncated.*

(2) *The functor $\mathcal{F}$ is a sheaf for the étale topology.*

(3) *The functor $\mathcal{F}$ is nilcomplete, infinitesimally cohesive, and integrable.*

(4) *The functor $\mathcal{F}$ admits a connective cotangent complex $L_{\mathcal{F}}$.*

(5) *The natural transformation f is locally almost of finite presentation.*

Remark 1. *Just as Theorem 1 can be used to construct moduli spaces in classical algebraic geometry, Theorem 2 can be used to construct moduli spaces in spectral algebraic geometry. However, the role of Theorem 2 is more essential than its classical counterpart. Most moduli spaces of interest in classical algebraic geometry can be constructed explicitly by other means. In the spectral setting, where "hands-on" presentations are not available, a result like Theorem 2 becomes indispensable.*

Of all of the hypotheses of Theorem 2, perhaps the most important is assumption (4): the existence of a cotangent complex $L_{\mathcal{F}}$. Proving the existence of $L_{\mathcal{F}}$ itself amounts to solving a certain representability problem, albeit in a much easier (linear) setting. In §4, we describe a variety of reformulations of condition (4), which amounts primarily to finite-dimensionality condition on the Zariski tangent spaces to the functor $\mathcal{F}$ (see Theorem 4.5.1 for a precise statement).

Remark 2. *Theorem 1 is really only a special case of Artin's result, which provides a more general criterion for a groupoid-valued functor to be representable by an Artin stack. There is a corresponding generalization of Theorem 2. If $\mathcal{F} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is a functor satisfying conditions* (1)*,* (2)*,* (3)*, and* (5) *which admits a (possibly nonconnective) cotangent complex $L_{\mathcal{F}}$, then $\mathcal{F}$ is representable by a (higher) Artin stack in the setting of spectral algebraic geometry. We plan to return to this point in a future work (see also [84] for a discussion of a closely related result).*

Notation and Terminology

We will use the language of ∞-categories freely throughout this paper. We refer the reader to [49] for a general introduction to the theory, and to [50] for a development of the theory of structured ring spectra from the ∞-categorical point of view. We will also assume that the reader is familiar with the formalism of spectral algebraic geometry developed in the earlier papers in this series. For convenience, we will adopt the following reference conventions:

(T) We will indicate references to [49] using the letter T.

(A) We will indicate references to [50] using the letter A.

(V) We will indicate references to [51] using the Roman numeral V.

(VII) We will indicate references to [52] using the Roman numeral VII.

($VIII$) We will indicate references to [53] using the Roman numeral VIII.

(IX) We will indicate references to [54] using the Roman numeral IX.

(X) We will indicate references to [55] using the Roman numeral X.

(XI) We will indicate references to [56] using the Roman numeral XI.

(XII) We will indicate references to [57] using the Roman numeral XII.

For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [49].

If $\mathcal{C}$ is an ∞-category, we let $\mathcal{C}^{\simeq}$ denote the largest Kan complex contained in $\mathcal{C}$: that is, the ∞-category obtained from $\mathcal{C}$ by discarding all non-invertible morphisms. We will say that a map of simplicial sets $f : S \to T$ is *left cofinal* if, for every right fibration $X \to T$, the induced map of simplicial sets $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ is a homotopy equivalence of Kan complexes (in [49], we referred to a map with this property as *cofinal*). We will say that f is *right cofinal* if the induced map $S^{op} \to T^{op}$ is left cofinal: that is, if f induces a homotopy equivalence $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ for every *left* fibration $X \to T$. If S and T are ∞-categories, then f is left cofinal if and only if for every object $t \in T$, the fiber product $S \times_T T_{t/}$ is weakly contractible (Theorem T.4.1.3.1).

Throughout this paper, we let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings. If R is an $\mathbb{E}_\infty$-ring, we let $\mathrm{CAlg}_R = \mathrm{CAlg}(\mathrm{Mod}_R)$ denote the ∞-category of $\mathbb{E}_\infty$-algebras over R. We let $\mathrm{Spec}\, R$ denote the affine spectral Deligne-Mumford stack associated to R. This can be identified with the pair $(\mathcal{S}\mathrm{hv}^{\text{ét}}_R, \mathcal{O})$, where $\mathcal{S}\mathrm{hv}^{\text{ét}}_R \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_R, \mathcal{S})$ is the full subcategory spanned by those functors which are sheaves with respect to the étale topology, and $\mathcal{O}$ is the sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{S}\mathrm{hv}^{\text{ét}}_R$ determined by the forgetful functor $\mathrm{CAlg}^{\text{ét}}_R \to \mathrm{CAlg}$. We let $\mathrm{Spec}^{\mathrm{f}}\, R : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by $\mathrm{Spec}\, R$: that is, the corepresentable functor $A \mapsto \mathrm{Map}_{\mathrm{CAlg}}(R, A)$.

Warning 0.0.3. If R is an ordinary commutative ring, we sometimes write $\mathrm{Spec}\, R$ for the affine scheme determined by R. There is some risk of confusion, since R can be viewed as a discrete $\mathbb{E}_\infty$-ring, and therefore has an associated affine spectral Deligne-Mumford stack which we indicate with the same notation. However, the danger is slight, since the two notions of spectrum are essentially interchangable.

Let $\mathcal{M}$ be an ∞-category which is left-tensored over an ∞-category $\mathcal{C}$. Given a pair of objects $M, N \in \mathcal{M}$, we let $\underline{\mathrm{Map}}_{\mathcal{M}}(M, N)$ denote a classifying object in $\mathcal{C}$ for morphisms from M to N (if such an object exists). That is, $\underline{\mathrm{Map}}_{\mathcal{M}}(M, N)$ is an object of $\mathcal{C}$ equipped with a map $\alpha : \underline{\mathrm{Map}}_{\mathcal{M}}(M, N) \otimes M \to N$ with the following universal property: for every object $C \in \mathcal{C}$, composition with α induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(C, \underline{\mathrm{Map}}_{\mathcal{M}}(M, N)) \to \mathrm{Map}_{\mathcal{M}}(C \otimes M, N)$. Note that if such a pair $(\underline{\mathrm{Map}}_{\mathcal{M}}(M, N), \alpha)$ exists, then it is well-defined up to a contractible space of choices. Moreover $\underline{\mathrm{Map}}_{\mathcal{M}}(M, N)$ is contravariantly functorial in M, and covariantly functorial in N. In the special case where $\mathcal{M} = \mathcal{C} = \mathrm{Mod}_A$ for an $\mathbb{E}_\infty$-ring A, a classifying object $\underline{\mathrm{Map}}_{\mathcal{M}}(M, N)$ exists for every pair of A-modules M and N. We will denote this classifying object by $\underline{\mathrm{Map}}_A(M, N)$.

Definition 0.0.4. Let $\phi : A \to B$ be a map of Noetherian commutative rings. We say that ϕ is *geometrically regular* if it is flat and, for every prime ideal $\mathfrak{p} \subseteq A$ and every finite extension κ of the residue field $\kappa(\mathfrak{p})$, the commutative ring $\kappa \otimes_A B$ is regular.

We say that a commutative ring A is a *Grothendieck ring* if it is Noetherian, and for every prime ideal $\mathfrak{p} \subseteq A$, the map $A_{\mathfrak{p}} \to \widehat{A}$ is geometrically regular, where $\widehat{A}$ denotes the completion of $A_{\mathfrak{p}}$ with respect to its maximal ideal.

We will need the following nontrivial theorems of commutative algebra.

Theorem 0.0.5 (Grothendieck). *The ring of integers* $\mathbf{Z}$ *is a Grothendieck ring. Moreover, if* R *is a Grothendieck ring, then every finitely generated* R*-algebra is also a Grothendieck ring. In particular, if* A *is finitely generated as a commutative ring, then* A *is a Grothendieck ring.*

Theorem 0.0.6 (Popescu). *Let* $\phi : A \to B$ *be a map of Noetherian commutative rings. Then following conditions are equivalent:*

(1) *The map* ϕ *is geometrically regular.*

(2) *The commutative ring* B *can be realized as a filtered colimit of smooth* A*-algebras.*

For proofs, we refer the reader to [62] and [73], respectively.

1 The Cotangent Complex

Let k be a commutative ring and let R be a commutative k-algebra. We let $\Omega_{R/k}$ denote the module of relative Kähler differentials of R over k, so that $\Omega_{R/k}$ is presented as an R-module by generators $\{dx\}_{x \in R}$ and relations

$$d(x+y) = dx + dy \qquad d(xy) = xdy + ydx \qquad dx = 0 \text{ if } x \in k.$$

This definition can be localized. Suppose that X is a topological space and that $\mathcal{O}_X$ is a sheaf of commutative k-algebras on X. We let Ω_X denote the sheafification of the presheaf of $\mathcal{O}_X$-modules given by $U \mapsto \Omega_{\mathcal{O}_X(U)/k}$. We will refer to Ω_X as the *sheaf of Kähler differentials of* X *relative to* k.

Sheaves of Kähler differentials play an fundamental role in algebraic geometry. If X is a scheme which is smooth of relative dimension n over $\operatorname{Spec} k$, then Ω_X is a locally free sheaf of rank n over X, which we can think of as a cotangent bundle of X (or, more precisely, a relative cotangent bundle for the structural map $\phi : X \to \operatorname{Spec} k$). Consequently, Ω_X encodes important information about the infinitesimal information about the fibers of ϕ.

In this section, we will study the analogue of the construction $X \mapsto \Omega_X$ in the setting of spectral algebraic geometry. We will begin in §1.1 by introducing the *relative cotangent complex* $L_{\mathfrak{X}/\mathfrak{Y}}$ associated to a map of spectrally ringed ∞-topoi $\phi : \mathfrak{X} \to \mathfrak{Y}$. We will then record some basic properties of the relative cotangent complex (Theorems 1.1.9 and 1.1.14), which follow easily from the general formalism developed in §A.8.4.

In §1.2, we will specialize to the case where $\phi : \mathfrak{X} \to \mathfrak{Y}$ is a map of spectral Deligne-Mumford stacks. Our main result is that the relative cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}}$ is quasi-coherent (Proposition 1.2.1), and that it "controls" the possible first-order thickenings of $\mathfrak{X}$ over $\mathfrak{Y}$ in the setting of spectral Deligne-Mumford stacks (Proposition 1.2.8).

Recall that a spectral Deligne-Mumford stack $\mathfrak{X}$ is determined by the functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ represented by $\mathfrak{X}$ (given by $X(R) = \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X})$). In §1.3, we will explain how to recover the relative cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}}$ of a map of spectral Deligne-Mumford stacks $\mathfrak{X} \to \mathfrak{Y}$ from the associated natural transformation of functors $X \to Y$. This leads us to the more general notion of the relative cotangent complex $L_{X/Y}$ associated to natural transformation of functors $X \to Y$, which can sometimes be defined even when the functors X and Y are not representable (see Definition 1.3.13). This notion will play an important role throughout the remainder of this paper.

Warning 1.0.7. Let $f : X \to Y$ be a map of schemes, which we will identify with the corresponding spectral Deligne-Mumford stacks. Our definition of the cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is based on a globalization of *topological* André-Quillen homology, rather than classical André-Quillen homology. Consequently, it generally does not agree with usual cotangent complex studied in algebraic geometry (for example, in [35] and [36]), which we will temporarily denote by $L^{\circ}_{X/Y}$. There is a canonical map $\theta : L_{X/Y} \to L^{\circ}_{X/Y}$, which is an equivalence if X is a $\mathbf{Q}$-scheme. In general, θ induces isomorphisms $\pi_n L_{X/Y} \to \pi_n L^{\circ}_{X/Y}$ for $n \leq 2$ and an epimorphism when $n = 2$.

To obtain the usual cotangent complex studied in algebraic geometry by Illusie and others, one should develop the ideas of this section in the setting of derived algebraic geometry, rather than spectral algebraic geometry. We will return to this point in a future work.

1.1 The Cotangent Complex of a Spectrally Ringed ∞-Topos

In §A.8.3, we defined the relative cotangent complex $L_{A/B}$ of a map of $\mathbb{E}_\infty$-rings $\phi : B \to A$. In this section, we will study a local version of the construction $\phi \mapsto L_{B/A}$, where we replace ϕ by a map of spectrally ringed ∞-topoi $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$. The purely algebraic situation can recovered as the special case in which we assume that $\mathcal{X} = \mathcal{Y} = \mathcal{S} = \mathcal{S}\mathrm{hv}(*)$.

We begin by introducing some terminology.

Notation 1.1.1. Let $\mathcal{X}$ be an ∞-topos. We let $\mathrm{Mod}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$ denote the ∞-category of pairs $(\mathcal{A}, \mathcal{F})$, where $\mathcal{A}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules. We have an evident forgetful functor $\theta : \mathrm{Mod}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})) \to \mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$, given by $\theta(\mathcal{A}, \mathcal{F}) = \mathcal{A}$. If $\mathcal{A}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, we let $\mathrm{Mod}_\mathcal{A}$ denote the fiber of θ over the object $\mathcal{A}$.

It follows from Theorem A.8.3.4.7 that θ exhibits $\mathrm{Mod}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$ as a tangent bundle to $\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$. In particular, for every object $\mathcal{A} \in \mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$, we have a canonical equivalence

$$\mathrm{Mod}_\mathcal{A} \simeq \mathrm{Sp}(\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{C})_{/\mathcal{A}})),$$

which determines a forgetful functor $\Omega^\infty : \mathrm{Mod}_\mathcal{A} \to \mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))_{/\mathcal{A}}$ which we will denote by $\mathcal{F} \mapsto \mathcal{A} \oplus \mathcal{F}$.

We let $L : \mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})) \to \mathrm{Mod}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$ denote an absolute cotangent complex functor (see §A.8.3.2). To each object $\mathcal{A} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, we let $L_\mathcal{A} \in \mathrm{Mod}_\mathcal{A}$ denote the image of $\mathcal{A}$ under the functor L. Then there exists a derivation $d \in \mathrm{Map}_{\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))_{/\mathcal{A}}}(\mathcal{A}, \mathcal{A} \oplus L_\mathcal{A})$ with the following universal property: for every object $\mathcal{F} \in \mathrm{Mod}_\mathcal{A}$, composition with d induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_\mathcal{A}}(L_\mathcal{A}, \mathcal{F}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))_{/\mathcal{A}}}(\mathcal{A}, \mathcal{A} \oplus \mathcal{F}).$$

We will sometimes refer to d as the *universal derivation.* If $\phi : \mathcal{A} \to \mathcal{B}$ is a morphism of $\mathbb{E}_\infty$-algebra objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, we will denote the relative cotangent complex of ϕ by $L_{\mathcal{B}/\mathcal{A}}$ (see §A.8.3.3). More explicitly, $L_{\mathcal{B}/\mathcal{A}}$ is defined to be the cofiber of the map $\mathcal{B} \otimes_\mathcal{A} L_\mathcal{A} \to L_\mathcal{B}$ induced by ϕ.

The formation of cotangent complexes is compatible with pullback, in the following sense:

Proposition 1.1.2. *Let $\phi^* : \mathcal{Y} \to \mathcal{X}$ be a geometric morphism of ∞-topoi, let $\mathcal{A} \in \mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}))$, and let $d : \mathcal{A} \to \mathcal{A} \oplus L_\mathcal{A}$ be the universal derivation. Then the induced map*

$$\phi^* \mathcal{A} \to \phi^*(\mathcal{A} \oplus L_\mathcal{A}) \simeq \phi^* \mathcal{A} \oplus \phi^* L_\mathcal{A}$$

is classified by an equivalence of absolute cotangent complexes $\phi^* L_\mathcal{A} \to L_{\phi^* \mathcal{A}}$.

Proof. Let $\mathcal{F} \in \mathrm{Mod}_{\phi^* \mathcal{A}}$; we wish to show that the pullback of d induces a homotopy equivalence $\theta : \mathrm{Map}_{\mathrm{Mod}_{\phi^* \mathcal{A}}}(\phi^* L_\mathcal{A}, \mathcal{F}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))_{/\phi^* \mathcal{A}}}(\phi^* \mathcal{A}, \phi^* \mathcal{A} \oplus \mathcal{F})$. Unwinding the definitions, we can identify θ with the composite map

$$\mathrm{Map}_{\mathrm{Mod}_\mathcal{A}}(L_\mathcal{A}, \phi_* \mathcal{F}) \xrightarrow{\theta'} \mathrm{Map}_{\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y})_{/\mathcal{A}}}(\mathcal{A}, \mathcal{A} \oplus \phi_* \mathcal{F}) \xrightarrow{\theta''} \mathrm{Map}_{\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y}))_{/\phi_* \phi^* \mathcal{A}}}(\mathcal{A}, \phi_*(\phi^* \mathcal{A} \oplus \mathcal{F})).$$

The universal property of d implies that θ' is a homotopy equivalence, and θ'' is a homotopy equivalence because the diagram

$$\begin{array}{ccc} \mathcal{A} \oplus \phi_* \mathcal{F} & \longrightarrow & \phi_*(\phi^* \mathcal{A} \oplus \mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \phi_* \phi^* \mathcal{A} \end{array}$$

is Cartesian. □

Example 1.1.3. Let $\mathcal{C}$ be a small ∞-category, and let $\mathcal{X} = \mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ be the ∞-category of presheaves on $\mathcal{C}$. We can identify $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ with the ∞-category $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ of presheaves of spectra on $\mathcal{C}$, and $\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$ with the ∞-category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{CAlg})$ of presheaves of $\mathbb{E}_\infty$-rings on $\mathcal{C}$. If $\mathcal{A} : \mathcal{C}^{op} \to \mathrm{CAlg}$ is such a presheaf, then Proposition 1.1.2 implies that $L_{\mathcal{A}}$ is given pointwise by the formula $L_{\mathcal{A}}(C) = L_{\mathcal{A}(C)}$ for $C \in \mathcal{C}$; here the right hand side denotes the $\mathcal{A}(C)$-module given by the absolute cotangent complex of the $\mathbb{E}_\infty$-ring $\mathcal{A}(C)$.

Example 1.1.4. Let $\mathcal{X}$ an arbitrary ∞-topos. Then there exists a small ∞-category $\mathcal{C}$ such that $\mathcal{X}$ is equivalent to an accessible left exact localization of $\mathcal{P}(\mathcal{C})$. Let us identify $\mathcal{X}$ with its image in $\mathcal{P}(\mathcal{C})$, and let $f^* : \mathcal{P}(\mathcal{C}) \to \mathcal{X}$ denote a left adjoint to the inclusion. Let $\mathcal{A}$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, so that we can identify $\mathcal{A}$ with a functor $\mathcal{C}^{op} \to \mathrm{CAlg}$. Combining Example 1.1.3 with Proposition 1.1.2, we deduce that $L_{\mathcal{A}} = f^* \mathcal{F}$, where $\mathcal{F} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sp}))$ is given by the formula $\mathcal{F}(C) = L_{\mathcal{A}(C)}$. In other words, $L_{\mathcal{A}}$ is the sheafification of the presheaf obtained by pointwise application of the algebraic cotangent complex functor $A \mapsto L_A$ defined in §A.8.3.2.

Definition 1.1.5. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectrally ringed ∞-topos. We let $L_{\mathfrak{X}}$ denote the absolute cotangent complex $L_{\mathcal{O}_{\mathcal{X}}} \in \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$. We will refer to $L_{\mathfrak{X}}$ as the *absolute cotangent complex* of $\mathfrak{X}$.

If $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is another spectrally ringed ∞-topos and $\phi : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of spectrally ringed ∞-topoi, we let $L_{\mathfrak{X} / \mathfrak{Y}} \in \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$ denote the relative cotangent complex of the morphism $\phi^* \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ in $\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$; we refer to $L_{\mathfrak{X} / \mathfrak{Y}}$ as the *relative cotangent complex* of the morphism ϕ.

Remark 1.1.6. Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectrally ringed ∞-topoi. If the structure sheaf of $\mathfrak{X}$ is the pullback of the structure sheaf of $\mathfrak{Y}$, then the relative cotangent complex $L_{\mathfrak{X} / \mathfrak{Y}}$ vanishes. In particular, if ϕ is étale, then $L_{\mathfrak{X} / \mathfrak{Y}} \simeq 0$.

Remark 1.1.7. Suppose we are given morphisms of spectrally ringed ∞-topoi

$$\mathfrak{X} \xrightarrow{\phi} \mathfrak{Y} \xrightarrow{\psi} \mathfrak{Z},$$

and write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Using Propositions 1.1.2 and A.8.3.3.5, we deduce that the diagram

$$\begin{array}{ccc} \phi^* L_{\mathfrak{Y} / \mathfrak{Z}} & \longrightarrow & L_{\mathfrak{X} / \mathfrak{Z}} \\ \downarrow & & \downarrow \\ \phi^* L_{\mathfrak{Y} / \mathfrak{Y}} & \longrightarrow & L_{\mathfrak{X} / \mathfrak{Y}} \end{array}$$

is a pushout square in the stable ∞-category $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$. Since $L_{\mathfrak{Y} / \mathfrak{Y}} \simeq 0$, we obtain a fiber sequence

$$\phi^* L_{\mathfrak{Y} / \mathfrak{Z}} \to L_{\mathfrak{X} / \mathfrak{Z}} \to L_{\mathfrak{X} / \mathfrak{Y}}.$$

Our next goal is to formulate "global" versions of some of the algebraic results of §A.8.4.

Notation 1.1.8. Let $\mathcal{X}$ be an ∞-topos, $\mathcal{A}$ a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and $\mathcal{F} \in \mathrm{Mod}_{\mathcal{A}}$. Let $\eta : L_{\mathcal{A}} \to \Sigma \mathcal{F}$ be a map of $\mathcal{A}$-modules. We will sometimes refer to η as a *derivation of* $\mathcal{A}$ *into* $\Sigma \mathcal{F}$. In this case, η classifies a map $\phi_\eta : \mathcal{A} \to \mathcal{A} \oplus \Sigma \mathcal{F}$ (in the ∞-category of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$). Similarly, the zero map $L_{\mathcal{A}} \to \Sigma \mathcal{F}$ classifies a map $\phi_0 : \mathcal{A} \to \mathcal{A} \oplus \Sigma \mathcal{F}$. Form a pullback diagram

$$\begin{array}{ccc} \mathcal{A}^\eta & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \phi_\eta \\ \mathcal{A} & \xrightarrow{\phi_0} & \mathcal{A} \oplus \Sigma \mathcal{F}. \end{array}$$

We will refer to $\mathcal{A}^\eta$ as the *square-zero extension of* $\mathcal{A}$ determined by η. There is a canonical fiber sequence

$$\mathcal{F} \to \mathcal{A}^\eta \to \mathcal{A}$$

in the ∞-category of sheaves of spectra on $\mathcal{X}$.

Let $\mathcal{X}$ be an ∞-topos and let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. The canonical map $\eta : L_{\mathcal{B}} \to L_{\mathcal{B}\,/\,\mathcal{A}}$ determines a square-zero extension $\mathcal{B}^\eta$ of $\mathcal{B}$ by $\Sigma^{-1} L_{\mathcal{B}\,/\,\mathcal{A}}$. Since the restriction of η to $L_{\mathcal{A}}$ vanishes, the associated square-zero extension of $\mathcal{A}$ is split: that is, the map f factors as a composition

$$\mathcal{A} \xrightarrow{f'} \mathcal{B}^\eta \xrightarrow{f''} \mathcal{B}\,.$$

In particular, we obtain a map of $\mathcal{A}$-modules $\mathrm{cofib}(f) \to \mathrm{cofib}(f'')$, which induces a map of $\mathcal{B}$-modules

$$\epsilon_f : \mathcal{B} \otimes_{\mathcal{A}} \mathrm{cofib}(f) \to \mathrm{cofib}(\phi'') \simeq \Sigma^{-1} L_{\mathcal{B}\,/\,\mathcal{A}}.$$

The following result is a special case of Theorem A.8.4.3.12:

Theorem 1.1.9. *Let $\mathcal{X}$ be an ∞-topos, let $f : \mathcal{A} \to \mathcal{B}$ be a morphism between sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and let $\epsilon_f : \mathcal{B} \otimes_{\mathcal{A}} \mathrm{cofib}(f) \to L_{\mathcal{B}\,/\,\mathcal{A}}$ be defined as in Notation 1.1.8. Assume that $\mathcal{A}$ and $\mathcal{B}$ are connective, and that* $\mathrm{cofib}(f)$ *is n-connective (as a sheaf of spectra on $\mathcal{X}$). Then the morphism ϵ_f is $2n$-connective: that is,* $\mathrm{fib}(\epsilon_f)$ *is a $2n$-connective sheaf of spectra on $\mathcal{X}$.*

Let us collect up some consequences of Theorem 1.1.9:

Corollary 1.1.10. *Let $\mathcal{X}$ be an ∞-topos and let $f : \mathcal{A} \to \mathcal{B}$ be a map of connective sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Assume that* $\mathrm{cofib}(f)$ *is n-connective for some $n \geq 0$. Then $L_{\mathcal{B}\,/\,\mathcal{A}}$ is n-connective. The converse holds provided that f induces an isomorphism $\pi_0\,\mathcal{A} \to \pi_0\,\mathcal{B}$.*

Proof. Let $\epsilon_f : \mathcal{B} \otimes_{\mathcal{A}} \mathrm{cofib}(f) \to L_{\mathcal{B}\,/\,\mathcal{A}}$ be as in Notation 1.1.8, so that we have a fiber sequence of $\mathcal{B}$-modules:

$$\mathrm{fib}(\epsilon_f) \to \mathcal{B} \otimes_{\mathcal{A}} \mathrm{cofib}(f) \to L_{\mathcal{B}\,/\,\mathcal{A}}$$

To prove that $L_{\mathcal{B}\,/\,\mathcal{A}}$ is n-connective, it suffices to show that $\mathcal{B} \otimes_{\mathcal{A}} \mathrm{cofib}(f)$ is n-connective and that $\mathrm{fib}(\epsilon_f)$ is $(n-1)$-connective. The first assertion is obvious, and the second follows from Theorem 1.1.9 since $2n \geq n-1$.

To prove the converse, let us suppose that $\mathrm{cofib}(f)$ is *not* n-connective. We wish to show that $L_{\mathcal{B}\,/\,\mathcal{A}}$ is not n-connective. Let us assume that n is chosen as small as possible, so that $\mathrm{cofib}(f)$ is $(n-1)$-connective. By assumption, f induces an isomorphism $\pi_0\,\mathcal{A} \to \pi_0\,\mathcal{B}$, so we must have $n \geq 2$. Applying Theorem 1.1.9, we conclude that ϵ_f is $(2n-2)$-connective. Since $n \geq 2$, we deduce in particular that ϵ_f is n-connective, so that the map $\pi_{n-1}(\mathcal{B} \otimes_{\mathcal{A}} \mathrm{cofib}(f)) \to \pi_{n-1} L_{\mathcal{B}\,/\,\mathcal{A}}$ is an isomorphism. Since $\mathrm{cofib}(f)$ is $(n-1)$-connective and $\pi_0\,\mathcal{A} \simeq \pi_0\,\mathcal{B}$, the map $\pi_{n-1}\,\mathrm{cofib}(f) \to \pi_{n-1}(\mathcal{B} \otimes_{\mathcal{A}} \mathrm{cofib}(f))$ is an isomorphism. It follows that $\pi_{n-1}\,\mathrm{cofib}(f) \to \pi_{n-1} L_{\mathcal{B}\,/\,\mathcal{A}}$ is also an isomorphism, so that $\pi_{n-1} L_{\mathcal{B}\,/\,\mathcal{A}}$ is nonzero. □

Corollary 1.1.11. *Let $\mathcal{X}$ be a sheaf of $\mathbb{E}_\infty$-rings and let $\mathcal{A}$ be a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then the absolute cotangent complex $L_{\mathcal{A}}$ is connective.*

Proof. Let $\mathbf{1}$ denote the initial object of $\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$, and apply Corollary 1.1.10 to the unit map $\mathbf{1} \to \mathcal{A}$. □

Corollary 1.1.12. *Let $f : \mathcal{A} \to \mathcal{B}$ be a map of connective sheaves of $\mathbb{E}_\infty$-rings on an ∞-topos $\mathcal{X}$. Assume that* $\mathrm{cofib}(f)$ *is n-connective for $n \geq 0$. Then the induced map $L_f : L_{\mathcal{A}} \to L_{\mathcal{B}}$ has n-connective cofiber. In particular, the canonical map $\pi_0 L_{\mathcal{A}} \to \pi_0 L_{\pi_0\,\mathcal{A}}$ is an isomorphism.*

Proof. The map L_f factors as a composition

$$L_{\mathcal{A}} \xrightarrow{g} \mathcal{B} \otimes_{\mathcal{A}} L_{\mathcal{A}} \xrightarrow{g'} L_{\mathcal{B}}.$$

We observe that $\mathrm{cofib}(g) \simeq \mathrm{cofib}(f) \otimes_{\mathcal{A}} L_{\mathcal{A}}$. Since the cotangent complex $L_{\mathcal{A}}$ is connective and $\mathrm{cofib}(f)$ is n-connective, we conclude that $\mathrm{cofib}(g)$ is n-connective. It will therefore suffice to show that $\mathrm{cofib}(g') \simeq L_{B/A}$ is n-connective. Let ϵ_f be as Notation 1.1.8, so we have a fiber sequence

$$\mathcal{B} \otimes_{\mathcal{A}} \mathrm{cofib}(f) \xrightarrow{\epsilon_f} L_{\mathcal{B}\,/\,\mathcal{A}} \to \mathrm{cofib}(\epsilon_f).$$

It therefore suffices to show that $\mathcal{B} \otimes_{\mathcal{A}} \mathrm{cofib}(f)$ and $\mathrm{cofib}(\epsilon_f)$ are n-connective. The first assertion follows immediately from the n-connectivity of $\mathrm{cofib}(f)$, and the second from Theorem 1.1.9 since $2n+1 \geq n$. □

Definition 1.1.13. Let $\mathcal{X}$ be an ∞-topos and let $n \geq 0$ be an integer. We will say that a morphism $\phi : \overline{\mathcal{A}} \to \mathcal{A}$ in $\mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$ is an *n-small extension* if the following conditions are satisfied:

(i) The sheaf $\mathcal{A}$ is connective.

(ii) The fiber $\mathcal{I} = \mathrm{fib}(\phi)$ is n-connective (from which it follows that $\overline{\mathcal{A}}$ is also connective).

(iii) The fiber $\mathcal{I}$ belongs to $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 2n}$.

(iv) The multiplication map $\mathcal{I} \otimes \mathcal{I} \to \mathcal{I}$ is nullhomotopic.

The following result is a special case of Theorem A.8.4.1.26:

Theorem 1.1.14. *Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{A}$ be a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and let $n \geq 0$ be an integer. Let $\mathcal{C}$ denote the full subcategory of $(\mathrm{Mod}_{\mathcal{A}})_{L_{\mathcal{A}}/}$ spanned by morphisms of the form $\eta : L_{\mathcal{A}} \to \Sigma \mathcal{I}$, where $\mathcal{I}$ is n-connective and $(2n)$-truncated. Then the construction $\eta \mapsto \mathcal{A}^\eta$ determines a fully faithful embedding from $\mathcal{C}$ to $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{/\mathcal{A}}$, whose essential image is the collection of n-small extensions $\overline{\mathcal{A}} \to \mathcal{A}$.*

1.2 The Cotangent Complex of a Spectral Deligne-Mumford Stack

In §1.1 we defined the absolute cotangent complex $L_{\mathfrak{X}}$ of a map of spectrally ringed ∞-topos $\mathfrak{X}$. In this section, we will restrict our attention to the case where $\mathfrak{X}$ is a spectral Deligne-Mumford stack. Our main results can be summarized as follows:

(a) If $\mathfrak{X}$ is a (possibly nonconnective) spectral Deligne-Mumford stack, then the absolute cotangent complex $L_{\mathfrak{X}}$ is a quasi-coherent sheaf on $\mathfrak{X}$ (Proposition 1.2.1).

(b) Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, and let $\mathcal{O}_{\mathcal{X}}^\eta$ be a square-zero extension of $\mathcal{O}_{\mathcal{X}}$ by a quasi-coherent sheaf $\mathcal{F}$ (classified by a map of quasi-coherent sheaves $\eta : L_{\mathcal{O}} \to \Sigma \mathcal{F}$). Then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^\eta)$ is also a spectral Deligne-Mumford stack (Proposition 1.2.8.

(c) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Under some mild hypotheses, the vanishing of the relative cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}}$ is equivalent to the requirement that f is étale (Proposition 1.2.13). Moreover, finiteness condition on the morphism f are closely related to finiteness conditions on the quasi-coherent sheaf $L_{\mathfrak{X}/\mathfrak{Y}}$ (Proposition 1.2.14).

We begin our discussion with (a).

Proposition 1.2.1. *Let $\mathfrak{X}$ be a nonconnective spectral Deligne-Mumford stack. Then the cotangent complex $L_{\mathfrak{X}}$ is a quasi-coherent sheaf on $\mathfrak{X}$.*

Corollary 1.2.2. *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of nonconnective spectral Deligne-Mumford stacks. Then the relative cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}}$ is a quasi-coherent sheaf on $\mathfrak{X}$.*

The proof of Proposition 1.2.1 will require some preliminary observations.

Remark 1.2.3. Let $\phi : A \to B$ be an étale morphism of $\mathbb{E}_\infty$-rings. Then the relative cotangent complex $L_{B/A}$ vanishes (Corollary A.8.5.4.5). It follows that, for every $\mathbb{E}_\infty$-ring R, every R-module M, and every map $\eta : L_R \to \Sigma M$, the diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(B, R^\eta) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(B, R) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(A, R^\eta) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(A, R) \end{array}$$

is a pullback square. In particular, taking $\eta = 0$, we obtain a pullback square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}}(B, R \oplus M) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(B, R) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}}(A, R \oplus M) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}}(A, R). \end{array}$$

Remark 1.2.4. Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{A}$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and let $\overline{\mathcal{A}}$ be a square-zero extension of $\mathcal{A}$. Using Remark 1.2.3, we deduce:

(a) If $\mathcal{A}$ is local (Henselian, strictly Henselian), then $\overline{\mathcal{A}}$ is also local (Henselian, strictly Henselian).

(b) Assume that $\mathcal{A}$ is local, and let $\mathcal{B}$ be another local sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then a morphism $\phi : \mathcal{B} \to \overline{\mathcal{A}}$ is local if and only if the composite map $\mathcal{B} \xrightarrow{\phi} \overline{\mathcal{A}} \to \mathcal{A}$ is local. In particular, the projection map $\overline{\mathcal{A}} \to \mathcal{A}$ is local.

Lemma 1.2.5. *Let A be an $\mathbb{E}_\infty$-ring and let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}) = \operatorname{Spec} A$ denote the corresponding nonconnective spectral Deligne-Mumford stack. Then the cotangent complex $L_{\mathcal{O}}$ is a quasi-coherent on $\mathfrak{X}$. The equivalence $\mathrm{Mod}_A \simeq \mathrm{QCoh}(\operatorname{Spec} A)$ carries L_A to the absolute cotangent complex $L_{\mathcal{O}}$.*

Proof. Let $\mathbf{1}$ denote the final object of $\mathcal{X}$. The universal derivation $\mathcal{O} \to \mathcal{O} \oplus L_{\mathcal{O}}$ induces a morphism

$$A \simeq \mathcal{O}(\mathbf{1}) \to (\mathcal{O} \oplus L_{\mathcal{O}})(\mathbf{1}) \simeq A \oplus L_{\mathcal{O}}(\mathbf{1})$$

in $\mathrm{CAlg}_{/A}$, which is classified by a map of A-modules $\epsilon : L_A \to L_{\mathcal{O}}(\mathbf{1})$. Let $\mathcal{F}$ denote a preimage of L_A under the equivalence $\mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{Mod}_A$. Then ϵ is adjoint to a morphism $\epsilon' : \mathcal{F} \to L_{\mathcal{O}}$ in $\mathrm{Mod}_{\mathcal{O}}$. We will prove that ϵ' is an equivalence. To prove this, let $\mathcal{F}' \in \mathrm{Mod}_{\mathcal{O}}$ be arbitrary. We wish to show that composition with ϵ' induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}(L_{\mathcal{O}}, \mathcal{F}') \to \mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}(\mathcal{F}, \mathcal{F}') \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_A, \mathcal{F}'(\mathbf{1})).$$

Invoking the universal properties of $L_{\mathcal{O}}$ and L_A, we can identify θ with the map

$$\theta' : \mathrm{Map}_{\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})_{/\mathcal{O}}}(\mathcal{O}, \mathcal{O} \oplus \mathcal{F}') \to \mathrm{Map}_{\mathrm{CAlg}_{/A}}(A, A \oplus \mathcal{F}(\mathbf{1})).$$

given by evaluation at $\mathbf{1}$. It follows from Remark 1.2.4 (and the universal property of $\mathfrak{X} = \operatorname{Spec} A$) that this map is a homotopy equivalence as desired. □

Proof of Proposition 1.2.1. The assertion is local on $\mathfrak{X}$ (Proposition 1.1.2). We may therefore assume without loss of generality that $\mathfrak{X}$ is affine, in which case the result follows from Lemma 1.2.5. □

Variant 1.2.6. Let $\mathfrak{X}$ be a nonconnective spectral scheme. Then the cotangent complex $L_{\mathfrak{X}}$ is a quasi-coherent sheaf on $\mathfrak{X}$. As in the proof of Proposition 1.2.1, we can assume that $\mathfrak{X} = \operatorname{Spec} A$ is affine. In this case, the proof of Lemma 1.2.5 gives a more precise assertion: $L_{\mathfrak{X}}$ is the quasi-coherent sheaf corresponding to $L_A \in \mathrm{Mod}_A$ under the equivalence $\mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{Mod}_A$.

Proposition 1.2.7. *Suppose we are given a pullback square*

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ \mathfrak{Y}' & \longrightarrow & \mathfrak{Y} \end{array}$$

of nonconnective spectral Deligne-Mumford stacks. Then the canonical map

$$\phi^* L_{\mathfrak{X}/\mathfrak{Y}} \to L_{\mathfrak{X}'/\mathfrak{Y}'}$$

is an equivalence in $\mathrm{QCoh}(\mathfrak{X}')$.

Proof. The assertion is local on $\mathfrak{Y}$; we may therefore assume without loss of generality that $\mathfrak{Y} = \operatorname{Spec} A$ is affine. Similarly, we can assume that $\mathfrak{Y}' = \operatorname{Spec} A'$ and $\mathfrak{X}' = \operatorname{Spec} B$ are affine. Then $\mathfrak{X}' \simeq \operatorname{Spec} B'$, where $B' = A' \otimes_A B$. Using Lemma 1.2.5, we are reduced to proving that the canonical amp $B' \otimes_B L_{B/A} \to L_{B'/A'}$ is an equivalence of B'-modules, which is a special case of Proposition A.8.3.3.7. □

We next study the behavior of structure sheaves of spectral Deligne-Mumford stacks under square-zero extensions.

Proposition 1.2.8. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, $\mathcal{F}$ a connective quasi-coherent sheaf on $\mathcal{X}$, and $\eta : L_{\mathfrak{X}} \to \Sigma \mathcal{F}$ a morphism in $\operatorname{QCoh}(\mathfrak{X})$. Then the pair $(\mathcal{X}, \mathcal{O}^{\eta}_{\mathcal{X}})$ is also a spectral Deligne-Mumford stack.*

The proof of Proposition 1.2.8 will require some preliminaries.

Lemma 1.2.9. *Let A be a connective $\mathbb{E}_\infty$-ring, M a connective A-module, and $\eta : L_A \to \Sigma M$ a map of A-modules which determines a square-zero extension A^η of A. Then the base change functor $\theta : \operatorname{CAlg}^{\acute{e}t}_{A^\eta} \to \operatorname{CAlg}^{\acute{e}t}_A$ is an equivalence of ∞-categories.*

Proof. We have a short exact sequence of abelian groups

$$\pi_0 A^\eta \to \pi_0 A \to \pi_{-1} M.$$

Since M is connective, the map $\pi_0 A^\eta \to \pi_0 A$ is a surjection. Using the structure theory of étale morphisms (Proposition VII.8.10), we deduce that θ is essentially surjective. It remains to show that θ is fully faithful. Let $\overline{B}$ and $\overline{B}'$ be étale A^η-algebras, and set $B = A \otimes A^\eta \overline{B}$ and $B' = A \otimes_{A^\eta} \overline{B}'$. We wish to show that θ induces a homotopy equivalence

$$\phi : \operatorname{Map}_{\operatorname{CAlg}_{A^\eta/}}(\overline{B}', \overline{B}) \to \operatorname{Map}_{\operatorname{CAlg}_{A^\eta/}}(\overline{B}', B) \simeq \operatorname{Map}_{\operatorname{CAlg}_{A/}}(B', B).$$

We have a pullback diagram of A^η-algebras

$$\begin{array}{ccc} \overline{B} & \longrightarrow & B \\ \downarrow & & \downarrow \\ B & \longrightarrow & (A \oplus \Sigma M) \otimes_{A^\eta} \overline{B}. \end{array}$$

We note that the lower right corner can be identified with the square-zero extension $B \oplus \Sigma N$, where $N = B \otimes_A M$. It follows that ϕ is a pullback of the map

$$\phi_0 : \operatorname{Map}_{\operatorname{CAlg}_{A^\eta/}}(\overline{B}', B) \to \operatorname{Map}_{\operatorname{CAlg}_{A^\eta/}}(\overline{B}', B \oplus \Sigma N).$$

It will therefore suffice to show that ϕ_0 is a homotopy equivalence. The projection $B \oplus \Sigma N \to B$ induces a map

$$\psi : \operatorname{Map}_{\operatorname{CAlg}_{A^\eta/}}(\overline{B}', B \oplus \Sigma N) \to \operatorname{Map}_{\operatorname{CAlg}_{A^\eta/}}(\overline{B}', B)$$

which is left homotopy inverse to ϕ_0. We claim that ψ is a homotopy equivalence. To prove this, fix a map of A^η-algebras $f : \overline{B}' \to B$. We will show that the homotopy fiber of ψ over f is contractible. This homotopy fiber is given by

$$\operatorname{Map}_{\operatorname{Mod}_{\overline{B}'}}(L_{\overline{B}'/A^\eta}, \Sigma N),$$

which vanishes by virtue of our assumption that $\overline{B}'$ is étale over A^η. □

Lemma 1.2.10. *Let A be a connective $\mathbb{E}_\infty$-ring, M a connective A-module, and $\eta : L_A \to \Sigma M$ a map of A-modules which determines a square-zero extension A^η of A. Then the induced map $\operatorname{Spec} A \to \operatorname{Spec} A^\eta$ induces an equivalence of the underlying ∞-topoi.*

Proof. According to Lemma 1.2.9, we have an equivalence of ∞-categories $\mathrm{CAlg}^{\text{ét}}_{A^\eta} \simeq \mathrm{CAlg}^{\text{ét}}_{A}$. Note that a morphism $\overline{f} : \overline{B}' \to \overline{B}$ in $\mathrm{CAlg}^{\text{ét}}_{A^\eta}$ is faithfully flat if and only if its image $f : B' \to B$ in $\mathrm{CAlg}^{\text{ét}}_{A}$ is faithfully flat. The "only if" direction is obvious, and the "if" direction follows from the observation that the map of commutative rings $\pi_0 \overline{B} \to \pi_0 B$ is a surjection with nilpotent kernel, and therefore induces a homeomorphism of Zariski spectra $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 B) \to \mathrm{Spec}^{\mathrm{Z}}(\pi_0 \overline{B})$. It follows that the equivalence $\mathrm{CAlg}^{\text{ét}}_{A^\eta} \simeq \mathrm{CAlg}^{\text{ét}}_{A}$ induces an equivalence after taking sheaves with respect to the étale topology, and therefore induces an equivalence between the underlying ∞-topoi of $\mathrm{Spec}\, A^\eta$ and $\mathrm{Spec}\, A$. □

Lemma 1.2.11. *Let A be a connective $\mathbb{E}_\infty$-ring, let M be an A-module which is connective as a spectrum, and let $\mathcal{F}$ be the corresponding quasi-coherent sheaf on* $\mathrm{Spec}\, A = (\mathcal{X}, \mathcal{O})$*. Suppose we are given a map $\eta : L_{\mathcal{O}} \to \Sigma\, \mathcal{F}$ which determines a square-zero extension $\mathcal{O}^\eta$ of $\mathcal{O}$. Passing to global sections (and using Lemma 1.2.5), we obtain a map of A-modules $\eta_0 : L_A \to \Sigma M$ which determines a square-zero extension A^{η_0} of A. Then there is a canonical equivalence $(\mathcal{X}, \mathcal{O}^\eta) \simeq$* $\mathrm{Spec}\, A^{\eta_0}$ *(in the ∞-category* RingTop *of spectrally ringed ∞-topoi).*

Proof. Remark 1.2.4 implies that $\mathcal{O}^\eta$ is strictly Henselian. Since A^{η_0} can be identified with the $\mathbb{E}_\infty$-ring of global sections of $\mathcal{O}^\eta$, the universal property of $\mathrm{Spec}\, A^{\eta_0}$ gives a map of spectrally ringed ∞-topoi $\phi : (\mathcal{X}, \mathcal{O}^\eta) \to \mathrm{Spec}\, A^{\eta_0}$. Lemma 1.2.10 implies that ϕ induces an equivalence at the level of the underlying ∞-topoi. Write $\mathrm{Spec}\, A^{\eta_0}$ as $(\mathcal{X}, \mathcal{O}')$. We can identify $\mathcal{O}'$ with the sheaf of $\mathbb{E}_\infty$-rings on $(\mathrm{CAlg}^{\text{ét}}_{A})^{op}$ given by a homotopy inverse of the equivalence $\mathrm{CAlg}^{\text{ét}}_{A^{\eta_0}} \to \mathrm{CAlg}^{\text{ét}}_{A}$ of Lemma 1.2.9. Then ϕ induces a map of sheaves $\mathcal{O}' \to \mathcal{O}^\eta$; we wish to show that this map is an equivalence. Unwinding the definitions, we are required to show that for every étale A-algebra B, if we let $\eta' : L_B \to B \otimes_A \Sigma M$ denote the map induced by η, then ϕ induces an equivalence of $\mathbb{E}_\infty$-rings $\mathcal{O}'(B) \to B^{\eta'}$. Using Lemma 1.2.9, we are reduced to proving that $B^{\eta'}$ is étale over A^{η_0}, and that the canonical map $A \otimes_{A^{\eta_0}} B^{\eta'} \to B$ is an equivalence. This is a special case of Proposition A.8.4.2.5. □

Proof of Proposition 1.2.8. The assertion is local on $\mathcal{X}$. We may therefore assume without loss of generality that $\mathfrak{X}$ is affine, in which case the desired result is a consequence of Lemma 1.2.11. □

Variant 1.2.12. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral scheme, let $\mathcal{F}$ a connective quasi-coherent sheaf on $\mathcal{X}$, and $\eta : L_{\mathfrak{X}} \to \Sigma\, \mathcal{F}$ a morphism in $\mathrm{QCoh}(\mathfrak{X})$. Then the pair $(\mathcal{X}, \mathcal{O}^\eta_{\mathcal{X}})$ is also a spectral scheme. This can be proven by slight modification of the arguments given above.

Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. If ϕ is étale, then $L_{\mathfrak{X}/\mathfrak{Y}}$ vanishes (Remark 1.1.6). Under some mild hypotheses, we have the following converse:

Proposition 1.2.13. *Let $\phi : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of spectral Deligne-Mumford stacks and suppose that the relative cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}}$ is trivial. The following conditions are equivalent:*

(1) *The map ϕ is étale.*

(2) *The map ϕ is locally of finite presentation.*

(3) *The induced map $(\mathfrak{X}, \pi_0\, \mathcal{O}_{\mathcal{X}}) \to (\mathfrak{Y}, \pi_0\, \mathcal{O}_{\mathcal{Y}})$ is locally of finite presentation to order* 1.

Proof. The assertion is local on $\mathfrak{X}$ and $\mathfrak{Y}$, so we may assume that both $\mathfrak{X}$ and $\mathfrak{Y}$ are affine. In this case, the desired result follows from Lemma VII.8.9. □

Proposition 1.2.14. *Let $\phi : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of spectral Deligne-Mumford stacks. Then:*

(1) *If the map ϕ is locally of finite presentation, then the relative cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}} \in$* $\mathrm{QCoh}(\mathfrak{X})$ *is perfect.*

(2) *If the map ϕ is locally almost of finite presentation, then the relative cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}} \in$* $\mathrm{QCoh}(\mathfrak{X})$ *is almost perfect.*

(3) *If the map ϕ is locally of finite presentation to order n, then the relative cotangent complex $L_{\mathfrak{X}\,/\,\mathfrak{Y}} \in \mathrm{QCoh}(\mathfrak{X})$ is perfect to order n.*

The converse assertions hold if we assume that the induced map $(\mathfrak{X}, \pi_0\,\mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \pi_0\,\mathcal{O}_{\mathfrak{Y}})$ is locally of finite presentation to order 1.

Proof. Assertion (1) and (2) (and their converses) follow from Theorem A.8.4.3.18, and assertion (3) (and its converse) from Proposition IX.8.8. $\square$

1.3 The Cotangent Complex of a Functor

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be spectral Deligne-Mumford stacks, representing functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose we are given a map of spectral Deligne-Mumford stacks $\mathfrak{X} \to \mathfrak{Y}$. The relative cotangent complex $L_{\mathfrak{X}\,/\,\mathfrak{Y}}$ is a quasi-coherent sheaf on $\mathfrak{X}$ (Proposition 1.2.1). According to Proposition VIII.2.7.18, we can identify quasi-coherent sheaves on $\mathfrak{X}$ with quasi-coherent sheaves on the functor represented by $\mathfrak{X}$. That is, $L_{\mathfrak{X}\,/\,\mathfrak{Y}}$ is determined by specifying an A-module $\eta^* L_{\mathfrak{X}\,/\,\mathfrak{Y}} \in \mathrm{QCoh}(\mathrm{Spec}\, A) \simeq \mathrm{Mod}_A$ for every map $\eta : \mathrm{Spec}\, A \to \mathfrak{X}$. Unwinding the definitions, we see that if N is a connective A-module, then we can identify A-module maps from $\eta^* L_{\mathfrak{X}\,/\,\mathfrak{Y}}$ into N with dotted arrows rendering commutative the diagram

$$\begin{array}{ccc} \mathrm{Spec}\, A & \xrightarrow{\eta} & \mathfrak{X} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A \oplus N) & \longrightarrow & \mathfrak{Y}, \end{array}$$

where the lower horizontal map is given by the composition

$$\mathrm{Spec}(A \oplus N) \to \mathrm{Spec}\, A \xrightarrow{\eta} \mathfrak{X} \to \mathfrak{Y}\,.$$

The above analysis suggests the possibility of defining the relative cotangent complex for a general natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Our goal in this section is to develop the theory of the cotangent complex in this setting, and to show that it agrees with Definition 1.1.5 when we restrict to functors which are represented by spectral Deligne-Mumford stacks (Proposition 1.3.17). To this end, suppose we are given a natural transformation of functors $f : X \to Y$. We would like to define an object $L_{X/Y} \in \mathrm{QCoh}(X)$, which we can think of as a rule which assigns to each point $\eta \in X(A)$ an A-module M_η, compatible with base change in A. Motivated by the discussion above, the module M_η should have the following property: for every connective A-module N, $\mathrm{Map}_{\mathrm{Mod}_A}(M_\eta, N)$ is given by the fiber of the canonical map

$$X(A \oplus N) \to X(A) \times_{Y(A)} Y(A \oplus N)$$

(over the base point determined by η). In the special case where M_η is connective, this mapping property determines M_η up to a contractible space of choices (by the ∞-categorical version of Yoneda's lemma). However, for some applications this is unnecessarily restrictive: the cotangent complex of an Artin stack (over a field of characteristic zero, say) is usually not connective. We will therefore need a mechanism for recovering M_η given the functor that it corepresents on the ∞-category $\mathrm{Mod}_A^{\mathrm{cn}}$ of connective A-modules.

Notation 1.3.1. Recall that if $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories which admit final objects, then a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *reduced* if it preserves final objects. If $\mathcal{C}$ admits finite colimits and $\mathcal{D}$ admits finite limits, we say that F is *excisive* if it carries pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$. We let $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are reduced and excisive.

Lemma 1.3.2. *Let $\mathcal{C}$ be a stable ∞-category equipped with a right-bounded t-structure. Then the restriction functor $\theta : \mathrm{Exc}_*(\mathcal{C}, \mathcal{S}) \to \mathrm{Exc}_*(\mathcal{C}_{\geq 0}, \mathcal{S})$ is a trivial Kan fibration.*

Proof. Since θ is obviously a categorical fibration, it will suffice to show that θ is a categorical equivalence. Note that $\mathrm{Exc}_*(\mathcal{C}, \mathcal{S})$ is the homotopy limit of the tower of ∞-categories

$$\cdots \to \mathrm{Exc}_*(\mathcal{C}_{\geq -2}, \mathcal{S}) \to \mathrm{Exc}_*(\mathcal{C}_{\geq -1}, \mathcal{S}) \to \mathrm{Exc}_*(\mathcal{C}_{\geq 0}, \mathcal{S}).$$

It will therefore suffice to show that each of the restriction maps

$$\mathrm{Exc}_*(\mathcal{C}_{\geq -n}, \mathcal{S}) \to \mathrm{Exc}_*(\mathcal{C}_{\geq 0}, \mathcal{S})$$

is an equivalence of ∞-categories. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Exc}_*(\mathcal{C}_{\geq -n}, \mathrm{Sp}) & \longrightarrow & \mathrm{Exc}_*(\mathcal{C}_{\geq 0}, \mathrm{Sp}) \\ \downarrow & & \downarrow \\ \mathrm{Exc}_*(\mathcal{C}_{\geq -n}, \mathcal{S}) & \longrightarrow & \mathrm{Exc}_*(\mathcal{C}_{\geq 0}, \mathcal{S}) \end{array}$$

where the vertical maps (given by composition with $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$) are equivalences of ∞-categories (Proposition A.1.4.2.22). It will therefore suffice to show that the forgetful functor

$$\theta : \mathrm{Exc}_*(\mathcal{C}_{\geq -n}, \mathrm{Sp}) \to \mathrm{Exc}_*(\mathcal{C}_{\geq 0}, \mathrm{Sp})$$

is an equivalence of ∞-categories. This is clear, since θ has a homotopy inverse given by the construction $F \mapsto \Omega^n \circ F \circ \Sigma^n$. □

Example 1.3.3. Let $\mathcal{C}$ be a stable ∞-category equipped with a right-bounded t-structure, let $C \in \mathcal{C}$ be an object, and let $F : \mathcal{C} \to \mathcal{S}$ be the functor corepresented by C. Then F is an excisive functor. It follows from Lemma 1.3.2 that F is determined by the restriction $F|\,\mathcal{C}_{\geq 0}$, up to a contractible space of choices. Combining this observation with Yoneda's lemma (Proposition T.5.1.3.2), we see that the object C can be recovered from $F|\,\mathcal{C}_{\geq 0}$ up to a contractible space of choices. More precisely, the construction $C \mapsto F|\,\mathcal{C}_{\geq 0}$ determines a fully faithful embedding

$$\mathcal{C}^{op} \to \mathrm{Exc}_*(\mathcal{C}_{\geq 0}, \mathcal{S}).$$

Example 1.3.4. Let A be a connective $\mathbb{E}_\infty$-ring. Recall that an A-module M is said to be *almost connective* if it is n-connective for some n, and let $\mathrm{Mod}_A^{\mathrm{acn}}$ denote the full subcategory of Mod_A spanned by the A-modules which are almost connective. Example 1.3.3 determines a fully faithful embedding

$$\theta : (\mathrm{Mod}_A^{\mathrm{acn}})^{op} \to \mathrm{Exc}_*(\mathrm{Mod}_A^{\mathrm{cn}}, \mathcal{S}).$$

We will say that a functor $\mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ is *almost corepresentable* if it belongs to the essential image of the functor θ.

Proposition 1.3.5. *Let A be connective $\mathbb{E}_\infty$-ring and let $F : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then F is almost corepresentable if and only if the following conditions are satisfied:*

(a) *The functor F is reduced and excisive.*

(b) *There exists an integer n such that the functor $M \mapsto \Omega^n F(M)$ commutes with small limits.*

(c) *The functor F is accessible: that is, F commutes with κ-filtered colimits for some regular cardinal κ.*

Proof. Assume that condition (a) is satisfied, so that F extends to an left exact $F^+ : \mathrm{Mod}_A^{\mathrm{acn}} \to \mathcal{S}$ (Lemma 1.3.2). Suppose that F^+ is represented by an almost connective A-module N. Choose n such that $\Sigma^n N$ is connective. Then the functor

$$M \mapsto \Omega^n F(M) \simeq F^+(\Omega^n M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(N, \Omega^n M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(\Sigma^n N, M)$$

is corepresented by the object $\Sigma^n N \in \mathrm{Mod}_A^{\mathrm{cn}}$, and therefore preserves small limits. If N is a κ-compact object of Mod_A, then F commutes with κ-filtered colimits, so that (c) is satisfied.

Conversely, suppose that (b) and (c) are satisfied. Choose $n \geq 0$ as in (b). Then the restriction $F^+|(\mathrm{Mod}_A)_{\geq -n}$ is given by the composition

$$(\mathrm{Mod}_A)_{\geq -n} \xrightarrow{\Sigma^n} \mathrm{Mod}_A^{\mathrm{cn}} \xrightarrow{F} \mathcal{S}_* \xrightarrow{\Omega^n} \mathcal{S},$$

and therefore commutes with small limits. Using Proposition T.5.5.2.7, we deduce that $F^+|(\mathrm{Mod}_A)_{\geq -n}$ is corepresented by an object $N \in (\mathrm{Mod}_A)_{\geq -n}$. Using Lemma 1.3.2, we deduce that F^+ is corepresented by N, so that F is almost corepresentable. □

Now suppose that $f : X \to Y$ is a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and η is a point of $X(A)$. Example 1.3.4 shows that if there exists an almost connective A-module M_η which corepresents the functor carrying $N \in \mathrm{Mod}_A^{\mathrm{cn}}$ to the fiber of the canonical map

$$X(A \oplus N) \to X(A) \times_{Y(A)} Y(A \oplus N),$$

then M_η is determined up to a contractible space of choices. However, we will need a stronger statement in what follows: namely, that M_η can be chosen to depend functorially on the pair (A, η). For this, we need to formulate a relative version of Lemma 1.3.2.

Lemma 1.3.6. *Let $q : X \to S$ be a coCartesian fibration of simplicial sets satisfying the following conditions:*

(1) *Each fiber X_s of q is a stable ∞-category equipped with a right-bounded t-structure $(X_{s,\geq 0}, X_{s,\leq 0})$.*

(2) *For every edge $e : s \to s'$ in S, the associated functor $X_s \to X_{s'}$ is exact and right t-exact.*

Let $X_{\geq 0}$ denote the full simplicial subset of X spanned by the vertices which belong to $X_{s,\geq 0}$ for some vertex $s \in S$. Let $\mathcal{E} \subseteq \mathrm{Fun}(X, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(X, \mathcal{S})$ spanned by those functors whose restriction to each fiber X_s is reduced and excisive, and define $\mathcal{E}_0 \subseteq \mathrm{Fun}(X_{\geq 0}, \mathcal{S})$ similarly. Then the restriction functor $\mathcal{E} \to \mathcal{E}_0$ is a trivial Kan fibration.

Proof. Since $\mathcal{E} \to \mathcal{E}_0$ is obviously a categorical fibration, it will suffice to show that it is an equivalence of ∞-categories. For every map of simplicial sets $\phi : T \to S$, let $\mathcal{E}(T) \subseteq \mathrm{Fun}(X \times_S T, \mathcal{S})$ denote the full subcategory spanned by those functors $F : X \times_S T \to \mathcal{S}$ whose restriction to $X_{\phi(t)}$ is reduced and excisive for each vertex $t \in T$, and define $\mathcal{E}_0(T)$ similarly. There is an evident restriction map $\psi(T) : \mathcal{E}(T) \to \mathcal{E}_0(T)$. We will prove that this map is an equivalence of ∞-categories for every map $\phi : T \to S$. Note that $\phi(T)$ is the homotopy limit of a tower of functors $\psi(\mathrm{sk}^n T)$ for $n \geq 0$. We may therefore assume that T is a simplicial set of finite dimension n. We proceed by induction on n, the case $n = -1$ being vacuous. Let K be the set of n-simplices of T. We have a pushout diagram of simplicial sets

$$\begin{array}{ccc} K \times \partial \Delta^n & \longrightarrow & K \times \Delta^n \\ \downarrow & & \downarrow \\ \mathrm{sk}^{n-1} T & \longrightarrow & T, \end{array}$$

which gives rise to a homotopy pullback diagram of functors

$$\begin{array}{ccc} \psi(K \times \partial \Delta^n) & \longleftarrow & \psi(K \times \Delta^n) \\ \uparrow & & \uparrow \\ \psi(\mathrm{sk}^{n-1} T) & \longleftarrow & \psi(T). \end{array}$$

It will therefore suffice to prove that $\psi(\mathrm{sk}^{n-1} T)$, $\psi(K \times \partial \Delta^n)$, and $\psi(K \times \Delta^n)$ are equivalences. In the first two cases, this follows from the inductive hypothesis. In the third case, we can write $\psi(K \times \Delta^n)$ as a product of functors $\psi(\{v\} \times \Delta^n)$ indexed by the elements of K. We are therefore reduced to proving the Lemma in the case $S = \Delta^n$.

For $0 \leq i \leq n$, let X_i denote the fiber of q over the ith vertex of $S = \Delta^n$. Using Proposition T.3.2.2.7, we can choose a composable sequence of maps

$$\theta : X_0^{op} \to X_1^{op} \to \cdots \to X_n^{op}$$

and a categorical equivalence $M(\theta)^{op} \to X$, where $M(\theta)$ denotes the mapping simplex of the diagram θ (see §T.3.2.2). Note that each of the maps in the above diagram is exact and right t-exact, so that θ restricts to a sequence of maps

$$\theta_0 : (X_{0,\geq 0})^{op} \to \cdots \to (X_{n,\geq 0})^{op}$$

and we have a categorical equivalence $M(\theta_0)^{op} \to X_{\geq 0}$. For every simplicial subset $T \subseteq S = \Delta^n$, let $\mathcal{E}'(T)$ denote the full subcategory of $\mathrm{Fun}(T \times_S M(\theta)^{op}, \mathcal{S})$ spanned by those functors whose restriction to each X_i is reduced and excisive, and define $\mathcal{E}'_0(T) \subseteq \mathrm{Fun}(T \times_S M(\theta_0)^{op}, \mathcal{S})$ similarly. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}(T) & \longrightarrow & \mathcal{E}_0(T) \\ \downarrow & & \downarrow \\ \mathcal{E}'(T) & \longrightarrow & \mathcal{E}'_0(T) \end{array}$$

where the vertical maps are categorical equivalences. It follows from the inductive hypothesis that the restriction map $\mathcal{E}'(T) \to \mathcal{E}'_0(T)$ is a categorical equivalence for every proper simplicial subset $T \subsetneq S$. To complete the proof, it will suffice to show that $\mathcal{E}'(S) \to \mathcal{E}'_0(S)$ is a categorical equivalence.

Let σ denote the face of $S = \Delta^n$ opposite the 0th vertex. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{E}'(S) & \longrightarrow & \mathcal{E}'_0(S) \\ \downarrow & & \downarrow \\ \mathcal{E}'(\sigma) & \longrightarrow & \mathcal{E}'_0(\sigma), \end{array}$$

where the bottom horizontal map is a categorical equivalence. It will therefore suffice to show that this diagram is a homotopy pullback square: that is, that the map

$$\rho : \mathcal{E}'(S) \to \mathcal{E}'_0(S) \times_{\mathcal{E}'_0(\sigma)} \mathcal{E}'(\sigma)$$

is a categorical equivalence. Let $\mathcal{C} = X_0$ and $\mathcal{C}_{\geq 0} = X_{0,\geq 0}$. Unwinding the definitions, we see that ρ is a pullback of the canonical map

$$\rho_0 : \mathrm{Fun}(S, \mathrm{Exc}_*(\mathcal{C}, \mathcal{S})) \to \mathrm{Fun}(S, \mathrm{Exc}_*(\mathcal{C}_{\geq 0}, \mathcal{S})) \times_{\mathrm{Fun}(\sigma, \mathrm{Exc}_*(\mathcal{C}_{\geq 0}, \mathcal{S}))} \mathrm{Fun}(\sigma, \mathrm{Exc}_*(\mathcal{C}, \mathcal{S})).$$

It follows from Lemma 1.3.2 that this map is a trivial Kan fibration. □

Definition 1.3.7. Let $p : X \to S$ be a coCartesian fibration of simplicial sets. We will say that a map $F : X \to \mathcal{S}$ is *locally corepresentable* (with respect to p) if the following conditions are satisfied:

(1) For every vertex $s \in S$, there exists an object x of the ∞-category X_s and a point $\eta \in F(x)$ which corepresents the functor $F|X_s$ in the following sense: for every object $y \in X_s$, evaluation on η induces a homotopy equivalence $\mathrm{Map}_{X_s}(x, y) \to F(y)$.

(2) Let $x \in X_s$ and $\eta \in F(x)$ be as in (1), let $e : x \to x'$ be a coCartesian edge of X covering an edge $s \to s'$ in S. Let $\eta' \in F(x')$ be the image of η under the map $F(x) \to F(x')$ determined by e. Then η' corepresents the functor $F|X_{s'}$ (that is, for every $y \in X_{s'}$, evaluation on η' induces a homotopy equivalence $\mathrm{Map}_{X_{s'}}(x', y) \to F(y)$).

In the situation of Definition 1.3.7, condition (2) guarantees that the object x_s representing the functors $F_s = F|X_s$ can be chosen to depend functorially on $s \in S$. We can articulate this idea more precisely as follows:

Lemma 1.3.8. *Let $p : X \to S$ be a coCartesian fibration of simplicial sets and let $\mathcal{C} \subseteq \mathrm{Fun}_S(S, X)$ denote the full subcategory of $\mathrm{Fun}_S(S, X)$ spanned by those maps $f : S \to X$ which carry each edge of S to a coCartesian edge of X. Then there is fully faithful embedding $\mathcal{C}^{op} \to \mathrm{Fun}(X, \mathcal{S})$, whose essential image is the full subcategory of $\mathrm{Fun}(X, \mathcal{S})$ spanned by the locally corepresentable functors.*

Proof. Let $\chi : S \to \mathcal{C}\mathrm{at}_\infty$ be a map classifying the coCartesian fibration p (given informally by the formula $\chi(s) = X_s$), so that $\mathcal{C}$ can be identified with the limit of the diagram χ in the ∞-category $\mathcal{C}\mathrm{at}_\infty$ (Proposition T.3.3.3.1). Let χ' be the result of composing χ with the "opposition" functor $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$.

Let $\mathrm{Dl}(p)$ and $\mathrm{Dl}^0(p)$ be defined as in Construction X.3.4.14 (so that $\mathrm{Dl}(p) \to S$ is a Cartesian fibration whose fibers are given by $\mathrm{Dl}(p)_s = \mathrm{Fun}(X_s, \mathcal{S})$), and $\mathrm{Dl}^0(p)$ is the full simplicial subset whose fibers $\mathrm{Dl}^0(p)_s$ are the full subcategories of $\mathrm{Fun}(X_s, \mathcal{S})$ spanned by the corepresentable functors. Then the projection $q : \mathrm{Dl}^0(p) \to S$ is a coCartesian fibration classified by the map χ' (Proposition X.3.4.17). We have an isomorphism of simplicial sets $\theta : \mathrm{Fun}_S(S, \mathrm{Dl}(p)) \simeq \mathrm{Fun}(X, \mathcal{S})$. A map $F : X \to \mathcal{S}$ is locally corepresentable if and only if $\theta^{-1}(F) : S \to \mathrm{Dl}(p)$ factors through $\mathrm{Dl}^0(p)$ and carries edges of S to q-coCartesian edges of $\mathrm{Dl}^0(p)$. Using Proposition T.3.3.3.1, we can identify the limit $\varprojlim \chi'$ with the full subcategory of $\mathrm{Fun}(X, \mathcal{S})$ spanned by the locally corepresentable functors. We conclude the proof by observing that there is a canonical equivalence of ∞-categories $(\varprojlim \chi)^{op} \simeq \varprojlim \chi'$. □

For our applications, we will need a variant of Lemma 1.3.8 where the functors $F_s = F|X_s$ are not quite assumed to be representable.

Definition 1.3.9. Let $p : X \to S$ be a coCartesian fibration of simplicial sets. Assume that:

(*i*) For each vertex $s \in S$, the ∞-category X_s is stable and equipped with a right-bounded t-structure $(X_{s,\geq 0}, X_{s,\leq 0})$.

(*ii*) For every edge $e : s \to s'$ in S, the associated functor $X_s \to X_{s'}$ is exact and right t-exact.

Let $X_{\geq 0}$ be the full simplicial subset of X spanned by those vertices which belong to $X_{s,\geq 0}$ for some vertex $s \in S$.

We will say that a map $F : X_{\geq 0} \to \mathcal{S}$ is *locally almost corepresentable* (with respect to p) if the following conditions are satisfied:

(1) For every vertex $s \in S$, the induced map $X_{s,\geq 0} \to \mathcal{S}$ is reduced and excisive.

(2) Let $F^+ : X \to \mathcal{S}$ be an extension of F such that $F^+|X_s$ is reduced and excisive for each $s \in S$ (it follows from Lemma 1.3.6 that F^+ exists and is unique up to a contractible space of choices). Then F^+ is locally corepresentable (in the sense of Definition 1.3.7).

Combining Lemmas 1.3.8 and 1.3.6, we obtain the following:

Proposition 1.3.10. *Let $p : X \to S$ be as in Definition 1.3.9, let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}_S(S, X)$ spanned by those maps which carry each edge of S to a p-coCartesian edge of X. Then there is a fully faithful functor $\mathcal{C}^{op} \to \mathrm{Fun}(X_{\geq 0}, \mathcal{S})$, whose essential image is the full subcategory of $\mathrm{Fun}(X_{\geq 0}, \mathcal{S})$ spanned by the locally almost corepresentable functors.*

Remark 1.3.11. In the situation of Proposition 1.3.10, the fully faithful functor $\mathcal{C}^{op} \to \mathrm{Fun}(X_{\geq 0}, \mathcal{S})$ is left exact. In particular, the essential image of this functor is closed under finite limits.

Example 1.3.12. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Let $\overline{\mathrm{CAlg}}^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{cn}}$ be a left fibration classified by X, and let Mod^X denote the fiber product $\mathrm{Mod}(\mathrm{Sp}) \times_{\mathrm{CAlg}} \overline{\mathrm{CAlg}}^{\mathrm{cn}}$. More informally, we let Mod^X denote the ∞-categories whose objects are triples (A, η, M), where A is a connective $\mathbb{E}_\infty$-ring, $\eta \in X(A)$ is a point, and

$M \in \mathrm{Mod}_A$ is an A-module spectrum. Let $\mathrm{Mod}^X_{\mathrm{acn}}$ denote the full subcategory of Mod^X spanned by those triples (A, η, M) where M is almost connective (that is, M is n-connective for $n \ll 0$). The forgetful functor $q : \mathrm{Mod}^X_{\mathrm{acn}} \to \overline{\mathrm{CAlg}}^{\mathrm{cn}}$ is a coCartesian fibration. Moreover, the ∞-category of coCartesian sections of q is canonically equivalent to $\mathrm{QCoh}(X)^{\mathrm{acn}}$, the full subcategory of $\mathrm{QCoh}(X)$ spanned by the almost connective quasi-coherent sheaves on X (see Remark VIII.2.7.12).

Let $\mathrm{Mod}^X_{\mathrm{cn}}$ denote the full subcategory of Mod^X spanned by those triples (A, η, M) where M is connective. Applying Proposition 1.3.10, we deduce that $\mathrm{QCoh}(X)^{\mathrm{acn}}$ is equivalent to the full subcategory of $\mathrm{Fun}(\mathrm{Mod}^X_{\mathrm{cn}}, \mathcal{S})^{op}$ spanned by those functors $\mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ which are locally almost corepresentable (relative to q).

Definition 1.3.13. Suppose we are given a natural transformation $\alpha : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We define a functor $F : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ by the formula

$$F(A, \eta, M) = \mathrm{fib}(X(A \oplus M) \to X(A) \times_{Y(A)} Y(A \oplus M)),$$

where the fiber is taken over the point of $X(A) \times_{Y(A)} Y(A \oplus M)$ determined by η. We will say that α *admits a cotangent complex* if the functor F is locally almost corepresentable relative to q. In this case, we let $L_{X/Y} \in \mathrm{QCoh}(X)$ denote a preimage for F under the fully faithful embedding $\mathrm{QCoh}(X)^{\mathrm{aperf}} \to \mathrm{Fun}(\mathrm{Mod}^X_{\mathrm{cn}}, \mathcal{S})^{op}$ given by Example 1.3.12. We will refer to $L_{X/Y}$ as the *relative cotangent complex of X over Y*. In the special case where Y is a final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, we will say that X *admits a cotangent complex* if the essentially unique map $\alpha : X \to Y$ admits a cotangent complex. In this case, we will denote the relative cotangent complex $L_{X/Y}$ by L_X and refer to it as the *absolute cotangent complex* of X.

Remark 1.3.14. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and $F : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ be functors. Unwinding the definitions, we see that F is locally almost corepresentable if and only if the following conditions are satisfied:

(a) For every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the induced functor $F_\eta : \mathrm{Mod}^{\mathrm{cn}}_A \to \mathcal{S}$ is corepresented by an almost connective A-module M_η (which is uniquely determined up to contractible ambiguity: see Example 1.3.4).

(b) Let $\eta \in X(A)$ be as in (a), and suppose we are given a map of connective $\mathbb{E}_\infty$-rings $A \to A'$. Let $\eta' \in X(A')$ denote the image of η. Then the functor $F_{\eta'}$ is corepresented by $A' \otimes_A M_\eta$. More precisely, for every A'-module N, the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_{A'}}(A' \otimes_A M_\eta, N) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(M_\eta, N) \simeq F_\eta(N) \to F_{\eta'}(N)$$

is a homotopy equivalence.

We can rephrase condition (b) as follows:

(b') The functor F carries p-Cartesian morphisms in $\mathrm{Mod}^X_{\mathrm{cn}}$ to homotopy equivalences, where $p : \mathrm{Mod}^X_{\mathrm{cn}} \to \overline{\mathrm{CAlg}}^{\mathrm{cn}}$ denotes the projection map (here $\overline{\mathrm{CAlg}}^{\mathrm{cn}}$ is defined as in Example 1.3.12).

Example 1.3.15. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then X admits a cotangent complex if and only if the following conditions are satisfied:

(a) For every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, define $F_\eta : \mathrm{Mod}^{\mathrm{cn}}_A \to \mathcal{S}$ by the formula $F_\eta(N) = X(A \oplus N) \times_{X(A)} \{\eta\}$. Then the functor F_η is corepresented by an almost connective A-module M_η.

(b) For every map of connective $\mathbb{E}_\infty$-rings $A \to B$ and every connective B-module M, the diagram of spaces

$$\begin{array}{ccc} X(A \oplus M) & \longrightarrow & X(B \oplus M) \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array}$$

is a pullback square.

In this case, the absolute cotangent complex $L_X \in \mathrm{QCoh}(X)$ is described by the formula $\eta^* L_X = M_\eta \in \mathrm{Mod}_A$ for $\eta \in X(A)$.

Using Proposition 1.3.5, we can reformulate condition (a) as follows:

- (a') For every point $\eta \in X(A)$, the functor $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ is reduced, excisive, and accessible. Moreover, there exists an integer $n \geq 0$ such that the functor $M \mapsto \Omega^n F_\eta(M)$ preserves small limits.

Remark 1.3.16. Fix an integer n and a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then $\mathrm{QCoh}(X)_{\geq n}$ is a full subcategory of $\mathrm{QCoh}(X)^{\mathrm{acn}}$ which is closed under small colimits. The construction of Example 1.3.12 determines a fully faithful embedding $\mathrm{QCoh}(X)^{op}_{\geq n} \to \mathrm{Fun}(\mathrm{Mod}^X_{\mathrm{cn}}, \mathcal{S})$ which commutes with small limits. It follows that the essential image of this embedding is closed under small limits. From this we deduce the following:

- $(*)$ Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the limit of a diagram of functors $\{X_\alpha : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}\}$. Assume that each X_α admits a cotangent complex which is n-connective. Then X admits a cotangent complex which is n-connective. Moreover, we have a canonical equivalence

$$L_X \simeq \varinjlim_{\alpha} f_\alpha^* L_{X_\alpha},$$

where $f_\alpha : X \to X_\alpha$ is the canonical map.

Proposition 1.3.17. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack, and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by $\mathfrak{X}$. Then X admits a cotangent complex. Moreover, we can identify L_X with the image of the cotangent complex $L_{\mathfrak{X}}$ under the equivalence of ∞-categories* $\mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{QCoh}(X)$.

Proof. Let $\mathrm{Mod}^X_{\mathrm{cn}}$ denote the ∞-category introduced in Definition 1.3.13. Let $F : \mathrm{Mod}^X_{\mathrm{cn}}$ be the functor given by

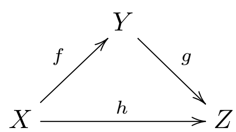
$$F(R, \eta, M) = X(R \oplus M) \times_{X(R)} \{\eta\} \simeq \mathrm{Map}_{\mathrm{Stk}_{\mathrm{Spec}\, R/}}(\mathrm{Spec}\, R \oplus M, \mathfrak{X}).$$

Let $\mathcal{Y}$ denote the underlying ∞-topos of $\mathrm{Spec}\, R$, let $\mathcal{O}'$ denote its structure sheaf, and let $\mathcal{F}_M$ denote the quasi-coherent sheaf on $\mathcal{Y}$ corresponding to the R-module M. Let $\eta^{-1} : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})$ denote the pullback functor induced by η, and let $\eta^* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathrm{Spec}\, R)$ be the usual pullback functor on quasi-coherent sheaves. According to Lemma 1.2.11, we can identify $\mathrm{Spec}\, R \oplus M$ with $(\mathcal{Y}, \mathcal{O}' \oplus \mathcal{F}_M)$. We have canonical homotopy equivalences

$$\begin{aligned} F(R, \eta, M) &\simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{Y})_{/\mathcal{O}'}}(\eta^{-1}\mathcal{O}, \mathcal{O}' \oplus \mathcal{F}_M) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\eta^{-1}\mathcal{O}}}(L_{\eta^{-1}\mathcal{O}}, \mathcal{F}_M) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}'}}(\eta^* L_{\mathfrak{X}}, \mathcal{F}_M). \end{aligned}$$

It follows that F is the image of $L_{\mathfrak{X}}$ under the composition of the equivalence $\mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{QCoh}(X)$ with the fully faithful functor $\mathrm{QCoh}(X) \to \mathrm{Fun}(\mathrm{Mod}^X_{\mathrm{cn}}, \mathcal{S})^{op}$ of Proposition 1.3.10. □

Proposition 1.3.18. *Suppose we are given a commutative diagram*



in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. *Assume that g and h admit cotangent complexes. Then f admits a cotangent complex. Moreover, we have a canonical fiber sequence*

$$f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$$

in the stable ∞-category $\mathrm{QCoh}(X)$.

Proof. Let $\mathrm{Mod}^X_{\mathrm{cn}}$ be the ∞-category introduced in Definition 1.3.13. We define functors $F', F, F'' : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ by the formulas

$$F'(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to Y(R \oplus M) \times_{Y(M)} X(M))$$
$$F(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to Z(R \oplus M) \times_{Z(M)} X(M))$$
$$F''(R, \eta, M) = \mathrm{fib}(Y(R \oplus M) \to Z(R \oplus M) \times_{Z(M)} Y(M)).$$

These functors fit into a fiber sequence

$$F' \to F \xrightarrow{\alpha} F''.$$

Let $\theta : \mathrm{QCoh}(X)^{op} \to \mathrm{Fun}(\mathrm{Mod}^X_{\mathrm{cn}}, \mathcal{S})$ be the fully faithful functor of Proposition 1.3.10. Since g and h admit cotangent complexes, we have equivalences

$$F \simeq \theta(L_{X/Z}) \qquad F'' = \theta(f^* L_{Y/Z}).$$

Since θ is fully faithful, the natural transformation α is induced by a map $\beta : f^* L_{Y/Z} \to L_{X/Z}$. It follows from Remark 1.3.11 that F' is equivalent to $\theta(\mathrm{cofib}(\beta))$. □

Corollary 1.3.19. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that X and Y admit cotangent complexes L_X and L_Y. Then f admits a cotangent complex. Moreover, we have a canonical fiber sequence*

$$f^* L_Y \to L_X \to L_{X/Y}$$

in the stable ∞-category $\mathrm{QCoh}(X)$.

Corollary 1.3.20. *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks, and let $f : X \to Y$ be the induced map between the functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ represented by $\mathfrak{X}$ and $\mathfrak{Y}$. Then f admits a cotangent complex. Moreover, we can identify $L_{X/Y}$ with the image of the relative cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}}$ under the equivalence of ∞-categories $\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(X)$.*

Proof. Combine Corollary 1.3.19 with Proposition 1.3.17. □

Remark 1.3.21. Suppose we are given a pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. If f admits a cotangent complex, then f' also admits a cotangent complex. Moreover, we have a canonical equivalence $L_{X'/Y'} \simeq g^* L_{X/Y}$ in the ∞-category $\mathrm{QCoh}(X')$.

We conclude this section by establishing a converse to Remark 1.3.21, which guarantees that the existence of a cotangent complex can be tested locally.

Proposition 1.3.22. *Let $f : X \to Y$ be a morphism in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Suppose that, for every corepresentable functor $Y' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and every natural transformation $\phi : Y' \to Y$, the projection map $Y' \times_Y X \to Y'$ admits a cotangent complex. Then f admits a cotangent complex.*

Proof. Let $\mathrm{Mod}^X_{\mathrm{cn}}$ be as in Definition 1.3.13, and let $F : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ be given by the formula $F(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to X(R) \times_{Y(R)} Y(R \oplus M))$. We wish to show that F is locally almost corepresentable. We will show that F satisfies conditions (a) and (b') of Remark 1.3.14.

To verify condition (a), let us fix a point $\eta \in X(R)$ and consider the functor $F_\eta : \mathrm{Mod}^{\mathrm{cn}}_R \to \mathcal{S}$ given by the restriction of F. Let $Y' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor corepresented by R. Then η determines a natural transformation $Y' \to Y$. Let $X' = Y' \times_Y X$, and let $F' : \mathrm{Mod}^{X'}_{\mathrm{cn}} \to \mathcal{S}$ be the functor given by the formula

$$F'(R_0, \eta_0, M_0) = \mathrm{fib}(X'(R_0 \oplus M_0) \to X'(R_0) \times_{Y'(R_0)} Y'(R_0 \oplus M_0)).$$

Since the projection map $X' \to Y'$ admits a cotangent complex, the functor F' is locally almost corepresentable and therefore satisfies condition (a) of Remark 1.3.14. We now observe that η lifts canonically to a point $\eta' \in X'(R)$. The restriction of F' to the fiber of $\mathrm{Mod}_{\mathrm{cn}}^{X'}$ over (R, η') agrees with F_η. It follows that F_η is corepresentable by an almost connective R-module, as desired.

We now verify condition (b'). Choose a morphism $\alpha : (R, \eta, M) \to (R', \eta', M')$ in $\mathrm{Mod}_{\mathrm{cn}}^{X}$ which induces an equivalence $R' \otimes_R M \to M'$. We wish to prove that $F(\alpha)$ is a homotopy equivalence. Let $F' : \mathrm{Mod}_{\mathrm{cn}}^{X'} \to \mathcal{S}$ be defined as above, and observe that α lifts canonically to a morphism $\overline{\alpha}$ in $\mathrm{Mod}_{\mathrm{cn}}^{X'}$. Since F' is locally almost corepresentable, it satisfies condition (b') of Remark 1.3.14. It follows that $F'(\overline{\alpha})$ is a homotopy equivalence. Since F' is the composition of F with the forgetful functor $\mathrm{Mod}_{\mathrm{cn}}^{X'} \to \mathrm{Mod}_{\mathrm{cn}}^{X}$, we deduce that $F(\alpha)$ is a homotopy equivalence. □

2 Properties of Moduli Functors

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Our goal is to formulate axioms which express that the idea that the functor X "behaves like a geometric object." More precisely, we will isolate some conditions on X which are automatically satisfied whenever X is representable by a spectral Deligne-Mumford stack. Ultimately, we would like to find conditions which are strong enough to admit some sort of converse: that any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ satisfying our axioms is automatically representable. We will prove two such results in §3 (Theorems 3.1.2 and 3.2.1).

We begin in §2.1 by introducing the notions of *cohesive*, *infinitesimally cohesive*, *nilcomplete*, and *integrable* functors from $\mathrm{CAlg}^{\mathrm{cn}}$ to $\mathcal{S}$ (Definitions 2.1.1, 2.1.9, 2.1.3, and 2.1.5). Taken together, these conditions express the idea that X has a well-behaved deformation theory, and can be studied effectively using "infinitesimal" methods. Like many ideas in algebraic geometry, the conditions studied in §2.1 can be relativized. That is, they can be formulated not as properties of a single functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, but instead as properties of a natural transformation between such functors. We will outline this reformulation in §2.2.

Ultimately, the classification of all functors $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which are representable by spectral Deligne-Mumford stacks is too difficult. To make the question more reasonable, we fix a connective $\mathbb{E}_\infty$-ring R, and ask instead to characterize those natural transformations $f : X \to \mathrm{Spec}^{\mathrm{f}} R$ for which X is representable by a spectral Deligne-Mumford n-stack which is almost of finite presentation over R. In §2.3, we will explain how to translate the (almost) finite presentation hypothesis as a condition on the natural transformation f.

In §2.4, we will specialize our attention to a particular functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, which carries a connective $\mathbb{E}_\infty$-ring R to the classifying space for spectral algebraic spaces which are proper, flat, and locally almost of finite presentation over R. In particular, we will show that this functor satisfies most of the axioms introduced in this section, and make contact between this observation and the classical deformation theory of algebraic varieties.

2.1 Nilcomplete, Cohesive, and Integrable Functors

For every spectral Deligne-Mumford stack $\mathfrak{X}$, let $h_{\mathfrak{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by $\mathfrak{X}$, given by the formula $h_{\mathfrak{X}}(R) = \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{X})$. According to Theorem V.2.4.1, the construction $\mathfrak{X} \mapsto h_{\mathfrak{X}}$ determines a fully faithful embedding from the ∞-category Stk of spectral Deligne-Mumford stacks to the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ of space-valued functors on $\mathrm{CAlg}^{\mathrm{cn}}$. If $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is a general functor, then it is often useful to think of X as a kind of generalized geometric object, where a "morphism" from a spectral Deligne-Mumford stack $\mathfrak{X}$ into X is given by a natural transformation of functors $h_{\mathfrak{X}} \to X$. However, this intuition can often be misleading. For example, it is natural to expect that a "morphism" from $\mathfrak{X}$ to X should be determined by its local behavior. To guarantee this, we need an extra assumption on X: namely, that X should satisfy étale descent.

Our goal in this section is to introduce some other conditions on a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, which articulate other properties which should be expected of any reasonably "geometric" functor X. We will say that X is *cohesive* if, whenever an affine spectral Deligne-Mumford stack $\mathfrak{X}$ is obtained by gluing closed substacks $\mathfrak{X}_0$ and $\mathfrak{X}_1$ along a common closed substack $\mathfrak{X}_{01}$, giving a map from $\mathfrak{X}$ to X is equivalent to giving maps from $\mathfrak{X}_0$ and $\mathfrak{X}_1$ into X, together with a compatibility along $\mathfrak{X}_{01}$ (Definition 2.1.1). It will be convenient to also study a weaker version of this condition, where $\mathfrak{X}_0$ and $\mathfrak{X}_1$ is required to be an infinitesimal thickening of $\mathfrak{X}_{01}$ (Definition 2.1.9). We will also consider the closely related notions of a *nilcomplete* and *integrable* functors (Definition 2.1.3 and 2.1.5).

Definition 2.1.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is *cohesive* if the following condition is satisfied:

$(*)$ For every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective, the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow{\scriptstyle f} \\ X(B') & \xrightarrow{g} & X(B) \end{array}$$

is a pullback square in $\mathcal{S}$.

Remark 2.1.2. Using the results of §IX.6, we can reformulate condition $(*)$ of Definition 2.1.1 as follows:

$(*')$ For every pushout diagram of spectral Deligne-Mumford stacks σ :

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}, \end{array}$$

where i and j are closed immersions and $\mathfrak{X}$ is affine, the associated diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\mathfrak{X}_{01}}, X) & \longleftarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\mathfrak{X}_0}, X) \\ \uparrow & & \uparrow \\ \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\mathfrak{X}_1}, X) & \longleftarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\mathfrak{X}}, X) \end{array}$$

is a pullback square of spaces.

If X is cohesive and satisfies étale descent, then condition $(*')$ holds more generally without the assumption that $\mathfrak{X}$ is affine.

Definition 2.1.3. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is *nilcomplete* if, for every connective $\mathbb{E}_\infty$-ring R, the canonical map $X(R) \to \varprojlim X(\tau_{\leq n} R)$ is a homotopy equivalence.

Remark 2.1.4. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be an arbitrary functor. For each $n \geq 0$, let $\tau_{\leq n} \mathfrak{X} = (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$. We then have a canonical map

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\mathfrak{X}}, X) \to \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(h_{\tau_{\leq n} \mathfrak{X}}, X).$$

If $\mathfrak{X}$ is affine and X is nilcomplete, then the map θ is a homotopy equivalence. It follows that if X is nilcomplete and satisfies étale descent, then θ is a homotopy equivalence for an arbitrary spectral Deligne-Mumford stack $\mathfrak{X}$.

Definition 2.1.5. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is *integrable* if the following condition is satisfied:

$(*)$ Let R be a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal $\mathfrak{m} \subseteq \pi_0 R$. Then the inclusion of functors $\mathrm{Spf}\, R \hookrightarrow \mathrm{Spec}^{\mathrm{f}}\, R$ induces a homotopy equivalence

$$X(R) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spec}^{\mathrm{f}}\, R, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, R, X).$$

Remark 2.1.6. The notions of cohesive, nilcomplete, and integrable functor extend in an evident way to the setting of functors $\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$.

The requirements of Definitions 2.1.1, 2.1.3, and 2.1.5 are satisfied for any representable functor:

Proposition 2.1.7. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectrally ringed ∞-topos, and assume that the sheaf $\mathcal{O}$ is connective and strictly Henselian. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be the functor represented by $\mathfrak{X}$ (so that X is given by the formula $X(R) = \mathrm{Map}_{\mathrm{RingTop}_{\text{ét}}}(\mathrm{Spec}\, R, \mathfrak{X})$). Then the functor X is cohesive and nilcomplete. If $\mathfrak{X}$ is a spectral Deligne-Mumford n-stack for some $n < \infty$, then X is integrable.*

Proof. To prove that X is cohesive, it suffices to note that for every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ which induces surjective maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$, the induced diagram

$$\begin{array}{ccc} \mathrm{Spec}\, A' & \longleftarrow & \mathrm{Spec}\, A \\ \uparrow & & \uparrow \\ \mathrm{Spec}\, B' & \longleftarrow & \mathrm{Spec}\, B \end{array}$$

is a pushout square in $\mathrm{RingTop}_{\text{ét}}$ (Corollary IX.6.5).

We now show that X is nilcomplete. Fix a connective $\mathbb{E}_\infty$-ring R, and write $\mathrm{Spec}\, R = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$. We note that for every integer $n \geq 0$, we have an equivalence $\mathrm{Spec}\, \tau_{\leq n} R \simeq (\mathcal{Y}, \tau_{\leq n}\, \mathcal{O}_\mathcal{Y})$. We wish to show that the canonical map

$$\mathrm{Map}_{\mathrm{RingTop}_{\text{ét}}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), \mathfrak{X}) \to \varprojlim_n \mathrm{Map}_{\mathrm{RingTop}_{\text{ét}}}((\mathcal{Y}, \tau_{\leq n}\, \mathcal{O}_\mathcal{Y}), \mathfrak{X})$$

is a homotopy equivalence. Note that a map of spectrally ringed ∞-topoi $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \to \mathfrak{X}$ is local if and only if the induced map $(\mathcal{Y}, \tau_{\leq 0}\, \mathcal{O}_\mathcal{Y}) \to \mathfrak{X}$ is local; it will therefore suffice to show that the map

$$\theta : \mathrm{Map}_{\mathrm{RingTop}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), \mathfrak{X}) \to \varprojlim_n \mathrm{Map}_{\mathrm{RingTop}}((\mathcal{Y}, \tau_{\leq n}\, \mathcal{O}_\mathcal{Y}), \mathfrak{X})$$

is a homotopy equivalence. Let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by the geometric morphisms $f^* : \mathcal{X} \to \mathcal{Y}$. To prove that θ is a homotopy equivalence, it will suffice to show that it induces a homotopy equivalence after passing to the homotopy fiber over any geometric morphism $f^* \in \mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})^{\simeq}$. In other words, we must show that the canonical map

$$\mathrm{Map}_{\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{Y})}(f^*\, \mathcal{O}, \mathcal{O}_\mathcal{Y}) \to \mathrm{Map}_{\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{Y})}(f^*\, \mathcal{O}, \tau_{\leq n}\, \mathcal{O}_\mathcal{Y})$$

is an equivalence. For this, it suffices to show that $\mathcal{O}_\mathcal{Y} \simeq \varprojlim \tau_{\leq n}\, \mathcal{O}_\mathcal{Y}$, which was established in the proof of Theorem VII.8.42.

Now suppose that $\mathfrak{X}$ is a spectral Deligne-Mumford stack; we will show that X is integrable. Let A be a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal. Choose a tower of A-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma XII.5.1.5, so that $\operatorname{Spf} A \simeq \varinjlim \operatorname{Spec}^{\mathrm{f}} A_n$. Each of the maps $\pi_0 A_i \to \pi_0 A_0$ is surjective with nilpotent kernel, and therefore induces an equivalence of ∞-categories $\mathrm{CAlg}^{\text{ét}}_{A_i} \to \mathrm{CAlg}^{\text{ét}}_{A_0}$. For every functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ satisfying étale descent, let $Y(m) \in \mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0}$ be the functor given by the composition

$$\mathrm{CAlg}^{\text{ét}}_{A_0} \simeq \mathrm{CAlg}^{\text{ét}}_{A_m} \to \mathrm{CAlg}^{\mathrm{cn}} \xrightarrow{Y} \mathcal{S},$$

and let $Y(\infty)$ denote the image of $(Y|\,\mathrm{CAlg}^{\text{ét}}_A) \in \mathcal{S}\mathrm{hv}^{\text{ét}}_A$ under the pullback map $\mathcal{S}\mathrm{hv}^{\text{ét}}_A \to \mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0}$. Then the canonical map $X(A) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(\operatorname{Spf} A, X) \to \varprojlim_{m \geq 0} X(A_m)$ can be identified with the composition

$$X(A) \xrightarrow{\theta} X(\infty)(A_0) \xrightarrow{\theta'} \varprojlim_m X(m)(A_0).$$

Proposition VII.7.16 implies that A is Henselian, so that θ is a homotopy equivalence by Proposition XI.3.22. To prove that θ' is a homotopy equivalence, it will suffice to verify the following assertion:

$(*)$ Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is representable by a spectral Deligne-Mumford n-stack. Then the canonical map $\phi_Y : Y(\infty) \to \varprojlim Y(m)$ is an equivalence in the ∞-topos $\mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0}$.

To prove $(*)$, choose an étale surjection $u : Y_0 \to Y$, where Y_0 is representable by a disjoint union of affine spectral Deligne-Mumford stacks. Let $Y_\bullet$ denote the Čech nerve of u, so that $Y \simeq |Y_\bullet|$. Then ϕ_Y can be identified with the composite map

$$Y(\infty) \simeq |Y_\bullet(\infty)| \xrightarrow{\phi} |\varprojlim_m Y_\bullet(m)| \xrightarrow{\phi'} \simeq \varprojlim_m Y(m).$$

We first claim that ϕ' is an equivalence. Note that the simplicial object $\varprojlim_m Y_\bullet$ is given by the Čech nerve of the map $v : \varprojlim_m Y_0(m) \to \varprojlim Y(m)$. Since $\mathcal{S}\mathrm{hv}^{\text{ét}}_{A_0}$ is an ∞-topos, the map ϕ' is an equivalence if and only if v is an effective epimorphism. Let B_0 be any étale A_0-algebra, so that B_0 admits an essentially unique lift to an étale A_m-algebra B_m for each m. Since u is étale , the canonical map $Y_0(B_m) \to Y_0(B_0) \times_{Y(B_0)} Y(B_m)$ is a homotopy equivalence for each m. It follows that v is a pullback of the map $Y_0 \to Y$, which is an effective epimorphism by virtue of our assumption that u is an étale surjective.

Using the above argument, we see that ϕ_Y is an equivalence if and only if ϕ is an equivalence. Consequently, to prove that ϕ_Y is an equivalence, it will suffice to show that ϕ_{Y_p} is an equivalence. We now proceed by induction on n. If $n > 1$, then each Y_p is representable by a spectral Deligne-Mumford $(n-1)$-stack, so that the desired result follows from the inductive hypothesis. If $n = 1$, then each Y_p is representable by a spectral algebraic space; it will therefore suffice to verify $(*)$ in the special case where Y is representable by a spectral algebraic space. In this case, for each $p \geq 0$, the canonical map $Y_p(R) \to Y_0(R)^p$ is injective for every discrete commutative ring R. It will therefore suffice to verify $(*)$ under the assumption that there exists a map $Y \to Z$ which induces a monomorphism $Y(R) \to Z(R)$ for every discrete commutative ring R, where Z is representable by a disjoint union of affine spectral Deligne-Mumford stacks. In this case, each Y_p is itself a disjoint union of affine spectral Deligne-Mumford stacks. It will therefore suffice to verify $(*)$ in the special case $Y = \coprod_\alpha Y_\alpha$, where each Y_α is corepresented by a connective $\mathbb{E}_\infty$-ring R_α.

Let B_0 be an étale A_0-algebra; we wish to show that the canonical map

$$\gamma : Y(\infty)(B_0) \to \varprojlim Y(m)(B_0)$$

is a homotopy equivalence. Without loss of generality, we may suppose that the spectrum of B_0 is connected. In this case, γ is given by a disjoint union of maps

$$\gamma_\alpha : Y_\alpha(\infty)(B_0) \to Y_\alpha(m)(B_0).$$

It will therefore suffice to show that each γ_α is a homotopy equivalence. Let B be a finite étale A-algebra satisfying $B_0 \simeq B \otimes_A A_0$ (see Proposition XI.3.20). We are then reduced to showing that the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}}(R_\alpha, B) \to \varprojlim_m \mathrm{Map}_{\mathrm{CAlg}}(R_\alpha, \varprojlim (B \otimes_A A_m))$$

is a homotopy equivalence. To prove this, it suffices to show that B is given by the limit of the diagram $\{B \otimes_A A_m\}$. Since B is a finite flat A-module, this follows from the identification $A \simeq \varprojlim_m A_m$. □

The following reformulation of Definition 2.1.3 is sometimes convenient:

Proposition 2.1.8. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *The functor X is nilcomplete.*

(2) *Suppose we are given a tower of connective $\mathbb{E}_\infty$-rings*

$$\cdots \to R(2) \to R(1) \to R(0)$$

satisfying the following condition: for every integer n, the tower of abelian groups

$$\cdots \to \pi_n R(2) \to \pi_n R(1) \to \pi_n R(0)$$

is eventually constant. Then the canonical map $X(\varprojlim R(n)) \to \varprojlim X(R(n))$ is a homotopy equivalence.

Proof. Let R be an arbitrary connective $\mathbb{E}_\infty$-ring. Then the Postnikov tower

$$\cdots \to \tau_{\leq 2} R \to \tau_{\leq 1} R \to \tau_{\leq 0} R$$

satisfies the hypothesis appearing in condition (2). It follows that (2) ⇒ (1). For the converse, let us assume that X is nilcomplete and let

$$\cdots \to R(2) \to R(1) \to R(0)$$

be a tower of connective $\mathbb{E}_\infty$-rings satisfying the hypothesis of (2). Set $R = \varprojlim R(n)$. We have a commutative diagram

$$\begin{array}{ccc} X(R) & \longrightarrow & \varprojlim_n X(R(n)) \\ \downarrow & & \downarrow \\ \varprojlim_m X(\tau_{\leq m} R) & \longrightarrow & \varprojlim_{n,m} X(\tau_{\leq m} R(n)) \end{array}$$

Since X is nilcomplete, the vertical maps in this diagram are homotopy equivalences. Consequently, to show that the upper horizontal map is a homotopy equivalence, it suffices to show that the lower horizontal map is a homotopy equivalence. For this, it suffices to show that for every $m \geq 0$, the map $X(\tau_{\leq m} R) \to \varprojlim_n X(\tau_{\leq m} R(n))$ is a homotopy equivalence. This is clear, since the tower

$$\cdots \to \tau_{\leq m} R(2) \to \tau_{\leq m} R(1) \to \tau_{\leq m} R(0)$$

is eventually constant (with value $\tau_{\leq m} R$). □

We now introduce a weaker version of Definition 2.1.1.

Definition 2.1.9. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is *infinitesimally cohesive* if the following condition is satisfied:

(∗) For every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{g} & B \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjections whose kernels are nilpotent ideals in $\pi_0 A$ and $\pi_0 B'$, respectively. Then the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow f \\ X(B') & \xrightarrow{g} & X(B) \end{array}$$

is a pullback square in $\mathcal{S}$.

Remark 2.1.10. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. If X is cohesive, then X is infinitesimally cohesive. In particular, if X is representable by a spectral Deligne-Mumford stack, then X is infinitesimally cohesive.

Remark 2.1.11. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ spanned by those functors which are cohesive (infinitesimally cohesive, nilcomplete, integrable). Then $\mathcal{C}$ is closed under small limits in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

Remark 2.1.12. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Let R be a connective $\mathbb{E}_\infty$-ring, and let $\widetilde{R}$ be a square-zero extension of R by a connective R-module M, classified by a map of R-modules $d : L_R \to M$, so that we have a commutative diagram of spaces σ :

$$\begin{array}{ccc} X(\widetilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ X(R) & \longrightarrow & X(R \oplus \Sigma M). \end{array}$$

Let η be a point of $X(R)$ and let $X(\widetilde{R})_\eta$ denote the fiber product $X(\widetilde{R}) \times_{X(R)} \{\eta\}$. Suppose that X admits a cotangent complex L_X, so that we can identify $\eta^* L_X$ with an R-module. Let ν denote the composite map

$$\eta^* L_X \to L_R \xrightarrow{d} \Sigma M.$$

Then the diagram σ determines a map $\theta : X(\widetilde{R})_\eta \to P$, where P denotes the space of paths from ν to the base point of the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_X, \Sigma M)$. If X is infinitesimally cohesive, then σ is a pullback diagram, so that θ is a homotopy equivalence. In this case, η can be lifted to a point of $X(\widetilde{R})$ if and only if ν represents the zero element of the abelian group $\mathrm{Ext}^1_R(\eta^* L_X, M)$.

We can summarize Remark 2.1.12 informally as follows: if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is an infinitesimally cohesive functor which admits a cotangent complex L_X, then L_X "controls" the deformation theory of the functor X. The following result provides a converse:

Proposition 2.1.13. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a nilcomplete functor which admits a cotangent complex. The following conditions are equivalent:*

(1) *For every pullback diagram*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \longrightarrow & B, \end{array}$$

of connective $\mathbb{E}_\infty$-rings, if the map f induces a surjection of commutative rings $\pi_0 A \to \pi_0 B$ with nilpotent kernel, then the diagram of spaces

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ X(B') & \longrightarrow & X(B) \end{array}$$

is a pullback square.

(2) *The functor X is infinitesimally cohesive.*

(3) *Let R be a connective $\mathbb{E}_\infty$-ring, M a connective R-module, $\eta : L_R \to \Sigma M$ a derivation, and R^η the corresponding square-zero extension of R by M, so that we have a pullback square*

$$\begin{array}{ccc} R^\eta & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \oplus \Sigma M. \end{array}$$

Then the diagram

$$\begin{array}{ccc} X(R^\eta) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ X(R) & \longrightarrow & X(R \oplus \Sigma M) \end{array}$$

is a pullback square in $\mathcal{S}$.

Lemma 2.1.14. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Suppose that f induces a surjection of commutative rings $\pi_0 A \to \pi_0 B$ whose kernel I is a nilpotent ideal of $\pi_0 A$. Then we can write A as the limit of a tower*

$$\cdots \to B(2) \to B(1) = B$$

in the ∞-category $\mathrm{CAlg}_{/B}$ with the following property: each $B(n+1)$ is a square-zero extension of $B(n)$ by a $B(n)$-module $M[k_n]$, where M is discrete and $k_n \geq 0$. Moreover, we can assume that the sequence of integers $\{k_n\}_{n \geq 0}$ tends to infinity as n grows.

Proof. Choose an integer m such that $I^m = 0$. For $k \leq m$, we define $B(k)$ by the formula $B \times_{\pi_0 B} (\pi_0 A / I^k)$. Since $\pi_0 A / I^{k+1}$ is a square-zero extension of $\pi_0 A / I^k$ by I^k / I^{k+1}, we deduce that $B(k+1)$ is a square-zero extension of $B(k)$ by the discrete module I^k / I^{k+1} for $0 < k < m$. We next define $B(k) \in \mathrm{CAlg}_A$ for $k > m$ using induction on k, so that the fiber of the map $A \to B(k)$ is $(k-m)$-connective. Assume that $B(k)$ has been defined for $k \geq m$, and let $M = \pi_{k-m} \operatorname{fib}(A \to B(k))$. Since the map $\pi_0 A \to \pi_0 B(k)$ is an isomorphism, Theorem A.8.4.3.1 implies that $L_{B(k)/A}$ is $(k-m+1)$-connective and that there is a canonical isomorphism $\pi_{k-m+1} L_{B(k)/A} \simeq M$. In particular, there exists a map of $B(k)$-modules $\eta : L_{B(k)/A} \to M[k-m+1]$ which induces an isomorphism $\pi_{k-m+1} L_{B(k)/A} \simeq M$. Let $B(k+1) = B(k)^\eta$ denote the square-zero extension of $B(k)$ by $M[k-m]$ classified by η. We now observe that by construction, the canonical map $A \to B(k+1)$ has $(k-m+1)$-connective fiber. □

Proof of Proposition 2.1.13. The implications (1) ⇒ (2) and (2) ⇒ (3) are obvious (and do not require any assumptions on X). Let us prove that (3) ⇒ (1). Suppose we are given a pullback square of connective $\mathbb{E}_\infty$-rings σ :

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B, \end{array}$$

where the maps $\pi_0 A \to \pi_0 B$ is a surjection with nilpotent kernel. We wish to show that $X(\sigma)$ is a pullback square in $\mathcal{S}$. Choose a tower

$$\cdots \to B(3) \to B(2) \to B(1) = B$$

satisfying the requirements of Lemma 2.1.14. For each integer $n \geq 1$, let $B'(n) = B(n) \times_B B'$, so that we have a pullback square $\sigma(n)$:

$$\begin{array}{ccc} B'(n) & \longrightarrow & B(n) \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B. \end{array}$$

Since X is nilcomplete, Proposition 2.1.8 implies that $X(\sigma)$ is a limit of the tower of diagrams $\{X(\sigma(n))\}_{n \geq 1}$. It will therefore suffice to show that each $X(\sigma(n))$ is a pullback square in $\mathcal{S}$. The proof proceeds by induction on n, the case $n = 1$ being trivial. If $n > 1$, we consider the commutative diagram

$$\begin{array}{ccc} X(B'(n)) & \longrightarrow & X(B(n)) \\ \downarrow & & \downarrow \\ X(B'(n-1)) & \longrightarrow & X(B(n-1)) \\ \downarrow & & \downarrow \\ X(B') & \longrightarrow & X(B). \end{array}$$

The inductive hypothesis implies that the lower square is a pullback diagram. To prove that the outer square is a pullback diagram, it suffices to show that the upper square is a pullback diagram. By hypothesis, $B(n)$ is a square-zero extension of $B(n-1)$ by a connective $B(n-1)$-module M. We therefore have a commutative diagram

$$\begin{array}{ccccc} X(B'(n)) & \longrightarrow & X(B(n)) & \longrightarrow & X(B(n-1)) \\ \downarrow & & \downarrow & & \downarrow \\ X(B'(n-1)) & \longrightarrow & X(B(n-1)) & \longrightarrow & X(B(n-1) \oplus \Sigma M) \end{array}$$

where the square on the right is a pullback diagram by virtue of assumption (3). To prove that the left square is a pullback, it will suffice to show that the outer rectangle is a pullback. Note that the bottom horizontal composite admits a factorization

$$B'(n-1) \to B'(n-1) \oplus \Sigma M \to B(n-1) \oplus \Sigma M.$$

We may therefore form a commutative diagram

$$\begin{array}{ccccc} B'(n) & \longrightarrow & R & \longrightarrow & B(n-1) \\ \downarrow & & \downarrow & & \downarrow \\ B'(n-1) & \longrightarrow & B'(n-1) \oplus \Sigma M & \longrightarrow & B(n-1) \oplus \Sigma M \\ & & \downarrow & & \downarrow \\ & & B'(n-1) & \longrightarrow & B(n-1) \end{array}$$

where every square is a pullback diagram. Since the vertical composition on the right is an equivalence, it follows that the vertical compositon in the middle is an equivalence: that is, we can identify R with $B(n-1)$.

Applying the functor X, we obtain a diagram of spaces

$$\begin{array}{ccccc}
X(B'(n)) & \longrightarrow & X(B'(n-1)) & \longrightarrow & X(B(n-1)) \\
\downarrow & & \downarrow & & \downarrow \\
X(B'(n-1)) & \longrightarrow & X(B'(n-1) \oplus \Sigma M) & \longrightarrow & X(B(n-1) \oplus \Sigma M) \\
 & & \downarrow & & \downarrow \\
 & & X(B'(n-1)) & \longrightarrow & X(B(n-1)).
\end{array}$$

The upper left square is a pullback diagram by assumption (3). Since X admits a cotangent complex, the lower right square is also a pullback diagram (Example 1.3.15). Since the vertical composite maps are equivalences, the rectangle on the right is a pullback diagram. It follows that the upper left square is a pullback square, so that the upper rectangle is a pullback square as desired. □

Here is a sample application:

Proposition 2.1.15. *Let $f : X \to Y$ be a natural transformations between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that X and Y are nilcomplete, infinitesimally cohesive and admit cotangent complexes, and that the relative cotangent complex $L_{X/Y}$ is $(n+2)$-connective for some integer $n \geq 0$. The following conditions are equivalent:*

(1) *For every commutative ring R (regarded as a discrete $\mathbb{E}_\infty$-ring), the map f induces a homotopy equivalence $X(R) \to Y(R)$.*

(2) *For every n-truncated connective $\mathbb{E}_\infty$-ring R, the map f induces a homotopy equivalence $X(R) \to Y(R)$.*

Proof. The implication (2) ⇒ (1) is obvious. Conversely, suppose that (1) is satisfied. We must show that for every n-truncated connective $\mathbb{E}_\infty$-ring A, the map f induces a homotopy equivalence $f_A : X(A) \to Y(A)$. The proof proceeds by induction on n. When $n = 0$, the desired result follows from (1). If $n > 0$, then $\tau_{\leq n} A$ is a square-zero extension of $\tau_{\leq n-1} A$ (Corollary A.8.4.1.28). We therefore have a pullback square of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc}
\tau_{\leq n} A & \longrightarrow & \tau_{\leq n-1} A \\
\downarrow & & \downarrow \\
\tau_{\leq n-1} A & \longrightarrow & \tau_{\leq n-1} A \oplus M,
\end{array}$$

where $M \simeq (\pi_n A)[n+1]$. Since X and Y are infinitesimally cohesive, to prove that $f_{\tau_{\leq n} A}$ is a homotopy equivalence, it will suffice to show that $f_{\tau_{\leq n-1} A}$ is a homotopy equivalence and $f_{\tau_{\leq n-1} A \oplus M}$ is (-1)-truncated (that is, it is equivalent to the inclusion of a summand). In the first case, this follows from the inductive hypothesis. For the second case, consider the commutative diagram

$$\begin{array}{ccc}
X(\tau_{\leq n-1} A \oplus M) & \longrightarrow & Y(\tau_{\leq n-1} A \oplus M) \\
\downarrow & & \downarrow \\
X(\tau_{\leq n-1} A) & \longrightarrow & Y(\tau_{\leq n-1} A).
\end{array}$$

We wish to prove that the upper horizontal map is (-1)-truncated. Since the bottom horizontal map is a homotopy equivalence, it will suffice to prove that we obtain a (-1)-truncated map after passing to the homotopy fibers over any point $\eta \in X(\tau_{\leq n-1} A)$. Unwinding the definitions, we are reduced to proving that the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_{\tau_{\leq n-1} A}}(\eta^* L_X, M) \to \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leq n-1} A}}(\eta^* f^* L_Y, M)$$

is (-1)-truncated. Using the fiber sequence,

$$\eta^* f^* L_Y \to \eta^* L_X \to \eta^* L_{X/Y},$$

we are reduced to proving that $\eta^* L_{X/Y}$ is $(n+2)$-truncated, which follows from our hypothesis. □

Corollary 2.1.16. *Let $f : X \to Y$ be a natural transformations between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that X and Y are infinitesimally cohesive, nilcomplete, and admit cotangent complexes. Then f is an equivalence if and only if the following conditions are satisfied:*

(1) *For every commutative ring R (regarded as a discrete $\mathbb{E}_\infty$-ring), the map f induces a homotopy equivalence $X(R) \to Y(R)$.*

(2) *The relative cotangent complex $L_{X/Y}$ is trivial.*

Proof. It is clear that if f is an equivalence then conditions (1) and (2) are satisfied. Conversely, suppose that (1) and (2) are satisfied. We wish to show that for every connective $\mathbb{E}_\infty$-ring R, the canonical map $\theta : X(R) \to Y(R)$ is a homotopy equivalence. Since X and Y are nilcomplete, the map θ is a limit of maps $\theta_n : X(\tau_{\leq n} R) \to Y(\tau_{\leq n} R)$. It will therefore suffice to show that each θ_n is a homotopy equivalence, which follows from Proposition 2.1.15. □

In good cases, the integrability of a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ can be tested at the level of discrete commutative rings:

Proposition 2.1.17. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. The following conditions are equivalent:*

(a) *The functor X is integrable.*

(b) *For every complete local Noetherian ring A, the canonical map*

$$X(A) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spec}^{\mathrm{f}} A, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, X)$$

is a homotopy equivalence.

(c) *For every complete local Noetherian ring A with maximal ideal $\mathfrak{m}$, the canonical map*

$$X(A) \to \varprojlim_n X(A/\mathfrak{m}^n)$$

is a homotopy equivalence.

Lemma 2.1.18. *Let A be a complete local Noetherian ring with maximal ideal $\mathfrak{m}$, and choose a tower of A-algebras*

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma XII.5.1.5. Then, for every integer $n \geq 0$, the tower $\{\tau_{\leq n} A_m\}_{m \geq 0}$ is equivalent (as a pro-object of CAlg*) to the tower $\{A/\mathfrak{m}^m\}_{m \geq 0}$.*

Proof. Let $k = A/\mathfrak{m}$ denote the residue field of A, and regard $\{\tau_{\leq n} A_m\}_{m \geq 0}$ and $\{A/\mathfrak{m}^m\}_{m \geq 0}$ as pro-objects of the ∞-category $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$ of Notation XII.6.1.3. It now suffices to show that both pro-objects corepresent the same functor $\mathrm{CAlg}^{\mathrm{sm}}_{/k} \to \mathcal{S}$, which follows from Lemma XII.6.3.3. □

Proof of Proposition 2.1.17. The implication (a) ⇒ (b) is obvious. We next prove that (b) ⇒ (a). Let A be a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to the maximal ideal $\mathfrak{m} \subseteq \pi_0 A$. Choose a tower of A-algebras

$$\cdots \to A_2 \to A_1 \to A_0$$

satisfying the requirements of Lemma XII.5.1.5, so that $\mathrm{Spf}\, A \simeq \varinjlim_m \mathrm{Spec}^{\mathrm{f}}\, A_m$. For every A-algebra B, we can identify the formal spectrum $\mathrm{Spf}\, B$ (taken with respect to the image of the maximal ideal of $\pi_0 A$) with the filtered colimit $\varinjlim_m \mathrm{Spec}^{\mathrm{f}}(A_m \otimes_A B)$. Let θ_B denote the canonical map $X(B) \to \varprojlim_m X(A_m \otimes_A B) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, B, X)$. We wish to show that θ_A is a homotopy equivalence.

Consider the diagram

$$\begin{array}{ccc} X(A) & \longrightarrow & \varprojlim_n X(\tau_{\leq n} A) \\ \downarrow & & \downarrow \\ \varprojlim_m X(A_m) & \longrightarrow & \varprojlim_{m,n} X(A_m \otimes_A \tau_{\leq n} A). \end{array}$$

The upper horizontal map is a homotopy equivalence since X is nilcomplete, and the bottom horizontal map is a homotopy equivalence by Proposition 2.1.8. It follows that θ_A can be identified with the limit of the tower of maps $\{\theta_{\tau_{\leq n} A}\}_{n \geq 0}$. It will therefore suffice to show that each $\theta_{\tau_{\leq n} A}$ is a homotopy equivalence. We may therefore replace A by $\tau_{\leq n} A$ and thereby reduce to the case where A is n-truncated for some integer n.

If $n = 0$, then A is discrete and the desired result follows from (b). Let us therefore assume that $n > 0$. Let $A' = \tau_{\leq n-1} A$ and let $M = \Sigma^{n+1}(\pi_n A)$, so that A fits into a pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A' \oplus M. \end{array}$$

Since X is infinitesimally cohesive, we obtain a pullback diagram

$$\begin{array}{ccc} \theta_A & \longrightarrow & \theta_{A'} \\ \downarrow & & \downarrow \\ \theta_{A'} & \longrightarrow & \theta_{A' \oplus M} \end{array}$$

in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{S})$. The inductive hypothesis implies that $\theta_{A'}$ is a homotopy equivalence. It will therefore suffice to show that $\theta_{A' \oplus M}$ is a homotopy equivalence. Using the inductive hypothesis, we are reduced to proving that the canonical map

$$\psi : X(A' \oplus M) \to X(A') \times_{\varprojlim X(A_m \otimes_A A')} \varprojlim X((A_m \otimes_A A') \oplus (A_m \otimes_A M)).$$

Using the assumption that X is infinitesimally cohesive, we can identify the right side with $\varprojlim_m X(A' \oplus (A_m \otimes_A M))$. To show that ψ is a homotopy equivalence, it will suffice to show that ψ induces a homotopy equivalence after passing to the fibers over any point $\eta \in X(A')$. Since X admits a cotangent complex, this is equivalent to the assertion that the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_{A'}}(\eta^* L_X, M) \to \varprojlim \mathrm{Map}_{\mathrm{Mod}_{A'}}(\eta^* L_X, A_m \otimes_A M).$$

For this, it suffices to show that the canonical map $M \to \varprojlim_m A_m \otimes_A M$ is an equivalence. Since M is connective, this is equivalent to the requirement that M is $\mathfrak{m}$-complete, where $\mathfrak{m}$ denotes the maximal ideal of $\pi_0 A$ (Remark XII.5.1.11). This completeness follows from our assumption that A is complete, since M is an almost perfect A-module (Proposition XII.4.3.8). This completes the proof that $(b) \Rightarrow (a)$.

To prove that (b) and (c) are equivalent, it will suffice to show that for every complete local Noetherian ring A with maximal ideal $\mathfrak{m}$, the canonical map $\rho : X(A) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, X)$ is a homotopy equivalence if and only if the canonical map $X(A) \to \varprojlim_n X(A/\mathfrak{m}^n)$ is a homotopy equivalence. To prove this, choose $\{A_m\}_{m \geq 0}$ as above. Since X is nilcomplete, we can identify ρ with the composite map

$$X(A) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, X) \simeq \varprojlim_m X(A_m) \simeq \varprojlim_{m,n} X(\tau_{\leq n} A_m).$$

The desired result now follows from Lemma 2.1.18. □

2.2 Relativized Properties of Functors

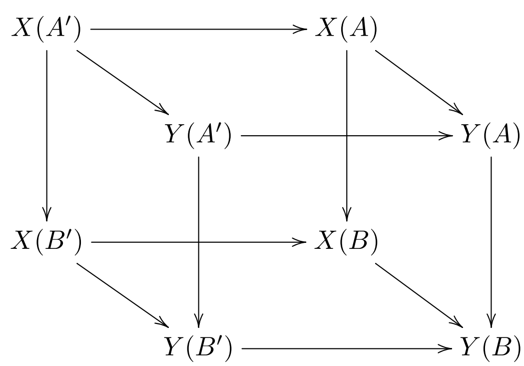
In §2.1, we introduced the definition of cohesive, infinitesimally cohesive, nilcomplete, and integrable functors from $\mathrm{CAlg}^{\mathrm{cn}}$ to $\mathcal{S}$. In this section, we will study relative versions of these definitions, which apply not to individual functors, but to natural transformations between functors.

Definition 2.2.1. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that f is:

(a) *cohesive* if, for every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

of connective $\mathbb{E}_\infty$-rings such that $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective, the cubical diagram of spaces

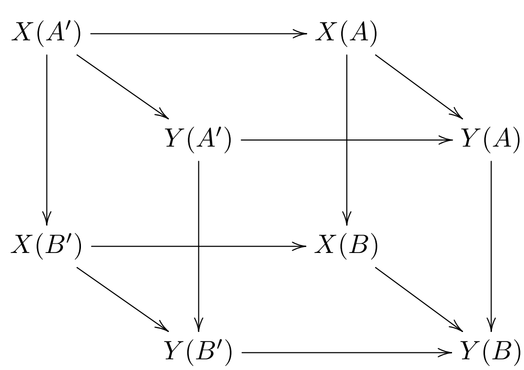


is a limit.

(b) *infinitesimally cohesive* if, for every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

of connective $\mathbb{E}_\infty$-rings such that $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjections with nilpotent kernel, the cubical diagram of spaces



is a limit.

(*c*) *nilcomplete* if, for every connective $\mathbb{E}_\infty$-ring A, the diagram

$$\begin{array}{ccc} X(A) & \longrightarrow & \varprojlim X(\tau_{\leq n} A) \\ \downarrow & & \downarrow \\ Y(A) & \longrightarrow & \varprojlim Y(\tau_{\leq n} A) \end{array}$$

is a pullback square.

(*d*) *integrable* if, for every local Noetherian $\mathbb{E}_\infty$-ring A, the diagram

$$\begin{array}{ccc} X(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, X) \\ \downarrow & & \downarrow \\ Y(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\mathrm{Spf}\, A, Y) \end{array}$$

is a pullback square.

Example 2.2.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be an arbitrary functor, and let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the constant functor taking the value $* \in \mathcal{S}$. Then there is a unique natural transformation $f : X \to Y$ (up to homotopy). Moreover, the natural transformation f is cohesive (infinitesimally cohesive, nilcomplete, integrable) if and only if X is cohesive (infinitesimally cohesive, nilcomplete, integrable).

Remark 2.2.3. Suppose we are given a diagram

$$\begin{array}{ccccc} & & Y & & \\ & {}^{f}\nearrow & & \searrow^{g} & \\ X & & \xrightarrow{\quad h \quad} & & Z \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, where g cohesive (infinitesimally cohesive, nilcomplete, integrable). Then f is cohesive (infinitesimally cohesive, nilcomplete, integrable) if and only if h is cohesive (infinitesimally cohesive, nilcomplete, integrable).

Taking Z to be the final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, we deduce that if $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is cohesive (infinitesimally cohesive, nilcomplete, integrable), then a morphism $f : X \to Y$ is cohesive (infinitesimally cohesive, nilcomplete, integrable) if and only if X is cohesive (infinitesimally cohesive, nilcomplete, integrable).

Notation 2.2.4. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Given a point $\eta \in Y(R)$, we let $X_\eta : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ given on objects by the formula $X_\eta(A) = X(A) \times_{Y(A)} \{\eta_A\}$, where $\eta_A \in Y(A)$ denotes the image of η.

Suppose now that η induces an equivalence $\mathrm{Spec}^{\mathrm{f}} R \to Y$: that is, for every $\mathbb{E}_\infty$-ring A, evaluation at η induces a homotopy equivalence $\mathrm{Map}_{\mathrm{CAlg}}(R, A) \to Y(A)$. In this case, the construction $X \mapsto X_\eta$ induces an equivalence of ∞-categories $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Y} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$ (Corollary T.5.1.6.12).

Now suppose that $F : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ is an arbitrary functor. The above discussion shows that there is an equivalence $F \simeq X_\eta$ for some natural transformation $f : X \to Y \simeq \mathrm{Spec}^{\mathrm{f}} R$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, which is determined uniquely up to a contractible space of choices. In this case, we will denote the functor X by $\underline{F}$. We will say that F is *cohesive* (*infinitesimally cohesive*, *nilcomplete*, *integrable*) if $\underline{F}$ is cohesive (infinitesimally cohesive, nilcomplete, *integrable*). Since $\mathrm{Spec}^{\mathrm{f}} R$ is cohesive and nilcomplete, we see from Remark 2.2.3 that F is cohesive (infinitesimally cohesive, nilcomplete, integrable) if and only if the natural transformation f is cohesive (infinitesimally cohesive, nilcomplete, integrable).

Remark 2.2.5. Let R be a connective $\mathbb{E}_\infty$-ring, let $F : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be a functor, and let $\underline{F} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be as in Notation 2.2.4. If F classifies a left fibration $p : \mathcal{C} \to \mathrm{CAlg}^{\mathrm{cn}}_R$, then $\overline{F}$ classifies the left fibration given by the composite map

$$\mathcal{C} \xrightarrow{p} \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{CAlg}^{\mathrm{cn}}.$$

More informally: we can identify $\overline{F}(A)$ with the space of all pairs (ϕ, η), where $\phi : R \to A$ is a map of $\mathbb{E}_\infty$-rings and $\eta \in F(A)$, where A is regarded as an R-algebra via the map ϕ.

Remark 2.2.6. Let R be a connective $\mathbb{E}_\infty$-ring and let $F : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be a functor. Unwinding the definitions, we deduce:

- The functor F is cohesive if and only if, for every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

  in $\mathrm{CAlg}^{\mathrm{cn}}_R$ for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective, the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow{\scriptstyle f} \\ X(B') & \xrightarrow{g} & X(B) \end{array}$$

  is a pullback square in $\mathcal{S}$.

- The functor F is infinitesimally cohesive if and only if, for every pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

  in $\mathrm{CAlg}^{\mathrm{cn}}_R$ for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective with nilpotent kernel, the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow{\scriptstyle f} \\ X(B') & \xrightarrow{g} & X(B) \end{array}$$

  is a pullback square in $\mathcal{S}$.

- The functor F is nilcomplete if and only if, for every connective R-algebra A, the canonical map $F(A) \to \varprojlim F(\tau_{\leq n} A)$ is a homotopy equivalence.

- The functor F is integrable if and only if, for every local Noetherian $\mathbb{E}_\infty$-algebra A over R which is complete with respect to its maximal ideal $\mathfrak{m}$, the canonical map $F(A) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})}(Y, F)$, where $Y : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ is the functor which assigns to each R-algebra B the full subcategory of $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$ spanned by those maps which annihilate some power of the maximal ideal $\mathfrak{m}$.

Proposition 2.2.7. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. The following conditions are equivalent:*

(1) *The map f is cohesive (infinitesimally cohesive, nilcomplete, integrable).*

(2) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$*, the map f' is cohesive (infinitesimally cohesive, nilcomplete,integrable).*

(3) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ *where Y' is a corepresentable functor, the map f' is cohesive (infinitesimally cohesive, nilcomplete,integrable).*

(4) *For every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in Y(R)$, the functor $X_\eta : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ is cohesive (infinitesimally cohesive, nilcomplete,integrable).*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious, and the equivalence (3) $\Leftrightarrow$ (4) follows from Remark 2.2.3. We will complete the proof by showing that (3) $\Rightarrow$ (1). For simplicity, let us treat the assertion concerning nilcomplete the functors; the proofs in the other cases are the same. Let R be a connective $\mathbb{E}_\infty$-ring; we wish to show that the diagram

$$\begin{array}{ccc} X(R) & \longrightarrow & \varprojlim X(\tau_{\leq n} R) \\ \downarrow & & \downarrow \\ Y(R) & \longrightarrow & Y(\tau_{\leq n} R) \end{array}$$

is a pullback square. Equivalently, we wish to show that for every point $\eta \in Y(R)$, the induced map

$$X(R) \times_{Y(R)} \{\eta\} \to \varprojlim (X(\tau_{\leq n} R) \times_{Y(\tau_{\leq n} R)} \{\eta\})$$

is a homotopy equivalence (here we abuse notation by identifying η with its image in $Y(\tau_{\leq n} R)$, for each $n \geq 0$). The point η determines a natural transformation $Y' \to Y$, where $Y' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is the functor corepresented by R. Since η lies in the essential image of the map $Y'(R) \to Y(R)$, we may replace f by the projection map $f' : X \times_Y Y' \to Y'$. In this case, the desired result follows from (3). ☐

Corollary 2.2.8. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and suppose that f is representable by spectral Deligne-Mumford stacks. Then f is cohesive, nilcomplete, and admits a cotangent complex. Moreover, the relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is connective.*

Proof. The first two assertions follow from Proposition 2.2.7 and 2.1.7. For the third, we combine Propositions 1.3.22 and 1.3.17. ☐

Proposition 2.2.9. *Let*

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ X & & \xrightarrow{\quad h \quad} & & Z \end{array}$$

be a commutative diagram in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$*. Assume that g is infinitesimally cohesive and admits a cotangent complex. Then f is infinitesimally cohesive and admits a cotangent complex if and only if h is infinitesimally cohesive and admits a cotangent complex.*

Proof. The "if" direction follows immediately from Remark 2.2.3 and Proposition 1.3.18. For the converse, let us suppose that f is infinitesimally cohesive and admits a cotangent complex. Remark 2.2.3 implies that h is infinitesimally cohesive. We will complete the proof by showing that h admits a cotangent complex.

Let $\mathrm{Mod}^X_{\mathrm{cn}}$ be the ∞-category defined in Example 1.3.12, and let $F : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ be defined by the formula

$$F(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to X(R) \times_{Z(R)} Z(R \oplus M)).$$

We wish to prove that F is locally almost corepresentable. Define $F', F'' : \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$ by the formulas

$$F'(R, \eta, M) = \mathrm{fib}(X(R \oplus M) \to X(R) \times_{Y(R)} Y(R \oplus M))$$

$$F''(R, \eta, M) = \mathrm{fib}(Y(R \oplus M) \to Y(R) \times_{Z(R)} Z(R \oplus M)),$$

so that we have a fiber sequence of functors

$$F' \to F \to F''.$$

Note that each of these functors is naturally pointed, so we get a fiber sequence

$$\Omega F \to \Omega F'' \to F'.$$

Since f and g admit cotangent complexes, the functors F' and F'' are locally almost corepresentable. It follows that ΩF is locally almost corepresentable (Remark 1.3.11). Since h is infinitesimally cohesive, the functor F is given by the formula

$$F(R, \eta, M) \simeq (\Omega F)(R, \eta, \Sigma M)$$

and is therefore also locally almost corepresentable. □

2.3 Finiteness Conditions on Functors

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks. Recall that f is said to be *locally of finite presentation* if, for every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\, B & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}\, A & \longrightarrow & \mathfrak{Y} \end{array}$$

where the horizontal maps are étale, the left vertical map exhibits B as a compact object of $\mathrm{CAlg}^{\mathrm{cn}}_A$ (see Definition IX.8.16). Our goal in this section is to formulate an analogous condition for an arbitrary natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Definition 2.3.1. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that f is *locally of finite presentation* if the following condition is satisfied:

(a) Let $\{A_\alpha\}$ be a filtered diagram of m-truncated connective $\mathbb{E}_\infty$-rings with colimit A. Then the canonical map

$$\theta : \varinjlim X(A_\alpha) \to X(A) \times_{Y(A)} (\varinjlim Y(A_\alpha))$$

is a homotopy equivalence.

We say that f is *locally almost of finite presentation* if it satisfies the following weaker condition:

(b) Let $m \geq 0$, and let $\{A_\alpha\}$ be a filtered diagram of m-truncated connective $\mathbb{E}_\infty$-rings with colimit A. Then the canonical map

$$\theta : \varinjlim X(A_\alpha) \to X(A) \times_{Y(A)} (\varinjlim Y(A_\alpha))$$

is a homotopy equivalence.

If $n \geq 0$, we say that f is *locally of finite presentation to order n* if the following even weaker condition is satisfied:

(c) Let $m \geq 0$, and let $\{A_\alpha\}$ be a filtered diagram of m-truncated connective $\mathbb{E}_\infty$-rings with colimit A. Then the canonical map
$$\theta : \varinjlim X(A_\alpha) \to X(A) \times_{Y(A)} (\varinjlim Y(A_\alpha))$$
is $(m-n-1)$-truncated (that is, the homotopy fibers of θ are $(m-n-1)$-truncated).

Remark 2.3.2. A morphism $f : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ is locally almost of finite presentation if and only if it is locally of finite presentation to order n for every integer $n \geq 0$.

Remark 2.3.3. Suppose we are given a commutative diagram

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{h} & & Z \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Suppose that g is locally of finite presentation (locally almost of finite presentation, locally of finite presentation to order n). Then f is locally of finite presentation (locally almost of finite presentation, locally of finite presentation to order n) if and only if h is locally of finite presentation (locally almost of finite presentation, locally of finite presentation to order n).

We have the following counterpart to Proposition 2.2.7:

Proposition 2.3.4. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. The following conditions are equivalent:*

(1) *The map f is locally of finite presentation (locally almost of finite presentation, locally of finite presentation to order n).*

(2) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, the map f' is locally of finite presentation (locally almost of finite presentation, locally of finite presentation to order n).

(3) *For every pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}$ where Y' is a corepresentable functor, locally of finite presentation (locally almost of finite presentation, locally of finite presentation to order n).

Remark 2.3.5. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that there exists a connective $\mathbb{E}_\infty$-ring R and a point $\eta \in Y(R)$ which exhibits Y as the functor corepresented by R. Let $X_\eta : \mathrm{CAlg}_R^{|conn} \to \mathcal{S}$ be as in Notation 2.2.4. Then:

(a) The map f is locally of finite presentation if and only if the functor X_η commutes with filtered colimits.

(b) The map f is locally of finite presentation if and only if, for every integer $m \geq 0$, the functor $X_\eta|(\mathrm{CAlg}_R^{\mathrm{cn}})_{\leq m}$ commutes with filtered colimits, where $(\mathrm{CAlg}_R^{\mathrm{cn}})_{\leq m}$ denotes the full subcategory of $\mathrm{CAlg}_R^{\mathrm{cn}}$ spanned by the connective, m-truncated R-algebras.

(c) The map f is locally of finite presentation to order n if and only if, for every integer $m \geq 0$ and every filtered diagram $\{A_\alpha\}$ of m-truncated connective $\mathbb{E}_\infty$-algebras over R having colimit A, the canonical map

$$\varinjlim X_\eta(A_\alpha) \to X_\eta(A)$$

is $(m-n-1)$-truncated.

We next show that the finiteness conditions of Definition 2.3.1 can often be reformulated as conditions on the relative cotangent complex $L_{X/Y}$.

Proposition 2.3.6. *Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which admits a cotangent complex. Then:*

(1) *If f is locally of finite presentation to order n, then the relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is perfect to order n.*

(2) *Assume that f is infinitesimally cohesive and satisfies the following additional condition:*

(∗) *For every filtered diagram $\{A_\alpha\}$ of commutative rings having colimit A, the diagram of spaces*

$$\begin{array}{ccc} \varinjlim X(A_\alpha) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ \varinjlim Y(A_\alpha) & \longrightarrow & Y(A) \end{array}$$

is a pullback square.

If the relative cotangent complex $L_{X/Y}$ is perfect to order n, then f is locally of finite presentation to order n.

Proof. Suppose first that f is locally of finite presentation to order n. Choose a connective $\mathbb{E}_\infty$-ring A and a point $\eta \in X(A)$; we wish to show that $\eta^* L_{X/Y} \in \mathrm{Mod}_A$ is perfect to order n. To prove this, we must show that if $\{M_\alpha\}$ is a filtered diagram in $(\mathrm{Mod}_A)_{\leq 0}$ with colimit M, then the canonical map

$$\varinjlim \mathrm{Ext}^i_A(\eta^* L_{X/Y}, M_\alpha) \to \mathrm{Ext}^i_A(\eta^* L_{X/Y}, M)$$

is injective when $i = n$ and bijective for $i < n$. Since $L_{X/Y}$ is locally almost connective, we can choose an integer $k \geq 0$ such that $\eta^* L_{X/Y} \in (\mathrm{Mod}_A)_{\geq -k}$. Note that replacing the diagram $\{M_\alpha\}$ with $\{\tau_{\geq -k-n} M_\alpha\}$ does not change the groups $\mathrm{Ext}^i_A(\eta^* L_{X/Y}, M_\alpha)$ and $\mathrm{Ext}^i_A(\eta^* L_{X/Y}, M)$ for $i \leq n$. We may therefore assume without loss of generality that each M_α belongs to $(\mathrm{Mod}_A)_{\geq -k-n}$.

Let $A' = \tau_{\leq n+k} A$, so that the forgetful functor $\mathrm{Mod}_{A'} \to \mathrm{Mod}_A$ induces an equivalence of ∞-categories

$$(\mathrm{Mod}_{A'})_{\leq 0} \cap (\mathrm{Mod}_{A'})_{\geq -n-k} \to (\mathrm{Mod}_A)_{\leq 0} \cap (\mathrm{Mod}_A)_{\geq -n-k}.$$

We may therefore assume that $\{M_\alpha\}$ is the image of a filtered system of R'-modules (which we will also denote by $\{M_\alpha\}$). Let $\eta' \in X(A')$ denote the image of η, so that we have a commutative diagram of abelian groups

$$\begin{array}{ccc} \varinjlim \mathrm{Ext}^i_A(\eta^* L_{X/Y}, M_\alpha) & \longrightarrow & \mathrm{Ext}^i_A(\eta^* L_{X/Y}, M) \\ \downarrow & & \downarrow \\ \varinjlim \mathrm{Ext}^i_{A'}(\eta'^* L_{X/Y}, M_\alpha) & \longrightarrow & \varinjlim \mathrm{Ext}^i_{A'}(\eta'^* L_{X/Y}, M) \end{array}$$

where the vertical maps are isomorphisms. We may therefore replace A by A' and thereby reduce to the case where A is $(k+n)$-truncated.

We wish to prove that the map

$$\theta_j : \varinjlim \pi_j \operatorname{Map}_{\operatorname{Mod}_A}(\eta^* L_{X/Y}, M_\alpha[k+n]) \to \pi_j \operatorname{Map}_{\operatorname{Mod}_A}(\eta^* L_{X/Y}, M[k+n])$$

is injective for $j = k$ and bijective for $j > k$. Since each $M_\alpha[k+n]$ is connective, we can identify θ_j with the canonical map from

$$\pi_j \varinjlim \operatorname{fib}(X(A \oplus M_\alpha[k+n]) \to X(A) \times_{Y(A)} Y(A \oplus M_\alpha[k+n]))$$

to

$$\pi_j \operatorname{fib}(X(A \oplus M[k+n]) \to X(A) \times_{Y(A)} Y(A \oplus M[k+n]))$$

We are therefore reduced to proving that the canonical map from $\varinjlim_\alpha \operatorname{fib}(X(A \oplus M_\alpha[k+n]) \to X(A) \times_{Y(A)} Y(A \oplus M_\alpha[k+n]))$ to the space $\operatorname{fib}(X(A \oplus M[k+n]) \to X(A) \times_{Y(A)} Y(A \oplus M[k+n]))$ has $(k-1)$-truncated homotopy fibers. Note that this map is a pullback of

$$\theta : \varinjlim X(A \oplus M_\alpha[k+n]) \to X(A \oplus M[k+n]) \times_{Y(A \oplus M[k+n])} \varinjlim Y(A \oplus M_\alpha[k+n])$$

Since each $A \oplus M_\alpha[k+n]$ is $(k+n)$-truncated, our assumption that f is locally of finite presentation to order n guarantees that the homotopy fibers of θ are $(k-1)$-truncated. This completes the proof of (1).

We now prove (2). Using Proposition 2.3.4, we can reduce to the case where the functor Y is corepresentable by a connective $\mathbb{E}_\infty$-ring R. The assumption that f is infinitesimally cohesive then implies that X is infinitesimally cohesive (Remark 2.2.3). Let $X_\eta : \operatorname{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be as in Remark 2.3.5. Assumption $(*)$ implies that the restriction of X_η to CAlg_R^0 commutes with filtered colimits, where CAlg_R^0 denotes the full subcategory of CAlg_R spanned by the discrete R-algebras. We wish to show that X_η satisfies condition (c) of Remark 2.3.5. FIx an integer $m \geq 0$, and suppose we are given a diagram $\{A_\alpha\}_\alpha$ of connective, m-truncated R-algebras indexed by a filtered partially ordered set P. Let $A = \varinjlim A_\alpha$; we wish to show that the canonical map

$$\phi : \varinjlim X_\eta(A_\alpha) \to X_\eta(R)$$

has $(m-n-1)$-truncated homotopy fibers. The proof proceeds by induction on m. If $m = 0$, then ϕ is a homotopy equivalence and there is nothing to prove. Let us therefore assume that $m > 0$. Choose a point $\nu \in X_\eta(A)$ and let F denote the homotopy fiber $\varinjlim_\alpha X_\eta(A_\alpha) \times_{X_\eta(A)} \{\nu\}$; we wish to show that F is $(m-n-1)$-truncated. For every index α, let $A'_\alpha = \tau_{\leq m-1} A_\alpha$ and let $M_\alpha = \pi_n A_\alpha$, so that we have a filtered system of pullback diagrams

$$\begin{array}{ccc} A_\alpha & \longrightarrow & A'_\alpha \\ \downarrow & & \downarrow \\ A'_\alpha & \longrightarrow & A'_\alpha \oplus M_\alpha[m+1]. \end{array}$$

Let $A' = \varinjlim_\alpha A'_\alpha \simeq \tau_{\leq m-1} A$, let ν' denote the image of ν in $X_0(A')$, and let F' denote the homotopy fiber $\varinjlim X_0(A'_\alpha) \times_{X_0(A')} \{\eta'\}$. The inductive hypothesis implies that F' is $(m-n-2)$-truncated. It will therefore suffice to prove that the map $\gamma : F \to F'$ is $(m-n-1)$-truncated.

Let ν'' denote the image of ν in $X_0(A' \oplus \pi_m A[m+1])$, and let F'' denote the homotopy fiber

$$\varinjlim X_0(A'_\alpha \oplus M_\alpha[m+1]) \times_{X_0(A' \oplus \pi_m R[m+1])} \{\nu''\}.$$

Since X is infinitesimally cohesive, we have a pullback diagram of spaces

$$\begin{array}{ccc} F & \xrightarrow{\gamma} & F' \\ \downarrow & & \downarrow \\ F' & \xrightarrow{\gamma_0} & F''. \end{array}$$

It will therefore suffice to show that γ_0 has $(m-n-1)$-truncated homotopy fibers. The map γ_0 admits a left homotopy inverse $\beta : F'' \to F'$. To prove that the γ_0 has $(m-n-1)$-truncated homotopy fibers, it will suffice to show that β has $(m-n)$-truncated homotopy fibers. A point of F' is given by a lifting of η' to $\eta'_\alpha \in X_0(A'_\alpha)$, for some $\alpha \in P$. Unwinding the definitions, we see that the homotopy fiber of β over the point η'_α is given by the fiber of the map

$$\varinjlim_{\alpha' \geq \alpha} \mathrm{Map}_{\mathrm{Mod}_{A'_\alpha}}(\eta'^*_\alpha L_{X/Y}, M_{\alpha'}[m+1]) \to \mathrm{Map}_{\mathrm{Mod}_{A'_\alpha}}(\eta'^*_\alpha L_{X/Y}, (\pi_m A)[m+1]).$$

Since $\eta'^*_\alpha L_{X/Y}$ is perfect to order n, this map has $(m-n)$-truncated homotopy fibers as desired. □

Corollary 2.3.7. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that f admits a cotangent complex. Then:*

(1) *If f is locally almost of finite presentation, then the relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is almost perfect.*

(2) *Assume that f is infinitesimally cohesive and satisfies condition $(*)$ of Proposition 2.3.6. If $L_{X/Y}$ is almost perfect, then f is locally almost of finite presentation.*

Proof. Combine Proposition 2.3.6 with Remark 2.3.2 □

Proposition 2.3.8. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and assume that f admits a cotangent complex. Then:*

(1) *If f is locally of finite presentation, the relative cotangent complex $L_{X/Y} \in \mathrm{QCoh}(X)$ is perfect.*

(2) *Assume that f is nilcomplete, infinitesimally cohesive, and satisfies condition $(*)$ of Proposition 2.3.6. If $L_{X/Y}$ is perfect, then f is locally of finite presentation.*

Proof. We first prove (1). Choose a connective $\mathbb{E}_\infty$-ring A and a point $\eta \in X(A)$; we wish to show that $\eta^* L_{X/Y} \in \mathrm{Mod}_A$ is perfect. Since $L_{X/Y}$ is locally almost connective, we can choose an integer k such that $\eta^* L_{X/Y} \in (\mathrm{Mod}_A)_{\geq -k}$. To prove that $\eta^* L_{X/Y}$ is perfect, it will suffice to show that it is a compact object of $(\mathrm{Mod}_A)_{\geq -k}$. For this, we note that the functor corepresented by $\eta^* L_{X/Y}$ is given by

$$M \mapsto \Omega^k \operatorname{fib}(X(R \oplus M[k]) \to X(R) \times_{Y(R)} Y(R \oplus M[k])),$$

which commutes with filtered colimits if f is locally of finite presentation.

We now prove (2). Using Proposition 2.3.4, we may assume without loss of generality that Y is corepresentable by a connective $\mathbb{E}_\infty$-ring R. Let $X_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be the functor given by the formula $X_0(A) = \operatorname{fib}(X(A) \to Y(A))$, as in Remark 2.3.5. We wish to prove that X_0 commutes with filtered colimits.

Let $\{A_\alpha\}$ be a diagram of connective $\mathbb{E}_\infty$-rings indexed by a filtered partially ordered set P, and set $A = \varinjlim A_\alpha$. We wish to prove that the canonical map $\varinjlim X_0(A_\alpha) \to X_0(A)$ is a homotopy equivalence. For this, it suffices to show that for every point $\eta \in X_0(\pi_0 A)$, the induced map

$$\theta : \varinjlim X_0(A_\alpha) \times_{X_0(\pi_0 A)} \{\eta\} \to X_0(A) \times_{X_0(\pi_0 A)} \{\eta\}$$

is a homotopy equivalence. Since $X_0 | \mathrm{CAlg}^0_R$ commutes with filtered colimits, we may assume that η is the image of a point $\eta_\alpha \in X_0(\pi_0 A_\alpha)$ for some $\alpha \in P$. For $\beta \geq \alpha$, let η_β denote the image of η in $X(\pi_0 A_\beta)$. Then we can identify θ with the canonical map

$$\varinjlim_{\beta \geq \alpha} X_0(A_\beta) \times_{X_0(\pi_0 A_\beta)} \{\eta_\beta\} \to X_0(A) \times_{X_0(\pi_0 A)} \{\eta\}.$$

To prove that this map is a homotopy equivalence, it will suffice to show that for every integer $n \geq 0$, the induced map

$$\theta_n : \tau_{\leq n} \varinjlim_{\beta \geq \alpha} (X_0(A_\beta) \times_{X_0(\pi_0 A_\beta)} \{\eta_\beta\}) \to \tau_{\leq n}(X_0(A) \times_{X_0(\pi_0 A)} \{\eta\})$$

is a homotopy equivalence.

For every map of $\mathbb{E}_\infty$-rings $A_\alpha \to B$, let η_B denote the image of η_α in $X_0(\pi_0 B)$. Our proof relies on the following assertion:

($\star$) There exists an integer $m \geq 0$ with the following property: for every map of connective $\mathbb{E}_\infty$-rings $A_\alpha \to B$, the canonical map

$$\tau_{\leq n}(X_0(B) \times_{X_0(\pi_0 B)} \{\eta_B\}) \to \tau_{\leq n}(X_0(\tau_{\leq m} B) \times_{X_0(\pi_0 B)} \{\eta_B\})$$

is a homotopy equivalence.

Let m satisfy the condition of ($\star$). We have a commutative diagram

$$\begin{array}{ccc} \tau_{\leq n} \varinjlim_{\beta \geq \alpha}(X_0(A_\beta) \times_{X_0(\pi_0 A_\beta)} \{\eta_\beta\}) & \xrightarrow{\theta_n} & \tau_{\leq n}(X_0(A) \times_{X_0(\pi_0 A)} \{\eta\}) \\ \downarrow & & \downarrow \\ \tau_{\leq n} \varinjlim_{\beta \geq \alpha}(X_0(\tau_{\leq m} A_\beta) \times_{X(\pi_0 A_\beta)} \{\eta_\beta\}) & \xrightarrow{\theta'_n} & \tau_{\leq n}(X_0(\tau_{\leq m} A) \times_{X_0(\pi_0 A)} \{\eta\}) \end{array}$$

where the vertical maps are homotopy equivalences. Consequently, to prove that θ_n is a homotopy equivalence, it suffices to show that θ'_n is a homotopy equivalence. This follows from the fact that the functor X is locally almost of finite presentation (which follows from Corollary 2.3.7).

It remains to prove ($\star$). Since $L_{X/Y}$ is perfect, $\eta_\alpha^* L_{X/Y}$ is a dualizable object of $\mathrm{Mod}_{\pi_0 A_\alpha}$. Let V denote a dual of $\eta_\alpha^* L_{X/Y}$, and choose an integer k such that V is k-connective. We claim that $m = n - k$ satisfies the condition of ($\star$). Choose a map of connective $\mathbb{E}_\infty$-rings $A_\alpha \to B$. We will prove that the map

$$X_0(B) \times_{X_0(\pi_0 B)} \{\eta_B\} \to X_0(\tau_{\leq m} B) \times_{X_0(\pi_0 B)} \{\eta_B\}$$

is $(n+1)$-connective.

Since the functor X is nilcomplete (Remark 2.2.3), $X_0(B) \times_{X_0(\pi_0 B)} \{\eta_B\}$ is the homotopy inverse limit of a tower of maps

$$\cdots \to X_0(\tau_{\leq m+2} B) \times_{X_0(\pi_0 B)} \{\eta_B\} \stackrel{\gamma(m+1)}{\to} X_0(\tau_{\leq m+1} B) \times_{X_0(\pi_0 B)} \{\eta_B\} \stackrel{\gamma(m)}{\to} X_0(\tau_{\leq m} B) \times_{X_0(\pi_0 B)} \{\eta_B\}.$$

It will therefore suffice to show that the maps $\gamma(m')$ is $(n+1)$-connective for each $m' \geq m$.

Let $M = \pi_{m'+1} B$, so that there is a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} \tau_{\leq m'+1} B & \longrightarrow & \tau_{\leq m'} B \\ \downarrow & & \downarrow \\ \tau_{\leq m'} B & \longrightarrow & \tau_{\leq m'} B \oplus M[m'+2]. \end{array}$$

Since X is infinitesimally cohesive (Remark 2.2.3), this diagram gives us a pullback square of spaces

$$\begin{array}{ccc} X_0(\tau_{\leq m'+1} B) \times_{X_0(\pi_0 B)} \{\eta_B\} & \xrightarrow{\gamma(m')} & X_0(\tau_{\leq m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\} \\ \downarrow & & \downarrow \\ X_0(\tau_{\leq m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\} & \xrightarrow{\gamma'} & X_0(\tau_{\leq m'} B \oplus M[m'+2]) \times_{X_0(\pi_0 B)} \{\eta_B\}. \end{array}$$

It will therefore suffice to show that the map γ' is $(n+1)$-connective. We note that γ' has a left homotopy inverse

$$\epsilon : X_0(\tau_{\leq m'} B \oplus M[m'+2]) \times_{X_0(\pi_0 B)} \{\eta_B\} \to X_0(\tau_{\leq m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\}.$$

Consequently, we are reduced to proving that the homotopy fibers of ϵ are $(n+2)$-connective. Choose a point of $X_0(\tau_{\leq m'} B) \times_{X_0(\pi_0 B)} \{\eta_B\}$, corresponding to a point $\eta' \in X_0(\tau_{\leq m'} B)$ lifting η_B. Unwinding the definitions, we see that the homotopy fiber of ϵ over this point is given by the mapping space

$$\begin{aligned} \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leq m'} B}}(\eta'^* L_{X/Y}, M[m'+2]) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\pi_0 B}}(\eta_B^* L_{X/Y}, M[m'+2]) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\pi_0 A_\alpha}}(\eta_\alpha^* L_{X/Y}, M[m'+2]) \\ &\simeq \Omega^\infty(V \otimes_{\pi_0 A_\alpha} M[m'+2]). \end{aligned}$$

Since V is k-connective, $V \otimes_{\pi_0 A_\alpha} M[m'+2]$ is $(k+m'+2)$-connective. It now suffices to observe that $k+m'+2 \geq n+2$, since $m' \geq m = n-k$. □

We conclude this section by studying the relationship of finiteness conditions of Definition 2.3.1 with the analogous conditions on morphisms of spectral Deligne-Mumford stacks.

Proposition 2.3.9. *Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Assume that ϕ is a relative Deligne-Mumford m-stack for some integer $m \gg 0$, and let $f : X \to Y$ denote the induced map of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then:*

(1) *For every integer $n \geq 0$, the map f is locally of finite presentation to order n if and only if ϕ is locally of finite presentation to order n.*

(2) *The map f is locally almost of finite presentation if and only if ϕ is locally almost of finite presentation.*

(3) *The map f is locally of finite presentation if and only if ϕ is locally of finite presentation.*

The proof of Proposition 2.3.9 will require some preliminaries.

Lemma 2.3.10. *Let R be a connective $\mathbb{E}_\infty$-ring, let $n \geq -2$ be an integer, and let $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor which satisfies the following condition:*

$(*_n)$ *For every diagram $\{A_\alpha\}$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$ indexed by a filtered partially ordered set P, the map $\varinjlim X(A_\alpha) \to X(A)$ is n-truncated, where $A = \varinjlim A_\alpha$.*

Assume that there exists an integer m such that $X(A)$ is m-truncated for each $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, and let Y be the sheafification of X with respect to the étale topology. Then the functor Y also satisfies condition $(*_n)$.

Proof. Without loss of generality, we may assume that $m \geq n$. Let us regard m as fixed and proceed by descending induction on n. If $n = m$, the result is obvious (since $Y(A)$ is m-truncated for each $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$). Let us therefore assume that $n < m$ and that the functor Y satisfies condition $(*_{n+1})$. We wish to prove that Y satisfies $(*_n)$. Fix a diagram $\{A_\alpha\}$ in $\mathrm{CAlg}_R^{\mathrm{cn}}$ indexed by a filtered partially ordered set P, let $A = \varinjlim A_\alpha$, and choose a point $\eta \in Y(A)$. We wish to prove that the space $Z = \{\eta\} \times_{Y(A)} \varinjlim_\alpha Y(A_\alpha)$ is n-truncated. Choose a map $\gamma : \partial \Delta^{n+2} \to Z$; we will prove that γ is nullhomotopic. Since $\partial \Delta^{n+2}$ is a finite simplicial set, the map γ factors as a composition

$$\partial \Delta^{n+2} \xrightarrow{\gamma_0} \{\eta\} \times_{Y(A)} Y(A_\alpha) \to Z$$

for some $\alpha \in P$.

Since Y is the sheafification of X with respect to the étale topology, there exists a finite collection of étale maps $\{A \to A(i)\}_{1 \leq i \leq n}$ with the following properties:

(a) The map $A \to \prod_{1 \leq i \leq n} A(i)$ is faithfully flat.

(b) For $1 \leq i \leq n$, let $\eta(i) \in Y(A(i))$ denote the image of η. Then $\eta(i)$ lifts (up to homotopy) to a point $\overline{\eta}(i) \in X(A(i))$.

Using the structure theory of étale morphisms (Proposition VII.8.10), we may choose an index $\alpha \in P$ and a collection of étale morphisms $A_\alpha \to A_\alpha(i)$, together with equivalences $A(i) \simeq A_\alpha(i) \otimes_{A_\alpha} A$. Since Y is the sheafification of X with respect to the étale topology, we may pass to a refinement of the étale cover $\{A_\alpha \to A_\alpha(i)\}$ and thereby guarantee that each of the composite maps

$$\partial \Delta^{n+2} \xrightarrow{\gamma_0} \{\eta\} \times_{Y(A)} Y(A_\alpha) \to \{\eta(i)\} \times_{Y(A(i))} Y(A_\alpha(i))$$

factors through a map $\gamma_i : \partial \Delta^{n+2} \to \{\overline{\eta}(i)\} \times_{X(A(i))} X(A_\alpha(i))$. Since the functor X satisfies $(*)$, we can enlarge α and reduce to the case where each $\gamma(i)$ is nullhomotopic.

Let $A^0_\alpha = \prod_{1 \leq i \leq n} A_\alpha(i)$, and let $A^\bullet_\alpha$ denote the Čech nerve of the map $A_\alpha \to A^0_\alpha$. Let $\eta^\bullet$ denote the image of η in $Y(A^\bullet)$. We define a cosimplicial space $Z^\bullet$ by the formula

$$Z^\bullet = \{\eta^\bullet\} \times_{Y(A \otimes_{A_\alpha} A^\bullet_\alpha)} \varinjlim_{\beta \geq \alpha} Y(A_\beta \otimes_{A_\alpha} A^\bullet_\alpha).$$

The inductive hypothesis implies that each of the spaces Z^p is $(n+1)$-truncated, and the assumption that Y is a sheaf with respect to the étale topology gives a homotopy equivalence $Z \simeq \operatorname{Tot} Z^\bullet$. It follows that Z is also $(n+1)$-truncated, and that a map $\partial \Delta^{n+2} \to Z$ is nullhomotopic if and only if the composite map $\partial \Delta^{n+2} \to Z \to Z^0$ is nullhomotopic. We complete the proof by observing that the map $\partial \Delta^{n+2} \xrightarrow{\gamma} Z \to Z^0$ is nullhomotopic by construction. □

Remark 2.3.11. In the statement of Lemma 2.3.10 (in in its applications given below), we can replace the étale topology by the Zariski or Nisnevich topologies: the proof carries over without essential change.

Lemma 2.3.12. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that there exists an integer k such that, for every m-truncated connective $\mathbb{E}_\infty$-ring A, the spaces $X(A)$ and $Y(A)$ are $(m+k)$-truncated. Let X' and Y' denote the sheafifications of X and Y with respect to the étale topology. If f is locally of finite presentation to order n, then the induced map $X' \to Y'$ is of finite presentation to order n.*

Proof. Using Proposition 2.3.4, we can reduce to the case where Y is corepresentable by an $\mathbb{E}_\infty$-ring R. Let $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be defined as in Remark 2.3.5 and let X'_0 be the sheafification of X_0 with respect to the étale topology. We wish to show that X'_0 satisfies condition (c) of Remark 2.3.5. Equivalently, we wish to show that for every integer m, the functor

$$A \mapsto X'_0(\tau_{\leq m} A)$$

satisfies condition $(*_{m-n-1})$ of Lemma 2.3.10. This follows from Lemma 2.3.10, since the functor $A \mapsto X_0(\tau_{\leq m} A)$ satisfies condition $(*_{m-n-1})$. □

Recall that if R is an $\mathbb{E}_\infty$-ring, we let $\mathcal{S}\mathrm{hv}^{\text{ét}}_R$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_R, \mathcal{S})$ spanned by those functors which are sheaves with respect to the étale topology.

Lemma 2.3.13. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{F}$ be a truncated object of $\mathcal{S}\mathrm{hv}^{\acute{e}t}_R$. For every map of $\mathbb{E}_\infty$-rings $R \to A$, let $\mathcal{F}_A$ denote the image of $\mathcal{F}$ in the ∞-category $\mathcal{S}\mathrm{hv}^{\acute{e}t}_A$ (in other words, the pullback of $\mathcal{F}$ along the map $\operatorname{Spec} A \to \operatorname{Spec} R$). Then the functor $A \mapsto \mathcal{F}_A(A)$ commutes with filtered colimits.*

Proof. Choose an integer m such that $\mathcal{F}$ is n-truncated. Let $X : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be a left Kan extension of $\mathcal{F} : \mathrm{CAlg}^{\text{ét}}_R \to \mathcal{S}$. For every connective R-algebra A, we have $X(A) \simeq \varinjlim_{R' \to A} \mathcal{F}(R')$, where the colimit is taken over the full subcategory of $\mathrm{CAlg}_{R//A}$ spanned by those R' which are étale over R. Since this ∞-category is filtered, we deduce that $X(A)$ is m-truncated for every object $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$. Because every étale R-algebra is a compact object of CAlg_R, the functor X commutes with filtered colimits. Note that the functor $A \mapsto \mathcal{F}_A(A)$ is the sheafification of X with respect to the étale topology. The desired result now follows from Lemma 2.3.10. □

Lemma 2.3.14. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Assume that f is a relative étale Deligne-Mumford n-stack: that is, for every morphism $Y' \to Y$ where Y' is representable by a connective $\mathbb{E}_\infty$-ring R, the fiber product $X \times_Y Y'$ is representable by a spectral Deligne-Mumford m-stack which is étale over* $\operatorname{Spec} R$*. Then f is locally of finite presentation.*

Proof. Using Propositon 2.3.4, we may suppose that Y is corepresentable. The desired result now follows by combining the criterion of Remark 2.3.5 with Lemma 2.3.13. $\square$

Lemma 2.3.15. *Let $f : R \to A$ be a map of connective $\mathbb{E}_\infty$-rings, and assume that the fiber of f is m-connective for $m \geq 0$. If B is an n-truncated $\mathbb{E}_\infty$-ring, then the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$ is $(n-m-1)$-truncated.*

Proof. We define a sequence of objects

$$R = R(0) \to R(1) \to \cdots$$

in the ∞-category $(\mathrm{CAlg}_R)_{/A}$ by induction. Assuming that $R(i)$ has been defined, let $K(i)$ denote the fiber of the map $R(i) \to A$, and form a pushout diagram

$$\begin{array}{ccc} \operatorname{Sym}^* K(i) & \longrightarrow & R \\ \downarrow & & \downarrow \\ R(i) & \longrightarrow & R(i+1). \end{array}$$

We first claim that the canonical map $\varinjlim R(i) \to A$ is an equivalence of $\mathbb{E}_\infty$-algebras over R. To prove this, it suffices to show that θ induces an equivalence in the ∞-category of R-modules. This is clear, since colimit $\{R(i)\}$ agrees with the colimit of

$$R(0) \to \mathrm{cofib}(K(0) \to R(0)) \to R(1) \to \mathrm{cofib}(K(1) \to R(1)\} \to \cdots,$$

which contains a cofinal subsequence taking the constant value A.

We next claim that each $K(i)$ is m-connective. Equivalently, we claim that each of the maps $\pi_j R(i) \to \pi_j A$ is surjective for $i = m$ and bijective for $i < m$. This is true by hypothesis when $i = 0$; we treat the general case using induction on i. Since $K(i)$ is m-connective, we have

$$\begin{aligned} \tau_{\leq m-1} R(i+1) &\simeq \tau_{\leq m-1}(R \otimes_{\operatorname{Sym}^* K(i)} R(i)) \\ &\simeq \tau_{\leq m-1}(\tau_{\leq m-1} R \otimes_{\tau_{\leq m-1} \operatorname{Sym}^* K(i)} \tau_{\leq m-1} R(i)) \\ &\simeq \tau_{\leq m-1}(\tau_{\leq m-1} R \otimes_{\tau_{\leq m-1} R} \tau_{\leq m-1} R(i)) \\ &\simeq \tau_{\leq m-1} R(i) \\ &\simeq \tau_{\leq m-1} A. \end{aligned}$$

The surjectivity of the map $\pi_m R(i) \to \pi_m A$ follows from the surjectivity of the map $\pi_m R \to \pi_m A$.

Observe that $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$ is the homotopy limit of the tower of spaces $\{\mathrm{Map}_{\mathrm{CAlg}_R}(R(i), B)\}_{i \geq 0}$, and that $\mathrm{Map}_{\mathrm{CAlg}_R}(R, B)$ is contractible. To prove that $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B)$ is $(n-1-m)$-truncated, it will suffice to show that each of the maps

$$\theta_i : \mathrm{Map}_{\mathrm{CAlg}_R}(R(i+1), B) \to \mathrm{Map}_{\mathrm{CAlg}_R}(R(i), B)$$

is $(n-m-1)$-truncated. Note that θ_i is a pullback of the map

$$\theta'_i : * \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(R, B) \to \mathrm{Map}_{\mathrm{CAlg}_R}(\operatorname{Sym}^* K(i), B) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(K(i), B).$$

It will therefore suffice to show that each of the mapping spaces $\mathrm{Map}_{\mathrm{Mod}_R}(K(i), B)$ is $(n-m)$-truncated. This follows from our assumption that $K(i)$ is m-connective and B is n-truncated. $\square$

Proof of Proposition 2.3.9. The implication (1) $\Rightarrow$ (2) is obvious. Note that f is locally of finite presentation if and only if it locally almost of finite presentation and $L_{X/Y}$ is perfect (Proposition 2.3.8). To prove that (2) $\Rightarrow$ (3), it will suffice to show that ϕ is locally of finite presentation if and only if it is locally almost of finite presentation and $L_{\mathfrak{X}/\mathfrak{Y}}$ is perfect. This assertion is local on $\mathfrak{X}$ and $\mathfrak{Y}$. We may therefore suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are affine, in which case the desired result follows from Theorem A.8.4.3.18.

It remains to prove (1). Using Proposition 2.3.4, we can reduce to the case where $\mathfrak{Y} = \operatorname{Spec} R$ is affine, so that $\mathfrak{X}$ is a spectral Deligne-Mumford m-stack. Suppose first that $\mathfrak{X} = \operatorname{Spec} A$ is also affine. If f is locally of finite presentation to order n, then the criterion of Remark 2.3.5 immediately implies that A is of finite presentation to order n over R. Conversely, suppose that A is of finite presentation to order n over R. Then $L_{A/R}$ is perfect to order n as an A-module (Proposition IX.8.8). If $n \geq 1$, then $\pi_0 A$ is finitely presented as a commutative algebra over $\pi_0 R$. Then $f : X \to Y$ satisfies condition $(*)$ of Proposition 2.3.6, so that f is locally of finite presentation to order n. If $n = 0$ we must work a little bit harder. The assumption that A is of finite presentation to order 0 over R implies that $\pi_0 A$ is finitely generated as an algebra over $\pi_0 R$. We may therefore choose a map of connective $\mathbb{E}_\infty$-rings $R' \to A$ which induces a surjection $\pi_0 R' \to \pi_0 A$, where $R' = \operatorname{Sym}^*_R(R^k)$ is free over R on finitely many generators. Let Y' denote the functor corepresented by R'. The map $Y' \to Y$ is locally of finite presentation. Invoking Remark 2.3.3, we are reduced to proving that $X \to Y'$ is locally of finite presentation to order 0. We may therefore replace R by R' and thereby reduce to the case where $\pi_0 R \to \pi_0 A$ is surjective. We prove that f is of finite presentation to order 0 by verifying the criterion of Remark 2.3.5: for every filtered diagram of q-truncated objects $\{B_\alpha\}$ in $\mathrm{CAlg}^{\mathrm{cn}}_R$ having colimit B, the canonical map

$$\varinjlim \operatorname{Map}_{\mathrm{CAlg}_R}(A, B_\alpha) \to \varinjlim \operatorname{Map}_{\mathrm{CAlg}_R}(A, B)$$

is $(q-1)$-truncated. In fact, the domain and codomain of this map are individually $(q-1)$-truncated, by Lemma 2.3.15. This completes the proof of (1) in the affine case.

We now prove the "if" direction of (1). Assume that f is locally of finite presentation to order n. Choose an affine map $\operatorname{Spec} A \to \mathfrak{X}$; we wish to show that A is locally of finite presentation to order n over R. Invoking the previous step of the proof, it will suffice to show that if X' denotes the functor corepresented by A, then the composite map $X' \xrightarrow{g} X \xrightarrow{f} Y$ is locally of finite presentation to order n. This follows from Remark 2.3.3, since f and g are both locally of finite presentation to order n (for the map g, this follows from Lemma 2.3.14).

We now treat the "only if" direction of (1). Assume first that $\mathfrak{X}$ is the coproduct of a collection of affine spectral Deligne-Mumford stacks $\{\mathfrak{X}_\alpha\}_{\alpha \in S}$, indexed by some finite set S. For every finite subset $T \subseteq S$, let $\mathfrak{X}_T = \coprod_{\alpha \in T} \mathfrak{X}_\alpha$, and let X_T denote the functor represented by $\mathfrak{X}_T$. If $\mathfrak{X}$ is locally of finite presentation to order n over $\mathfrak{Y}$, then each $\mathfrak{X}_T$ has the same property. Since each $\mathfrak{X}_T$ is affine, we conclude that the map $X_T \to Y$ is locally of of finite presentation to order n, so that the induced map $\varinjlim_{T \subseteq S} X_T \to Y$ is locally of finite presentation to order n (here the colimit is taken over the filtered partially ordered set of finite subsets of S). Note that X is the sheafification of $\varinjlim_{T \subseteq S} X_T$ with respect to the étale topology. It follows from Lemma 2.3.12 that X is locally of finite presentation to order n over Y.

We now treat the general case. Choose an étale surjection $u : \mathfrak{X}_0 \to \mathfrak{X}$, where $\mathfrak{X}_0$ is a coproduct of affine spectral Deligne-Mumford stacks. Let $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by $\mathfrak{X}_0$. For every connective $\mathbb{E}_\infty$-ring A, let $X'(A)$ denote the essential image of the map $X_0(A) \to X(A)$. Since u is an étale surjection, the inclusion $X' \hookrightarrow X$ exhibits X as a sheafificaton of X' with respect to the étale topology. Using Lemma 2.3.12, we are reduced to proving that the map $X' \to Y$ is locally of finite presentation to order n. Choose a filtered diagram of m-truncated connective $\mathbb{E}_\infty$-rings $\{A_\alpha\}$ having colimit A; we wish to show that the map

$$\theta : \varinjlim X'(A_\alpha) \to X'(A) \times_{Y(A)} \varinjlim Y(A_\alpha)$$

is $(m-n-1)$-truncated. This map fits into a commutative diagram

$$\begin{array}{ccc} \varinjlim X_0(A_\alpha) & \longrightarrow & X_0(A) \times_{Y(A)} \varinjlim Y(A_\alpha) \\ \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \psi} \\ \varinjlim X'(A_\alpha) & \xrightarrow{\theta} & X'(A) \times_{Y(A)} \varinjlim Y(A_\alpha). \end{array}$$

Lemma 2.3.14 implies that this diagram is a pullback square, and the fact that $\mathfrak{X}_0$ is a coproduct of affine spectral Deligne-Mumford stacks guarantees that θ_0 is $(m-n-1)$-truncated. To complete the proof, it suffices to observe that ψ is surjective on connected components (because the map $X_0(A) \to X'(A)$ is surjective on connected components, by construction). □

2.4 Moduli of Spectral Deligne-Mumford Stacks

Let X be a proper smooth algebraic variety over the field $\mathbf{C}$ of complex numbers, and let T_X be the tangent bundle of X. A standard argument in deformation theory establishes a bijection of the set of isomorphism classes of first order deformations of X with the cohomology group $\mathrm{H}^1(X; T_X)$, where T_X denotes the tangent bundle of X. Informally, this allows us to interpret the group $\mathrm{H}^1(X; T_X)$ as the Zariski tangent space to "moduli space" $\mathcal{M}$ of all proper smooth varieties (taken at the point $\eta \in \mathcal{M}(\mathbf{C})$ classifying the variety X). Our goal in this section is to make this idea precise. We will define the "moduli space" $\mathcal{M}$ as a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and show that it has a well-behaved deformation theory. In particular, it admits a cotangent complex $L_{\mathcal{M}}$, whose fiber at the point η is given by the formula

$$\eta^* L_{\mathcal{M}} \simeq \Sigma^{-1} f_* (L^{\vee}_{X/\operatorname{Spec}\mathbf{C}})^{\vee},$$

where f denotes the projection map $X \to \operatorname{Spec}\mathbf{C}$. Note that the smoothness of X guarantees that the cotangent complex $L_{X/\operatorname{Spec}\mathbf{C}}$ is perfect (in fact, it is locally free of finite rank), so that the right hand side in the above expression is well-defined. Alternatively, if we let f_+ denote the left adjoint to the pullback functor f^* (see Proposition XII.3.3.23), then for every perfect complex $\mathcal{F}$ on X we have a canonical equivalence $f_* \mathcal{F}^{\vee} = (f_+ \mathcal{F})^{\vee}$, so that we can rewrite the above equivalence as

$$\eta^* L_{\mathcal{M}} \simeq \Sigma^{-1} f_+ L_{X/\operatorname{Spec}\mathbf{C}}.$$

We will show that this formula is valid more generally for any proper algebraic variety X over $\mathbf{C}$ (or, more generally, and proper flat map of spectral algebraic spaces $\mathfrak{X} \to \operatorname{Spec}^{\text{ét}} R$): see Theorem 2.4.3 and Remark 2.4.14.

In §2.1, we studied a variety of deformation theoretic properties of an arbitrary functor $\mathcal{M} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Here it is useful to think of $\mathcal{M}$ as some sort of moduli problem: that is, for every connective $\mathbb{E}_\infty$-ring R, we should think of $\mathcal{M}(R)$ as a classifying space for a family of geometric objects parametrized by the spectral Deligne-Mumford stack $\operatorname{Spec} R$. We are interested in the situation where the functor $\mathcal{M}$ is given by $R \mapsto \mathrm{Stk}_{/\operatorname{Spec} R}$. Since this functor takes values in (large) ∞-categories rather than (small) spaces, we will need a few preliminary terminological remarks.

Definition 2.4.1. Let $\mathcal{C}$ be an ∞-category (which we do not assume to be locally small) and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}$ be a functor. We will say that X is *cohesive* (*infinitesimally cohesive*, *nilcomplete*, *integrable*) if, for every corepresentable functor $e : \mathcal{C} \to \mathcal{S}$, the composite functor $e \circ X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is cohesive (infinitesimally cohesive, nilcomplete); see Remark 2.1.6.

We will say that a natural transformation $\alpha : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}$ is *cohesive* (*infinitesimally cohesive*, *nilcomplete*, *integrable*) if, for every corepresentable functor $e : \mathcal{C} \to \widehat{\mathcal{S}}$, the induced natural transformation $e \circ X \to e \circ Y$ is cohesive (infinitesimally cohesive, nilcomplete), in the sense of Definition 2.2.1.

Remark 2.4.2. Let $\alpha : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then α is cohesive (infinitesimally cohesive, nilcomplete) in the sense of Definition 2.4.1 if and only if it is cohesive (infinitesimally cohesive, nilcomplete) in the sense of Definition 2.2.1.

We can now state our main result, which we will prove at the end of this section:

Theorem 2.4.3. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor which assigns to each connective $\mathbb{E}_\infty$-ring R the full subcategory of $\mathrm{Stk}_{/ \operatorname{Spec} R}$ spanned by those maps $f : \mathfrak{X} \to \operatorname{Spec} R$ which exhibit $\mathfrak{X}$ as a spectral algebraic space which is proper, flat, and locally almost of finite presentation over R. Then:*

(1) *The functor X is cohesive.*

(2) *The functor X is nilcomplete.*

(3) *The functor X is locally almost of finite presentation: that is, X commutes with filtered colimits when restricted to $\tau_{\leq n} \mathrm{CAlg}^{\mathrm{cn}}$, for every integer $n \geq 0$.*

(4) *For every connective $\mathbb{E}_\infty$-ring X, the ∞-category $X(R)$ is essentially small. Consequently, we can identify X with a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$.*

(5) *For every simplicial set K, define a functor $X_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ by the formula $X_K(R) = \operatorname{Fun}(X, F(R))^{\simeq}$. Then the functor X_K admits a (-1)-connective cotangent complex.*

(6) *Suppose that K is a simplicial set having only finitely many simplices of each dimension. Then the functor X_K is almost of finite presentation, and the cotangent complex to X_K is almost perfect.*

Warning 2.4.4. The functor X of Theorem 2.4.3 is not integrable. If A is a local Noetherian ring which is complete with respect to its maximal ideal $\mathfrak{m}$, then Corollary XII.5.4.2 shows that the functor $X(A) \to \varprojlim X(A/\mathfrak{m}^n)$ is a fully faithful embedding. However, it is generally not essentially surjective: that is, not every proper flat spectral algebraic space which is locally almost of finite presentation over $\operatorname{Spf} A$ arises as the algebraization of a proper flat spectral algebraic space over A.

As a first step towards proving Theorem 2.4.3, let us see what we can say about the classification of spectral Deligne-Mumford stacks in general.

Notation 2.4.5. For every integer $n \geq 0$, let $\mathrm{Stk}^{n-\mathrm{trun}}$ denote the full subcategory of Stk spanned by those spectral Deligne-Mumford stacks $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ where the structure sheaf $\mathcal{O}_\mathcal{X}$ is n-truncated. The inclusion $\mathrm{Stk}^{n-\mathrm{trun}} \subseteq \mathrm{Stk}$ admits a right adjoint which we will denote by $Tr_n : \mathrm{Stk} \to \mathrm{Stk}^{n-\mathrm{trun}}$, given on objects by

$$(\mathcal{X}, \mathcal{O}_\mathcal{X}) \mapsto (\mathcal{X}, \tau_{\leq n} \mathcal{O}_\mathcal{X}).$$

Proposition 2.4.6. *The ∞-category* Stk *is equivalent to the limit of the tower*

$$\cdots \to \mathrm{Stk}^{2-\mathrm{trun}} \stackrel{Tr_1}{\to} \mathrm{Stk}^{1-\mathrm{trun}} \stackrel{Tr_0}{\to} \mathrm{Stk}^{0-\mathrm{trun}}.$$

Corollary 2.4.7. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral Deligne-Mumford stack. For each $n \geq 0$, set $\mathfrak{X}_n = (\mathcal{X}, \tau_{\leq n} \mathcal{O}_\mathcal{X})$. Then the canonical map*

$$\mathrm{Stk}_{/ \mathfrak{X}} \to \varprojlim \mathrm{Stk}_{/ \mathfrak{X}_n}$$

is an equivalence of ∞-categories.

Proof. For every spectral Deligne-Mumford stack $\mathfrak{Y}$, let $\mathrm{Stk}^{m-\mathrm{trun}}_{/ \mathfrak{Y}}$ denote the full subcategory of $\mathrm{Stk}_{/ \mathfrak{Y}}$ spanned by those morphisms $\mathfrak{Z} \to \mathfrak{Y}$ where $\mathfrak{Z} \in \mathrm{Stk}^{m-\mathrm{trun}}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Stk}_{/ \mathfrak{X}} & \longrightarrow & \varprojlim_n \mathrm{Stk}_{/ \mathfrak{X}_n} \\ \downarrow & & \downarrow \\ \varprojlim_m \mathrm{Stk}^{m-\mathrm{trun}}_{/ \mathfrak{X}} & \longrightarrow & \varprojlim_n \varprojlim_m \mathrm{Stk}^{m-\mathrm{trun}}_{/ \mathfrak{X}_n} . \end{array}$$

It follows from Proposition 2.4.6 that the vertical maps are equivalences. It will therefore suffice to show that the lower horizontal map is an equivalence. For this, we prove that for every $m \geq 0$, the map

$$\mathrm{Stk}^{m-\mathrm{trun}}_{/\mathfrak{X}} \to \varprojlim_n \mathrm{Stk}^{m-\mathrm{trun}}_{/\mathfrak{X}_n}$$

is an equivalence. This is clear, because $\mathrm{Stk}^{m-\mathrm{trun}}_{/\mathfrak{X}} \simeq \mathrm{Stk}^{m-\mathrm{trun}}_{/\mathfrak{X}_n}$ for $n \geq m$. □

Corollary 2.4.8. *The construction $R \mapsto \mathrm{Stk}_{/\operatorname{Spec} R}$ determines a cohesive and nilcomplete functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$.*

Proof. The nilcompleteness follows from Corollary 2.4.7, and the cohesiveness is a consequence of Theorem IX.9.1. □

Proof of Proposition 2.4.6. Let G denote the evident functor $\mathrm{Stk} \to \varprojlim \mathrm{Stk}^{n-\mathrm{trun}}$. We first claim that G is fully faithful. Unwinding the definitions, we must show that if $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ are spectral Deligne-Mumford stacks, then the canonical map

$$\theta : \mathrm{Map}_{\mathrm{Stk}}(\mathfrak{X}, \mathfrak{Y}) \to \varprojlim \mathrm{Map}_{\mathrm{Stk}}((\mathcal{X}, \tau_{\leq n} \mathcal{O}_\mathcal{X}), \mathfrak{Y})$$

is a homotopy equivalence. Let $K = \mathrm{Map}_{\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y})$ denote the space of maps from the underlying ∞-topos of $\mathfrak{X}$ to the underlying ∞-topos of $\mathfrak{Y}$. We will show that θ induces a homotopy equivalence after passing to the homotopy fiber over any point of K, corresponding to a geometric morphims $f^* : \mathcal{Y} \to \mathcal{X}$. In this case, we wish to show that the canonical map

$$\phi : \mathrm{Map}_{\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})}(f^* \mathcal{O}_\mathcal{Y}, \mathcal{O}_\mathcal{X}) \to \varprojlim \mathrm{Map}_{\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})}(f^* \mathcal{O}_\mathcal{Y}, \tau_{\leq n} \mathcal{O}_\mathcal{X})$$

induces a homotopy equivalence on the summands corresponding to *local* maps between strictly Henselian sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. This follows from the following pair of assertions:

- (a) The map ϕ is a homotopy equivalence. In fact, the canonical map $\mathcal{O}_\mathcal{X} \to \varprojlim \tau_{\leq n} \mathcal{O}_\mathcal{X}$ is an equivalence of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$: this follows from the proof of Theorem VII.8.42.
- (b) A map $f^* \mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$ is local if and only if each of the induced maps $f^* \mathcal{O}_\mathcal{Y} \to \tau_{\leq n} \mathcal{O}_\mathcal{X}$ is local. Both conditions are equivalent to the assertion that the underlying map

$$\pi_0 f^* \mathcal{O}_\mathcal{Y} \to \pi_0 \mathcal{O}_\mathcal{X}$$

is local (Definition VII.2.5).

It remains to prove that G is essentially surjective. Suppose we are given an object of $\varprojlim_n \mathrm{Stk}^{n-\mathrm{trun}}$, given by a sequence of spectral Deligne-Mumford stacks

$$(\mathcal{X}_0, \mathcal{O}_0) \to (\mathcal{X}_1, \mathcal{O}_1) \to (\mathcal{X}_2, \mathcal{O}_2) \to \cdots$$

with the following property: each of the maps $(\mathcal{X}_i, \mathcal{O}_i) \to (\mathcal{X}_{i+1}, \mathcal{O}_{i+1})$ induces an equivalence $(\mathcal{X}_i, \mathcal{O}_i) \simeq Tr_i(\mathcal{X}_{i+1}, \mathcal{O}_{i+1})$. It follows that the sequence of ∞-topoi $\mathcal{X}_0 \to \mathcal{X}_1 \to \cdots$ is equivalent to the constant sequence taking the value $\mathcal{X} = \mathcal{X}_0$. To complete the proof, it will suffice to verify the following:

- (c) The spectrally ringed ∞-topos $\mathfrak{X} = (\mathcal{X}, \varprojlim \mathcal{O}_n)$ is a spectral Deligne-Mumford stack.
- (d) For every integer n, the canonical map $(\mathcal{X}, \mathcal{O}_n) \to \mathfrak{X}$ induces an equivalence $(\mathcal{X}, \mathcal{O}_n) \to Tr_n \mathfrak{X}$.

Both of these assertions are local on $\mathfrak{X}$. We may therefore assume without loss of generality that $(\mathcal{X}, \mathcal{O}_0)$ is affine. It follows that each pair $(\mathfrak{X}, \mathcal{O}_n)$ is affine, so that the sequence of spectral Deligne-Mumford stacks above is determined by a tower of connective $\mathbb{E}_\infty$-rings

$$\cdots A_2 \to A_1 \to A_0$$

which induces equivalences $\tau_{\leq n} A_{n+1} \to A_n$ for each $n \geq 0$. Since Postnikov towers in $\mathrm{CAlg}^{\mathrm{cn}}$ are convergent (Proposition A.8.1.3.19), we can the limit $A = \varprojlim_n A_n$ is a connective $\mathbb{E}_\infty$-ring with $A_n \simeq \tau_{\leq n} A$ for every integer n. A simple calculation now shows that $\mathfrak{X} \simeq \operatorname{Spec} A$, from which assertions (c) and (d) follow easily. □

Corollary 2.4.8 does not apply directly in the situation of Theorem 2.4.3, because the classification of proper flat spectral algebraic spaces is different from the classification of general spectral Deligne-Mumford stacks. Nevertheless, our next result guarantees that the deformation theory of the former is controlled by the deformation theory of the latter:

Proposition 2.4.9. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Corollary 2.4.8, and let $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor which assigns to each connective $\mathbb{E}_\infty$-ring R the full subcategory of $X(R) \simeq \mathrm{Stk}_{/\operatorname{Spec} R}$ spanned by those maps $f : \mathfrak{X} \to \operatorname{Spec} R$ satisfying any one of the following conditions:*

(1) *The map f is locally of finite presentation to order n (where $n \geq 0$ is some fixed integer).*

(2) *The map f is locally almost of finite presentation.*

(3) *The map f is locally of finite presentation.*

(4) *The spectral Deligne-Mumford stack $\mathfrak{X}$ is n-quasi-compact (where $0 \leq n \leq \infty$).*

(5) *The spectral Deligne-Mumford stack $\mathfrak{X}$ is a separated spectral algebraic space.*

(6) *The map f is flat.*

(7) *The map f exhibits $\mathfrak{X}$ as a spectral algebraic space which is proper over R.*

Then the inclusion $j : X_0 \to X$ is cohesive and nilcomplete.

Remark 2.4.10. In the situation of Proposition 2.4.9, let $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor which assigns to a connective $\mathbb{E}_\infty$-ring R the full subcategory of $X(R) = \mathrm{Stk}_{/\operatorname{Spec} R}$ spanned by those maps of spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \operatorname{Spec} R$ which satisfy some combination of the conditions (1) through (7) appearing in Proposition 2.4.9. Then the inclusion $j : X_0 \to X$ is cohesive and nilcomplete. In particular, the functor X_0 is cohesive and nilcomplete.

The proof of Proposition 2.4.9 will require some preliminaries.

Lemma 2.4.11. *Let $f : A \to A'$ be a map of connective $\mathbb{E}_{frm[o]--}$-rings which induces an isomorphism $\pi_0 A \to \pi_0 A'$, and let M be a connective left A-module. Then M is perfect to order n (almost perfect, perfect) as a left A-module if and only if $A' \otimes_A M$ is perfect to order n (almost perfect, perfect) as a left A'-module.*

Proof. The "only if" directions are obvious. To prove the converse, let us first suppose that $M' = A' \otimes_A M$ is perfect to order n over A'. We will prove that M is perfect to order n over A using induction on n. If $n < 0$, there is nothing to prove (since M is assumed to be connective). If $n = 0$, we must show that $\pi_0 M$ is finitely generated as a module over $\pi_0 A$. Since $\pi_0 M' \simeq \pi_0 M$ and $\pi_0 A' \simeq \pi_0 A$, this follows from our assumption that M' is perfect to order n over A'.

Assume now that $n > 0$ and that M' is perfect to order n over A. The argument above shows that $\pi_0 M$ is finitely generated as a module over $\pi_0 A$, so we can choose a fiber sequence

$$N \to A^k \to M$$

where N is connective. Applying Proposition VIII.2.6.12, we deduce that $A' \otimes_A N$ is perfect to order $(n-1)$ over A'. The inductive hypothesis now shows that N is perfect to order $(n-1)$ over A, so that M is perfect to order n over A (by Proposition VIII.2.6.12 again).

To complete the proof, it will suffice to show that if M' is perfect over A', then M is perfect over A. The reasoning above shows that M is almost perfect over A. According to Proposition A.8.2.5.23, it will suffice to show that M has finite Tor-amplitude over A. Suppose that M' has Tor-amplitude $\leq k$ over A', and let N be a discrete right A-module. Then N has the structure of a right A'-module (since $\pi_0 A \simeq \pi_0 A'$), and the canonical equivalence $N \otimes_A M \simeq N \otimes_{A'} M'$ shows that $N \otimes_A M$ is k-truncated. It follows that M has Tor-amplitude $\leq k$ over A. □

Lemma 2.4.12. *Let $f : \mathfrak{X} \to \operatorname{Spec} R$ be a map of spectral Deligne-Mumford stacks, and let*

$$f_0 : \mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \pi_0 R \to \operatorname{Spec} \pi_0 R$$

be the induced map. Suppose that f_0 satisfies one of the following conditions:

(1) *The map f_0 is locally of finite presentation to order n (where $n \geq 0$ is some fixed integer).*

(2) *The map f_0 is locally almost of finite presentation.*

(3) *The map f_0 is locally of finite presentation.*

(4) *The map f_0 is n-quasi-compact (where $0 \leq n \leq \infty$).*

(5) *The map f_0 is strongly separated.*

(6) *The map f_0 is flat.*

(7) *The map f_0 exhibits $\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \pi_0 R$ as a proper spectral algebraic space over $\pi_0 R$.*

Then f has the same property.

Proof. We first treat cases (1), (2), and (3). The assertions are local on $\mathfrak{X}$, so we may assume that $\mathfrak{X} = \operatorname{Spec} A$ is affine. Let $A' = A \otimes_R \pi_0 R$. Assertion (1) is obvious in the case $n = 0$ (see Remark IX.8.3). In all other cases, we may assume that f_0 is of finite presentation to order 1, so that $\pi_0 A \simeq \pi_0 A'$ is finitely presented as a commutative ring over $\pi_0 R$. It will therefore suffice to show that if $L_{A'/\pi_0 R}$ is perfect to order n (almost perfect, perfect) as a module over A', then $L_{A/R}$ is perfect to order n (almost perfect, perfect) as a module over A (see Proposition IX.8.8 and Theorem A.8.4.3.18). This follows from Lemma 2.4.11.

Case (4) is easy, since the underlying ∞-topoi of $\mathfrak{X}$ and $\mathfrak{X}' = \mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \pi_0 R$ are the same. To treat case (5), we must show that the diagonal map $\delta : \mathfrak{X} \to \mathfrak{X} \times_{\operatorname{Spec} R} \mathfrak{X}$ is a closed immersion of spectral Deligne-Mumford stacks. Since δ admits a left homotopy inverse (given by the projection onto either fiber), it suffices to show that δ induces a closed immersion between the underlying ∞-topoi of $\mathfrak{X}$ and $\mathfrak{X} \times_{\operatorname{Spec} R} \mathfrak{X}$. This follows from our assumption on f_0, since the underlying ∞-topos of $\mathfrak{X} \times_{\operatorname{Spec} R} \mathfrak{X}$ is equivalent to the underlying ∞-topos of $\mathfrak{X}' \times_{\operatorname{Spec} \pi_0 R} \mathfrak{X}'$.

We now consider case (6). The assertion is local on $\mathfrak{X}$, so we may assume that $\mathfrak{X} = \operatorname{Spec} A$ is affine. We wish to show that A is flat over R. Let M be a discrete R-module; we wish to show that $A \otimes_R M$ is discrete. This is clear, since M has the structure of a module over $\pi_0 R$, and the tensor product $A \otimes_R \simeq A' \otimes_{\pi_0 R} M$ is discrete by virtue of our assumption that A' is flat over $\pi_0 R$.

It remains to treat case (7). Assume that f_0 exhibits $\mathfrak{X}'$ as a proper spectral algebraic space over $\pi_0 R$. It follows from (1), (4) and (5) that $\mathfrak{X}$ is a quasi-compact separated spectral algebraic space which is locally of finite presentation to order 0 over R. It will therefore suffice to show that, for every map of connective $\mathbb{E}_\infty$-rings $R \to R'$, the map of topological spaces $\phi : |\,\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R'| \to \operatorname{Spec}^{\mathrm{Z}} R'$ is closed. This follows from our assumption on f_0, since ϕ can be identified with the map of topological spaces

$$|\,\mathfrak{X}' \times_{\operatorname{Spec} \pi_0 R} \operatorname{Spec}(\pi_0 R \otimes_R R')| \to \operatorname{Spec}^{\mathrm{Z}}(\pi_0 R \otimes_R R').$$

□

Proof of Proposition 2.4.9. The nilcompleteness of j is a consequence of Lemma 2.4.12. We will prove that j is cohesive. In cases (1) through (6), this follows from Proposition IX.9.3. Let us consider (7). Suppose we are given a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

where the maps $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$ are surjective. Fix a map $f : \mathfrak{X} \to \operatorname{Spec} R$, and assume that $\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R_0$ and $\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R_1$ are spectral algebraic spaces which are proper over R_0 and R_1, respectively. It follows from (1), (4), and (5) that $\mathfrak{X}$ is a quasi-compact separated spectral algebraic space which is locally of finite presentation to order 0 over R. To complete the proof, it will suffice to show that for every connective R-algebra R', the map of topological spaces $|\,\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R'| \to \operatorname{Spec}^{\mathrm{Z}} R'$ is closed. Replacing R by R', it suffices to show that the map $|\,\mathfrak{X}\,| \to \operatorname{Spec}^{\mathrm{Z}} R$ is closed. Fix a closed subset $K \subseteq |\,\mathfrak{X}\,|$. Then K is the image of a closed immersion $i : \mathfrak{Y} \to \mathfrak{X}$ (where we can take the spectral Deligne-Mumford stack $\mathfrak{Y}$ to be reduced, if so desired). Let $\mathfrak{Y}_0 = \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}_0$, and define $\mathfrak{Y}_1$ and $\mathfrak{Y}_{01}$ similarly. We have a pushout diagram of spectral Deligne-Mumford stacks and closed immersions

$$\begin{array}{ccc} \mathfrak{Y}_{01} & \longrightarrow & \mathfrak{Y}_0 \\ \downarrow & & \downarrow \\ \mathfrak{Y}_1 & \longrightarrow & \mathfrak{Y}, \end{array}$$

hence a pushout diagram of topological spaces

$$\begin{array}{ccc} |\,\mathfrak{Y}_{01}\,| & \longrightarrow & |\,\mathfrak{Y}_0\,| \\ \downarrow & & \downarrow \\ |\,\mathfrak{Y}_1\,| & \longrightarrow & |\,\mathfrak{Y}\,|. \end{array}$$

It follows that the image of K in $\operatorname{Spec}^{\mathrm{Z}} R$ is the union of the images of the maps

$$|\,\mathfrak{Y}_0\,| \to |\,\mathfrak{X}_0\,| \to \operatorname{Spec}^{\mathrm{Z}} R_0 \hookrightarrow \operatorname{Spec}^{\mathrm{Z}} R$$

$$|\,\mathfrak{Y}_1\,| \to |\,\mathfrak{X}_1\,| \to \operatorname{Spec}^{\mathrm{Z}} R_1 \hookrightarrow \operatorname{Spec}^{\mathrm{Z}} R.$$

Each of these sets is closed, since $\mathfrak{X}_0$ is proper over R_0 and $\mathfrak{X}_1$ is proper over R_1. □

We are now almost ready to give the proof of Theorem 2.4.3.

Lemma 2.4.13. *Suppose we are given a commutative diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} R, & \end{array}$$

and suppose that the underlying map $\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \pi_0 R \to \mathfrak{Y} \times_{\operatorname{Spec} R} \operatorname{Spec} \pi_0 R$ is an equivalence. Then f is an equivalence.

Proof. The assertion is local on $\mathfrak{Y}$; we may therefore assume without loss of generality that $\mathfrak{Y} = \operatorname{Spec} A$ is affine. In this case, $\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \pi_0 R$. It follows that $Tr_0\,\mathfrak{X}$ is affine, so that $\mathfrak{X} = \operatorname{Spec} B$ is affine (see the proof of Theorem VII.8.42). Let K denote the cofiber of the map $A \to B$ (formed in the ∞-category Mod_R). We wish to prove that $K \simeq 0$. Assume otherwise. Since K is connective, there exists a smallest integer n such that $\pi_n K$ is nontrivial. In this case, we have

$$\pi_n K \simeq \operatorname{Tor}_0^{\pi_0 R}(\pi_0 R, \pi_n K) \simeq \pi_n(\pi_0 R \otimes_R K) \simeq \pi_n \operatorname{cofib}(\pi_0 R \otimes_R A \to \pi_0 R \otimes_R B) \simeq 0,$$

and we obtain a contradiction. □

Proof of Theorem 2.4.3. Assertions (1) and (2) follow from Proposition 2.4.9 (and Remark 2.4.10). Assertion (3) follows from Theorem XII.2.3.2 and Proposition XII.3.1.10. We now prove (4). Since M is nilcomplete, we have $M(R) \simeq \varprojlim_n M(\tau_{\leq n} R)$, so it will suffice to prove that each of the ∞-categories $M(\tau_{\leq n} R)$ is essentially small. This follows from Theorem XII.2.3.2.

Let us now prove (5). We begin with the case $K = \Delta^0$, verifying the conditions of Example 1.3.15:

(*a*) Let R be a connective $\mathbb{E}_\infty$-ring and let $\eta \in X_K(R)$, corresponding to a spectral Deligne-Mumford stack $\mathfrak{X}$ over R. Let $G : \operatorname{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ be the functor defined by the formula $G(M) = X_K(R \oplus M) \times_{X_K(R)} \{\eta\}$. We wish to show that G is almost corepresentable. Since the functor X_K is cohesive, the pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R \oplus M & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \oplus \Sigma M \end{array}$$

gives rise to a pullback diagram of spaces

$$\begin{array}{ccc} X_K(R \oplus M) & \longrightarrow & X_K(R) \\ \downarrow & & \downarrow \\ X_K(R) & \longrightarrow & X_K(R \oplus \Sigma M). \end{array}$$

For every R-algebra R', let $Y(R')$ denote the mapping space

$$\operatorname{Map}_{\operatorname{Stk}_{/\operatorname{Spec} R'}}(\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R', \mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R').$$

Then we can identify $G(M)$ with the summand of the fiber $\operatorname{fib}(Y(R \oplus \Sigma M) \to Y(R))$ consisting of the equivalences of $\mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec}(R \oplus \Sigma M)$ with itself. Lemma 2.4.13 implies that this summand is the entirety of the fiber $\operatorname{fib}(Y(R \oplus \Sigma M) \to Y(R))$. It follows from Proposition 3.3.6 (and its proof) that the functor G is corepresented by almost perfect R-module $f_+ \Sigma^{-1} L_{\mathfrak{X}/\operatorname{Spec} R}$, where f_+ is a left adjoint to f^*. Since f is flat, the pullback functor f^* is left t-exact, so that its left adjoint f_+ is right t-exact. It follows that $f_+ L_{\mathfrak{X}/\operatorname{Spec} R}$ is connective, so that G is corepresented by a (-1)-connective object of Mod_R.

(*b*) For every map of connective $\mathbb{E}_\infty$-rings $R \to R'$ and every connective R'-module M, the diagram of spaces

$$\begin{array}{ccc} X_K(R \oplus M) & \longrightarrow & X_K(R' \oplus M) \\ \downarrow & & \downarrow \\ X_K(R) & \longrightarrow & X_K(R') \end{array}$$

is a pullback square. This follows from the proof of (*a*) and the compatibility of the construction $f \mapsto f_+$ with base change (see Proposition XII.3.3.23).

We now treat the case of a general simplicial set K. Writing K as the union of its finite simplicial subsets and applying Remark 1.3.16, we may reduce to the case where K is finite. We will prove more generally that for any inclusion $K' \subseteq K$ of finite simplicial sets, the restriction map $F_K \to F_{K'}$ admits a (-1)-connective relative cotangent complex. We proceed by induction on the dimension of K. Using Proposition 2.2.9 repeatedly, we can reduce to the case where K is obtained from K' by adjoining a single nondegenerate simplex, so that we have a pushout diagram of simplicial sets

$$\begin{array}{ccc} \partial \Delta^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ K' & \longrightarrow & K \end{array}$$

and therefore a pullback diagram of functors

$$\begin{array}{ccc} X_K & \longrightarrow & X_{K'} \\ \downarrow & & \downarrow \\ X_{\Delta^n} & \longrightarrow & X_{\partial \Delta^n}. \end{array}$$

We may therefore replace K by Δ^n and K' by $\partial \Delta^n$. If $n \geq 2$, we have a commutative diagram of functors

$$\begin{array}{ccccc} & & X_{K'} & & \\ & \nearrow & & \searrow & \\ X_K & & \longrightarrow & & X_{\Lambda^n_1} \end{array}$$

where the lower horizontal map is an equivalence. Since the diagonal map on the right admits a (-1)-connective cotangent complex by the inductive hypothesis, Proposition 1.3.18 implies that the restriction map $X_K \to X_{K'}$ admits a cotangent complex which is a pullback of $\Sigma L_{X_{K'}/X_{\Lambda^n_1}}$, and therefore connective. We may therefore assume that $n \leq 1$. If $n = 0$, then we are in the case $K = \Delta^0$ treated above. Let us therefore assume that $n = 1$. According to Proposition 1.3.22, it will suffice to show that for every pullback diagram of functors

$$\begin{array}{ccc} U & \longrightarrow & X_{\Delta^1} \\ \downarrow{\scriptstyle q} & & \downarrow \\ \operatorname{Spec}^{\mathrm{f}} R & \xrightarrow{\eta} & X_{\partial \Delta^0}, \end{array}$$

the natural transformation q admits a relative cotangent complex (which is (-1)-connective). The map η classifies a pair of spectral Deligne-Mumford stacks $\mathfrak{X}$ and $\mathfrak{Y}$ which are proper, flat, and locally almost of finite presentation over R. Unwinding the definitions, we see that the functor U is given by the formula $U(R') = \operatorname{Map}_{\mathrm{Stk}_{/\operatorname{Spec} R'}}(\mathfrak{X} \times_{\operatorname{Spec} R} R', \mathfrak{Y} \times_{\operatorname{Spec} R} R')$. The existence of a relative cotangent complex of q now follows from Proposition 3.3.6. Moreover, if $g \in U(R')$, then $L_{U/\operatorname{Spec} R}(g)$ can be identified with the R'-module given by $f_+ g^* L_{\mathfrak{Y}/\operatorname{Spec} R}$, where $f : \mathfrak{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R' \to \operatorname{Spec} R'$ denotes the projection onto the second factor. Since f is flat, the functor f_+ is right t-exact. From this we deduce that $L_{U/\operatorname{Spec} R}$ is connective (and, in particular, (-1)-connective). This completes the proof of (5).

We now prove (6). Let K be a simplicial set with finitely many simplices of each dimension. Fix an integer n; we wish to show that the X_K commutes with filtered colimits when restricted to $\tau_{\leq n} \mathrm{CAlg}^{\mathrm{cn}}$. Let R be an n-truncated connective $\mathbb{E}_\infty$-ring. We may assume without loss of generality that $n \geq 1$, so that every object of $X(R)$ is n-localic (Corollary VIII.1.3.8). Note that $\mathfrak{X}$ is a spectral algebraic space which is flat over R, then the structure sheaf of $\mathfrak{X}$ is also n-truncated. It follows from Lemma VIII.1.3.6 that $X(R)$ is equivalent to an $(n+1)$-category (that is, the mapping spaces in $X(R)$ as n-truncated). Consequently, the restriction map $X_K \to X_{\mathrm{sk}^{n+2} K}$ is an equivalence of functors. To prove that X_K commutes with filtered colimits when restricted to $\tau_{\leq n} \mathrm{CAlg}^{\mathrm{cn}}$, we may replace K by the skeleton $\mathrm{sk}^{n+2} K$ and thereby reduce to the case where the simplicial set K is finite. The desired result then follows immediately from (3). The assertion that the cotangent complex L_{X_K} is almost perfect follows from Corollary 2.3.7. $\square$

Remark 2.4.14. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ be the functor of Theorem 2.4.3, and let $Y = X_{\Delta^0}$ be the functor from $\mathrm{CAlg}^{\mathrm{cn}}$ to $\mathcal{S}$ given by the formula $Y(R) = X(R)^{\simeq}$. Our proof of Theorem 2.4.3 supplies an explicit description of the cotangent complex of the functor Y. Given a point $\eta \in Y(R)$ classifying a proper flat morphism $f : \mathfrak{X} \to \mathrm{Spec}\, R$ which is locally almost of finite presentation, the R-module $\eta^* L_Y$ is given by $\Sigma^{-1}(f_+ L_{\mathfrak{X} / \mathrm{Spec}\, R})$, where f_+ denotes a left adjoint to the pullback functor f^*. In particular, if M is a connective R-module, then equivalence deformations of f over the trivial square-zero extension $R \oplus M$ are parametrized by the abelian group $\mathrm{Ext}^1_{\mathrm{QCoh}(\mathfrak{X})}(L_{\mathfrak{X} / \mathrm{Spec}^{\mathrm{f}} R}, f^* M)$.

3 Representability Theorems

Our goal in this section is to address the following question: given a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, when is X representable by a spectral Deligne-Mumford stack? We have the following necessary conditions:

(a) If X is representable by a spectral Deligne-Mumford stack, then X has a well-behaved deformation theory. More precisely, X must be nilcomplete, infinitesimally cohesive, and admit a cotangent complex (Propositions 1.3.17 and 2.1.7).

(b) Let CAlg^0 denote the full subcategory of $\mathrm{CAlg}^{\mathrm{cn}}$ spanned by the discrete $\mathbb{E}_\infty$-rings (so that we can identify CAlg^0 with the ordinary category of commutative rings). If X is representable by a spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O})$, then the restriction $X_0 = X | \mathrm{CAlg}^0$ also representable by the 0-truncated Deligne-Mumford stack $(\mathcal{X}, \tau_{\leq 0} \mathcal{O})$.

In §3.1, we will prove that conditions (a) and (b) are also sufficient (Theorem 3.1.2). This can be regarded as an illustration of the heuristic principle

$$\{\text{Spectral Algebraic Geometry}\} = \{\text{Classical Algebraic Geometry}\} + \{\text{Deformation Theory}\}.$$

Using Theorem 3.1.2, we can reduce many representability questions in spectral algebraic geometry to the analogous questions in classical algebraic geometry. These classical questions can then be addressed using Artin's representability theorem (Theorem 1). However, this sort of reasoning is unnecessarily circuitous: the hypotheses of Artin's theorem are closely related to our assumption (a), and are somewhat clarified in the setting of spectral algebraic geometry. In §3.2, we will apply Artin's method to give a direct proof of an analogous representability theorem in the setting of spectral algebraic geometry (Theorem 3.2.1).

The remainder of this section is devoted to applications of Theorems 3.1.2 and 3.2.1. In §3.3, we will use them to construct Weil restrictions of spectral Deligne-Mumford stacks along arbitrary maps $\mathfrak{X} \to \mathfrak{Y}$ which are strongly proper, flat, and locally almost of finite presentation (Theorem 3.3.1). In §3.4, we will use them to prove the representability of the (rigidified) relative Picard functor for a large class of maps $\mathfrak{X} \to \mathfrak{Y}$ (Theorem 3.4.6).

3.1 From Classical Algebraic Geometry to Spectral Algebraic Geometry

Let X be a scheme, and let $F_X : \mathrm{Ring} \to \mathcal{S}\mathrm{et}$ be the functor represented by X. We might ask if there exists a spectral scheme $\mathfrak{X}$ whose truncation coincides with X (under the equivalence between schemes and 0-truncated, 0-localic spectral schemes supplied by Proposition VII.2.37. In this case, $\mathfrak{X}$ represents a functor $F_{\mathfrak{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ whose restriction to discrete commutative rings agrees with F_X. Using Theorem VII.9.1, we can identify $F_{\mathfrak{X}}$ with the functor represented by the spectral Deligne-Mumford stack $\mathrm{Spec}^{\text{ét}}_{\mathrm{Zar}} \mathfrak{X}$. It follows from Propositions 1.3.17 and 2.1.7 that the functor $F_{\mathfrak{X}}$ is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. In this section, we will prove the following converse:

Theorem 3.1.1. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. Suppose further that the restriction $X|\,\mathrm{CAlg}^0$ is represented by a scheme (so that, in particular, $X(R)$ is discrete whenever R is discrete). Then X is representable by a 0-localic spectral scheme.*

By virtue Theorem VII.9.1 and Corollary VII.9.9, Theorem 3.1.1 is an immediate consequence of the following more general result:

Theorem 3.1.2. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then X is representable by a spectral Deligne-Mumford stack if and only if the following conditions are satisfied:*

(1) *There exists a spectral Deligne-Mumford stack $\mathfrak{Y}$ representing a functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and an equivalence of functors $X|\,\mathrm{CAlg}^0 \simeq Y|\,\mathrm{CAlg}^0$.*

(2) *The functor X admits a cotangent complex.*

(3) *The functor X is nilcomplete.*

(4) *The functor X is infinitesimally cohesive.*

Remark 3.1.3. A version of Theorem 3.1.2 appears in the third appendix of [84].

We will give a proof of Theorem 3.1.2 at the end of this section. The main point is to show that any map $Y \to X$ which is close to being étale can be approximated by another map $Y' \to X$ which is actually étale. More precisely, we have the following:

Proposition 3.1.4. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. Let Y_0 be a functor which is representable by a spectral Deligne-Mumford stack $(\mathcal{Y}, \mathcal{O}_0)$, and suppose we are given a map $f_0 : Y_0 \to X$ for which the relative cotangent complex $L_{Y_0/X}$ is 2-connective. Assume either that $(\mathcal{Y}, \mathcal{O}_0)$ is affine or that X satisfies étale descent. Then the map f_0 factors as a composition*

$$Y_0 \xrightarrow{g} Y \xrightarrow{f} X$$

where $L_{Y/X} \simeq 0$, Y is representable by a spectral Deligne-Mumford stack $(\mathcal{Y}, \mathcal{O})$, and g is induced by a 1-connective map $\mathcal{O} \to \mathcal{O}_0$.

The proof of Proposition 3.1.4 will require some preliminaries.

Lemma 3.1.5. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. The following conditions are equivalent:*

(1) *The functor X is a sheaf with respect to the étale topology.*

(2) *The functor $X|\,\mathrm{CAlg}^0$ is a sheaf with respect to the étale topology.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. To prove the converse, let us suppose that $X|\,\mathrm{CAlg}^0$ is a sheaf with respect to the étale topology. We wish to prove that, for every connective $\mathbb{E}_\infty$-ring R, the restriction $X_R = X|\,\mathrm{CAlg}_R^{\text{ét}}$ is a sheaf with respect to the étale topology. Since X is nilcomplete, X_R is the limit of a tower of functors $\{X_R^n\}_{n \geq 0}$ given by the formula $X_R^n(A) = X(\tau_{\leq n} A)$. It will therefore suffice to show that each X_R^n is a sheaf with respect to the étale topology. Replacing X by $\tau_{\leq n} R$, we may suppose that R is n-truncated. We proceed by induction on n. When $n = 0$, the desired result follows from assumption (2). Let us therefore assume that $n > 0$, so that R is a square-zero extension of $R' = \tau_{\leq n-1} R$ by $M = \Sigma^n(\pi_n R)$. We have a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ R' & \longrightarrow & R' \oplus \Sigma M. \end{array}$$

Define functors $Y_R, Z_R : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ by the formulas

$$Y_R(A) = X(A \otimes_R R') \qquad Z_R(A) = X(A \otimes_R R' \oplus M).$$

Since X is infinitesimally cohesive, we have a pullback diagram of functors

$$\begin{array}{ccc} X_R & \longrightarrow & Y_R \\ \downarrow & & \downarrow \\ Y_R & \longrightarrow & Z_R. \end{array}$$

It follows from the inductive hypothesis that Y_R is a sheaf with respect to the étale topology. To complete the proof, it will suffice to show that Z_R is a sheaf with respect to the étale topology. Applying Lemma VIII.3.1.20 to the projection map $Z_R \to Y_R$, we are reduced to proving the following:

(∗) For every étale R-algebra A and every point $\eta \in X(\tau_{\leq n-1} A)$, the formula

$$B \mapsto X(\tau_{\leq n-1} B \oplus (B \otimes_R M) \times_{X(\tau_{\leq n-1} B)} \{\eta\}$$

defines an étale sheaf $F : \mathrm{CAlg}_A^{\text{ét}} \to \mathcal{S}$.

Invoking the definition of the cotangent complex L_X, we see that the functor F is given by the formula

$$F(B) = \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leq n-1} A}}(\eta^* L_X, B \otimes_R M).$$

It follows from Corollary VII.6.14 (and Proposition VII.5.12) that F is a hypercomplete sheaf with respect to the flat topology. □

Remark 3.1.6. Lemma 3.1.5 has many variants, which can be proven by the same argument. Suppose that the functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ infinitesimally cohesive, nilcomplete, and has a cotangent complex. Then X is a (hypercomplete) sheaf with respect to the Zariski topology (flat topology, Nisnevich topology) if and only if the restriction $X | \mathrm{CAlg}^0$ has the same property.

Lemma 3.1.7. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and which admits a cotangent complex. Let $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O})$ be a spectral Deligne-Mumford stack, let $\mathcal{F}$ be a connective quasi-coherent sheaf on $\mathfrak{Y}$, and let $\mathfrak{Y}'$ denote the spectral Deligne-Mumford stack $(\mathcal{Y}, \mathcal{O} \oplus \mathcal{F})$. Let $Y, Y' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functors represented by Y and Y', respectively, and let $\alpha : Y \to Y'$ be the canonical map. Suppose we are given a map $\eta : Y \to X$. If $X | \mathrm{CAlg}^0$ is a sheaf with respect to the étale topology, then the canonical map*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{Y/}}(Y', X) \to \mathrm{Map}_{\mathrm{QCoh}(Y)}(\eta^* L_X, \alpha^* L_{Y'}) \to \mathrm{Map}_{\mathrm{QCoh}(Y)}(L_X, \mathcal{F})$$

is a homotopy equivalence.

Proof. It follows from Lemma 3.1.5 that X is a sheaf with respect to the étale topology. The assertion is therefore local on $\mathfrak{Y}$, so we may reduce to the case where $\mathfrak{Y}$ is affine. In this case, the desired result follows immediately from the definition of L_X. □

Remark 3.1.8. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack, let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ be connective, and let $\eta : L_{\mathfrak{X}} \to \Sigma \mathcal{F}$ be a map of quasi-coherent sheaves, classifying a square-zero extension $\mathcal{O}^\eta$ of $\mathcal{O}$. We have a commutative diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{O} \oplus \Sigma \mathcal{F}) & \longrightarrow & (\mathcal{X}, \mathcal{O}) \\ \downarrow & & \downarrow \\ (\mathcal{X}, \mathcal{O}) & \longrightarrow & (\mathcal{X}, \mathcal{O}^\eta), \end{array}$$

giving rise to a commutative diagram

$$\begin{array}{ccc} X^+ & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^\eta \end{array}$$

in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Suppose that $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is an infinitesimally cohesive functor which is a sheaf with respect to the étale topology. Then the induced diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(X^+, Y) & \longleftarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(X, Y) \\ \uparrow & & \uparrow \\ \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(X, Y) & \longleftarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}},\mathcal{S})}(X^\eta, Y) \end{array}$$

is a pullback square. To prove this, we use the fact that Y is a sheaf with respect to the étale topology to reduce to the case where X is affine, in which case it follows from the definition of an infinitesimally cohesive functor.

Proof of Proposition 3.1.4. We will give the proof under the assumption that X satisfies étale descent; the same argument works in general when $(\mathcal{Y}, \mathcal{O}_0)$ is affine. We will construct $\mathcal{O}$ as the inverse limit of a tower of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{Y}$

$$\cdots \to \mathcal{O}_2 \to \mathcal{O}_1 \to \mathcal{O}_0,$$

where each pair $(\mathcal{Y}, \mathcal{O}_k)$ is a spectral Deligne-Mumford stack representing a functor $Y_k : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, equipped with a map $f_k : Y_k \to X$ for which the relative cotangent complex $L_{Y_i/X}$ is $(2^k + 1)$-connective.

Let us assume that $\mathcal{O}_k$ has been constructed and that the relative cotangent complex $L_{Y_k/X}$ is $(1 + 2^k)$-connective. Let $\mathcal{O}_{k+1}$ denote the square-zero extension of $\mathcal{O}_k$ classified by the map $u : L_{Y_k} \to L_{Y_k/X}$ in $\mathrm{QCoh}(Y_k) \simeq \mathrm{QCoh}(\mathcal{Y}, \mathcal{O}_k)$. Let Z_k be the functor represented by $(\mathcal{Y}, \mathcal{O}_k \oplus L_{Y_k/X})$. We have a pushout diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} (\mathcal{Y}, \mathcal{O}_k \oplus L_{Y_k/X}) & \longrightarrow & (\mathcal{Y}, \mathcal{O}_k) \\ \downarrow & & \downarrow \\ (\mathcal{Y}, \mathcal{O}_k) & \longrightarrow & (\mathcal{Y}, \mathcal{O}_{k+1}) \end{array}$$

(see Proposition 1.2.8), giving rise to a diagram of functors

$$\begin{array}{ccc} Z_k & \xrightarrow{\delta} & Y_k \\ \downarrow{\scriptstyle \delta'} & & \downarrow \\ Y_k & \longrightarrow & Y_{k+1}. \end{array}$$

We have a canonical nullhomotopy of the restriction of u to $f_k^* L_X$, which gives a homotopy between $f_k \circ \delta$ and $f_k \circ \delta'$ (Lemma 3.1.7). Using Remark 3.1.8, we see that this homotopy gives rise to a map $f_{k+1} : Y_{k+1} \to X$ extending f_k. We wish to prove that the relative cotangent complex $L_{Y_{k+1}/X}$ is $(2^{k+1} + 1)$-connective. Let $i : Y_k \to Y_{k+1}$ denote the canonical map. Since $L_{Y_k/X}$ is $(2^k + 1)$-connective, the projection map $q : \mathcal{O}_{k+1} \to \mathcal{O}_k$ is 2^k connective, and therefore induces an isomorphism $\pi_0 \mathcal{O}_{k+1} \to \pi_0 \mathcal{O}_k$. It will therefore suffice to show that $i^* L_{Y_{k+1}/X}$ is $(2^{k+1} + 1)$-connective. This pullback fits into a fiber sequence

$$i^* L_{Y_{k+1}/X} \to L_{Y_k/X} \xrightarrow{\phi} L_{Y_k/Y_{k+1}}.$$

We will prove that the map ϕ is $(2^{k+1}+1)$-connective. Unwinding the definitions, we see that ϕ factors as a composition

$$L_{Y_k/X} \simeq \mathrm{cofib}(q) \xrightarrow{\phi'} \mathcal{O}_k \otimes_{\mathcal{O}_{k+1}} \mathrm{cofib}(q) \xrightarrow{\epsilon_q} L_{Y_k/Y_{k+1}},$$

where ϵ_q is as in Lemma 1.1.9. Since $\mathrm{cofib}(q)$ is 2^k+1-connective, the map q is 2^k-connective, so that the map ϕ' is $(2^{k+1}+1)$-connective. Lemma 1.1.9 implies that ϵ_q is $(2^{k+1}+2)$-connective, so that the composition ϕ is $(2^{k+1}+1)$-connective as desired.

Let $\mathcal{O}$ denote the sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{Y}$ given by $\varprojlim \mathcal{O}_k$. Since each $\mathcal{O}_k$ is hypercomplete, the inverse limit $\mathcal{O}$ is hypercomplete. For any affine object $U \in \mathcal{Y}$, we have a tower of connective $\mathbb{E}_\infty$-rings

$$\cdots \to \mathcal{O}_2(U) \to \mathcal{O}_1(U) \to \mathcal{O}_0(U),$$

where the map $\mathcal{O}_{k+1}(U) \to \mathcal{O}_k(U)$ is 2^k-connective. It follows that that the projection map $\mathcal{O}(U) \to \mathcal{O}_k(U)$ is 2^k-connective for each k, so that the projection map $\mathcal{O} \to \mathcal{O}_k$ induces an equivalence $\tau_{\leq 2^k-1}\mathcal{O} \to \tau_{\leq 2^k-1}\mathcal{O}_k$ for each $k \geq 0$. Applying the criterion of Theorem VII.8.42, we deduce that $(\mathcal{Y}, \mathcal{O})$ is a spectral Deligne-Mumford stack. Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor represented by Y. Using the fact that X is a nilcomplete étale sheaf, we deduce that the natural transformations $\beta_i : Y_i \to X$ induce a natural transformation $\beta : Y \to X$.

It is clear from the construction that the map $\mathcal{O} \to \mathcal{O}_0$ is 1-connective. We will complete the proof by showing that $L_{Y/X} \simeq 0$. Fix an integer k; we will show that $L_{Y/X}$ is 2^k-connective. Let $i : Y_k \to Y$ denote the canonical map; since $\mathcal{O} \to \mathcal{O}_k$ is an equivalence, it will suffice to show that $i^* L_{Y/X}$ is 2^k-connective. We have a fiber sequence

$$i^* L_{Y/X} \to L_{Y_k/X} \to L_{Y_k/Y}.$$

Since $L_{Y_k/X}$ is 2^k-connective, it will suffice to show that $L_{Y_k/Y}$ is (2^k+1)-connective. This follows from Corollary 1.1.10, since the map $\mathcal{O} \to \mathcal{O}_k$ is 2^k-connective. □

To apply Proposition 3.1.4, we will need the following technical result.

Lemma 3.1.9. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack, and assume that $\mathcal{O}$ is discrete. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by $\mathfrak{X}$, and let $X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a left Kan extension of $X | \mathrm{CAlg}^0$. Then the canonical map $X' \to X$ exhibits X as a sheafification of X' with respect to the étale topology.*

Proof. For every object $U \in \mathcal{X}$, let X_U denote the functor represented by the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}|U)$, and let X'_U be a left Kan extension of $X_U | \mathrm{CAlg}^0$. Let $\mathcal{X}_0$ denote the full subcategory of $\mathcal{X}$ spanned by those objects for which the canonical map $X'_U \to X_U$ exhibits X_U as a sheafification of X'_U with respect to the étale topology. To complete the proof, it will suffice to show that $\mathcal{X}_0 = \mathcal{X}$. If U is affine, then X_U is corepresented by an object of CAlg^0, so the canonical map $X'_U \to X_U$ is an equivalence; it follows that $U \in \mathcal{X}_0$. Since $\mathcal{X}$ is generated by affine objects under small colimits (Lemma V.2.3.11), it will suffice to show that $\mathcal{X}_0$ is closed under small colimits. Suppose that $U \in \mathcal{X}$ is given as a colimit of a small diagram $\{U_\alpha\}$ of objects of $\mathcal{X}_0$. To prove that $U \in \mathcal{X}_0$, it will suffice to show that for every functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the canonical map

$$\theta_U : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X_U, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X'_U, Y) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^0, \mathcal{S})}(X_U | \mathrm{CAlg}^0, Y | \mathrm{CAlg}^0)$$

is a homotopy equivalence. This map fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X_U, Y) & \xrightarrow{\theta_U} & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^0, \mathcal{S})}(X_U | \mathrm{CAlg}^0, Y | \mathrm{CAlg}^0) \\ \downarrow & & \downarrow \\ \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(X_{U_\alpha}, Y) & \xrightarrow{\varprojlim \theta_{U_\alpha}} & \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^0, \mathcal{S})}(X_{U_\alpha} | \mathrm{CAlg}^0, Y | \mathrm{CAlg}^0). \end{array}$$

Since each U_α belongs to $\mathcal{X}_0$, the lower horizontal map is a homotopy equivalence. It will therefore suffice to show that the vertical maps are homotopy equivalences. In other words, we are reduced to proving that X_U is a sheafification of $\varinjlim X_{U_\alpha}$ with respect to the étale topology. This follows from Lemma V.2.4.13. □

We now return to our main result.

Proof of Theorem 3.1.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor which is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. Suppose that there exists a spectral Deligne-Mumford stack $(\mathcal{Y}, \mathcal{O}_0)$ which represents a functor Y_0 such that $X|\,\mathrm{CAlg}^0 \simeq Y_0|\,\mathrm{CAlg}^0$. We wish to prove that X is representable by a spectral Deligne-Mumford stack.

Since Y_0 is a sheaf for the étale topology, $X|\,\mathrm{CAlg}^0$ is also a sheaf for the étale topology. Applying Lemma 3.1.5, we deduce that X is a sheaf for the étale topology. Replacing $\mathcal{O}_0$ by $\tau_{\leq 0}\,\mathcal{O}_0$, we may assume without loss of generality that the structure sheaf $\mathcal{O}_0$ is discrete. Let $Y_0' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a left Kan extension of $Y_0|\,\mathrm{CAlg}^0$, so that the equivalence $Y_0|\,\mathrm{CAlg}^0 \simeq X|\,\mathrm{CAlg}^0$ extends to a natural transformation $\alpha : Y_0' \to X$. It follows from Lemma 3.1.9 that the canonical map $Y_0' \to Y_0$ exhibits Y_0 as a sheafification of Y_0' with respect to the étale topology. Since X is an étale sheaf, the map α factors as a composition

$$Y_0' \to Y_0 \stackrel{f_0}{\to} X.$$

We next prove:

$(*)$ The quasi-coherent sheaf $f_0^* L_X$ is connective, and the canonical map $\pi_0 L_{Y_0} \to \pi_0 f_0^* L_X$ is an isomorphism.

To prove $(*)$, choose an étale map $\eta : \operatorname{Spec} R \to (\mathcal{Y}, \mathcal{O}_0)$; we will show that $\eta^* f_0^* L_X$ is connective and the map $\pi_0 \eta^* L_X \to \pi_0 \eta^* f_0^* L_X$ is an isomorphism. Note that $\eta^* f_0^* L_X$ is almost connective; if it is not connective, then there exists a discrete R-module M and a nonzero map $\eta^* \beta_0^* L_X \to M[k]$ for some integer $k < 0$. It then follows that the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* f_0^* L_X, M)$ is non-discrete. The quasi-coherent sheaf L_{Y_0} is connective, so that for any discrete R-module M $\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_{ay_0}, M)$ is a discrete space, homotopy equivalent to the abelian group of R-module maps from $\pi_0 \eta^* L_{Y_0}$ to M. We are therefore reduced to proving that the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* f_0^* L_X, M) \to \mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_{Y_0}, M)$$

is a homotopy equivalence. This map is obtained by passing to vertical homotopy fibers in the diagram

$$\begin{array}{ccc} Y_0(R \oplus M) & \longrightarrow & X(R \oplus M) \\ \downarrow & & \downarrow \\ Y_0(R) & \longrightarrow & X(R). \end{array}$$

This diagram is a homotopy pullback square because the horizontal maps are homotopy equivalences (the map $f_0 : Y_0 \to X$ induces a homotopy equivalence after evaluation on any commutative ring R, by assumption).

Since X and Y_0 admit cotangent complexes, the morphism $f_0 : Y_0 \to X$ admits a cotangent complex, which fits into a fiber sequence

$$f_0^* L_X \to L_{Y_0} \to L_{Y_0/X}$$

(see Corollary 1.3.19). Using $(*)$, we deduce that $L_{Y_0/X}$ is 1-connective. We will need the following slightly stronger assertion:

$(*')$ The relative cotangent complex $L_{Y_0/X}$ is 2-connective.

To prove $(*')$, we note that $(*)$ gives a short exact sequence

$$\pi_1 f_0^* L_X \to \pi_1 L_{Y_0} \to \pi_1 L_{Y_0/X} \to 0$$

in the abelian category $\mathrm{QCoh}(Y_0)^{\heartsuit}$. Let $\mathcal{F} = \pi_1 L_{Y_0/X}$. If $\mathcal{F}$ is nonzero, then we obtain a nonzero map $\gamma : L_{Y_0} \to L_{Y_0/X} \to \Sigma\,\mathcal{F}$ whose restriction to $f_0^* L_X$ vanishes. Choose an étale map $\eta : \operatorname{Spec} R \to \mathfrak{X}$ such that $M = \eta^*\,\mathcal{F}$ is nonzero. Then γ determines a derivation $L_R \to \Sigma M$ which classifies a square-zero extension

R^γ of R by M. Since R and M are discrete, the $\mathbb{E}_\infty$-ring R^γ is discrete. Since the derivation γ is nonzero, the point $\eta \in Y_0(R)$ cannot be lifted to a point of $Y_0(R^\gamma)$. However, the restriction of γ to $f_0^* L_X$ vanishes, so that $f_0(\eta)$ can be lifted to a point of $X(R^\gamma)$. This is a contradiction, since the map $Y_0(R^\gamma) \to X(R^\gamma)$ is a homotopy equivalence.

Combining $(*')$ with Proposition 3.1.4, we deduce that there exists a sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O}$ on $\mathcal{Y}$ equipped with a 1-connective map $q : \mathcal{O} \to \mathcal{O}_0$, such that $(\mathcal{Y}, \mathcal{O})$ represents a functor Y and f_0 factors as a composition

$$Y_0 \to Y \xrightarrow{f} X$$

where $L_{Y/X} \simeq 0$. The map q induces an isomorphism $\pi_0 \mathcal{O} \to \pi_0 \mathcal{O}_0$, so that f induces a homotopy equivalence $Y(R) \to X(R)$ whenever R is discrete. Applying Corollary 2.1.16, we deduce that f is an equivalence, so that X is representable by the spectral Deligne-Mumford stack $(\mathcal{Y}, \mathcal{O})$. □

3.2 Artin's Representability Theorem

Let R be a well-behaved Noetherian ring, and let X be a functor from the category of commutative R-algebras to the category of sets. In [1], Artin supplied necessary and sufficient conditions for X to be representable by an algebraic space which is locally of finite presentation over R. Our goal in this section is to prove the following analogue of Artin's result:

Theorem 3.2.1 (Spectral Artin Representability Theorem)**.** *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor, and suppose we are given a natural transformation $X \to \mathrm{Spec}^{\mathrm{f}} R$, where R is a Noetherian $\mathbb{E}_\infty$-ring and $\pi_0 R$ is a Grothendieck ring. Let $n \geq 0$. Then X is representable by a spectral Deligne-Mumford n-stack which is locally almost of finite presentation over R if and only if the following conditions are satisfied:*

(1) *For every discrete commutative ring A, the space $X(A)$ is n-truncated.*

(2) *The functor X is a sheaf for the étale topology.*

(3) *The functor X is nilcomplete, infinitesimally cohesive, and integrable.*

(4) *The natural transformation $X \to \mathrm{Spec}^{\mathrm{f}} R$ admits a connective cotangent complex $L_{X/\mathrm{Spec}^{\mathrm{f}} R}$.*

(5) *The natural transformation f is locally almost of finite presentation.*

We will give the proof of Theorem 3.2.1 at the end of this section. The main point is to show that if X is a functor satisfying conditions (1) through (5), then there is a good supply of étale maps $\mathrm{Spec}^{\mathrm{f}} B \to X$, where A is almost of finite presentation over R. We begin by looking for maps u which are approximately étale at some point of $\mathrm{Spec}^{\mathrm{f}} B$.

Proposition 3.2.2. *Let R be an $\mathbb{E}_\infty$-ring, let $Y = \mathrm{Spec}^{\mathrm{f}} R$, and suppose we are given a natural transformation $q : X \to Y$ of functors $X, Y \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ satisfying the following conditions: Assume that the following conditions are satisfied:*

(1) *The functor X is infinitesimally cohesive, nilcomplete, and integrable.*

(2) *The $\mathbb{E}_\infty$-ring R is Noetherian and $\pi_0 R$ is a Grothendieck commutative ring.*

(3) *The natural transformation q is locally almost of finite presentation.*

(4) *The map q admits a cotangent complex $L_{X/Y}$.*

Suppose we are given a field κ and a map $f : \mathrm{Spec}^{\mathrm{f}} \kappa \to X$ which exhibits κ as a finitely generated field extension of some residue field of R. Then the map f factors as a composition

$$\mathrm{Spec}^{\mathrm{f}} \kappa \to \mathrm{Spec}^{\mathrm{f}} B \to X,$$

where B is almost of finite presentation over R and the vector space $\pi_1(\kappa \otimes_B L_{\mathrm{Spec}^{\mathrm{f}} B/X})$ vanishes.

Proof. Since κ is a finitely generated field extension of some residue field of R, the relative cotangent complex $L_{\kappa/R}$ is an almost perfect κ-module. Since $L_{X/\operatorname{Spec}^{\mathrm{f}} R}$ is almost perfect (Corollary 2.3.7), the fiber sequence

$$f^* L_{X/\operatorname{Spec}^{\mathrm{f}} R} \to L_{\kappa/R} \to L_{\operatorname{Spec}^{\mathrm{f}} \kappa/X}$$

shows that the relative cotangent complex $L_{\operatorname{Spec}^{\mathrm{f}} \kappa/X}$ is almost perfect.

Let $\mathrm{CAlg}^{\mathrm{sm}}_{/k}$ be as in Notation XII.6.1.3, and define $\widehat{X} : \mathrm{CAlg}^{\mathrm{sm}}_{/k} \to \mathcal{S}$ by the formula $\widehat{X}(C) = X(C) \times_{X(k)} \{f\}$. Since X is infinitesimally cohesive, the functor $\widehat{X}$ is a formal moduli problem (see Proposition XII.6.1.5). Let $T^{\mathrm{red}}_{\widehat{X}}$ denote the reduced tangent complex of $\widehat{X}$ (see Notation XII.6.1.8). Unwinding the definitions, we see that $T^{\mathrm{red}}_{\widehat{X}}$ is given by the κ-linear dual of the relative cotangent complex $L_{\operatorname{Spec}^{\mathrm{f}} \kappa/X}$. Since $L_{\operatorname{Spec}^{\mathrm{f}} \kappa/X}$ is almost perfect over κ, it follows that each homotopy group $\pi_n T^{\mathrm{red}}_{\widehat{X}}$ is a finite-dimensional vector space over κ. Applying Theorem XII.6.2.5, we deduce the existence of a local Noetherian $\mathbb{E}_\infty$-ring A with residue field κ, and a formally smooth map $u : \operatorname{Spec}_{\mathrm{f}} A \to \widehat{X}$. Since X is integrable and nilcomplete, the map u is induced by a map $\overline{f} : \operatorname{Spec}^{\mathrm{f}} A \to X$ which fits into a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}^{\mathrm{f}} \kappa & \longrightarrow & \operatorname{Spec}^{\mathrm{f}} A \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \overline{f}} & \\ & X. & \end{array}$$

Since u is formally smooth, we have $\pi_n(\kappa \otimes_A L_{\operatorname{Spec}^{\mathrm{f}} A/X}) \simeq 0$ for $n > 0$.

Let $A' = \pi_0 A$. The canonical map $A \to A'$ is 1-connective, so that $L_{A'/A}$ is 2-connective (Corollary A.8.4.3.2). Using the fiber sequence

$$\kappa \otimes_A L_{\operatorname{Spec}^{\mathrm{f}} A/X} \to \kappa \otimes_{A'} L_{\operatorname{Spec}^{\mathrm{f}} A'/X} \to \kappa \otimes_{A'} L_{A'/A},$$

we deduce that $\pi_1(\kappa \otimes_{A'} L_{\operatorname{Spec}^{\mathrm{f}} A'/X}) \simeq 0$.

Let $\mathfrak{m}$ denote the maximal ideal of A', so that $A'/\mathfrak{m} \simeq \kappa$. Let A'' be a subalgebra of A' which is finitely generated over $\pi_0 R$ with the following properties:

- The subalgebra A'' contains generators of the field κ over $\pi_0 R$: that is, κ is the fraction field of the integral domain $A''/(A'' \cap \mathfrak{m})$.
- The subalgebra A'' contains a basis for the κ-vector space $\mathfrak{m}/\mathfrak{m}^2$.

Let $\mathfrak{p}$ denote the intersection $A'' \cap \mathfrak{m}$, and let $\widehat{A}''$ denote the completion of the Noetherian local ring $A''_{\mathfrak{p}}$. The conditions above guarantee that the map $v : A''_{\mathfrak{p}} \to A'$ induces an isomorphism of residue fields and a surjection of Zariski cotangent spaces. Since A' is complete with respect to its maximal ideal, v extends to a surjective map $\widehat{v} : \widehat{A}'' \to A'$. In particular, as an $\mathbb{E}_\infty$-algebra over $\widehat{A}''$, A' is almost of finite presentation.

Since A'' is finitely generated over $\pi_0 R$, it is a Grothendieck commutative ring (Theorem 0.0.5). It follows that the map $A''_{\mathfrak{p}} \to \widehat{A}''$ is geometrically regular. Applying Theorem 0.0.6, we deduce that $\widehat{A}''$ can be written as a colimit $\varinjlim_{\alpha \in P} A''_\alpha$ indexed by a filtered partially ordered set P, where each A''_α is a smooth A''-algebra (in the sense of classical commutative algebra). Using Theorem XII.2.3.2 (and Proposition XII.2.5.1), we see that there exists an index α and an equivalence $A' \simeq \tau_{\leq 1}(C_\alpha \otimes_{A''_\alpha} \widehat{A}'')$, where C_α is an $\mathbb{E}_\infty$-ring which is finitely 1-presented over A''_α. For $\beta \geq \alpha$, let $C_\beta = \tau_{\leq 1}(C_\alpha \otimes_{A''_\alpha} A''_\beta)$, so that

$$\varinjlim_{\beta \geq \alpha} C_\beta \simeq \tau_{\leq 1}(C_\alpha \otimes_{A''_\alpha} \varinjlim_{\beta \geq \alpha} A''_\beta) \simeq \tau_{\leq 1}(C_\alpha \otimes_{A''_\alpha} \widehat{A}'') \simeq A'.$$

Since the map $X \to \operatorname{Spec}^{\mathrm{f}} R$ is locally almost of finite presentation, the map $\operatorname{Spec}^{\mathrm{f}} A' \to X$ factors as a composition

$$\operatorname{Spec}^{\mathrm{f}} A' \to \operatorname{Spec}^{\mathrm{f}} A''_\beta \xrightarrow{u'} X$$

for some $\beta \geq \alpha$. Set $B = C_\beta$. We will complete the proof by showing that the relative cotangent complex of u' satisfies $\pi_1(\kappa \otimes_B L_{\mathrm{Spec}^{\mathrm{f}} B/X}) \simeq 0$.

We have an exact sequence

$$\pi_2(\kappa \otimes_{A'} L_{A'/B}) \to \pi_1(\kappa \otimes_B L_{\mathrm{Spec}^{\mathrm{f}} B/X}) \to \pi_1(\kappa \otimes_{A'} L_{\mathrm{Spec}^{\mathrm{f}} A'/X}.$$

Since the third term vanishes, it will suffice to show that $\pi_2(\kappa \otimes_{A'} L_{A'/B})$ vanishes. Using the pushout diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A''_\beta & \longrightarrow & \widehat{A}'' \\ \downarrow & & \downarrow \\ B & \longrightarrow & A', \end{array}$$

we obtain an isomorphism

$$\pi_2(\kappa \otimes_{A'} L_{A'/B}) \simeq \pi_2(\kappa \otimes_{\widehat{A}''} L_{\widehat{A}''/A''_\beta}.$$

To compute the right hand side, we use the exact sequence

$$\pi_2(\kappa \otimes_{\widehat{A}''} L_{\widehat{A}''/A''}) \to \pi_2(\kappa \otimes_{\widehat{A}''} L_{\widehat{A}''/A''_\beta}) \to \pi_1(\kappa \otimes_{A''_\beta} L_{A''_\beta/A''}).$$

Note that the tensor product $\kappa' = \widehat{A}'' \otimes_{A''} \kappa$ is equivalent to κ, so that

$$\kappa \otimes_{\widehat{A}''} L_{\widehat{A}''/A''} \simeq \kappa \otimes_{\kappa'} L_{\kappa'/\kappa} \simeq 0.$$

It will therefore suffice to show that the homotopy group $\pi_1(\kappa \otimes_{A''_\beta} L_{A''_\beta/A''})$ vanishes. Let $D = A''_\beta \otimes_{A''} \kappa$. Then D is a commutative algebra over κ which is smooth (in the sense of classical commutative algebra) and equipped with an augmentation $D \to \kappa$. We have a canonical isomorphism

$$\pi_1(\kappa \otimes_{A''_\beta} L_{A''_\beta/A''}) \simeq \pi_1(\kappa \otimes_D L_{D/\kappa}).$$

Note that the dual of the vector space $\pi_1(\kappa \otimes_D L_{D/\kappa})$ is the set of homotopy classes of D-module maps from $L_{D/\kappa}$ to $\Sigma(\kappa)$. This set classifies the collection of all isomorphism classes of square-zero extensions

$$0 \to I \to \widetilde{D} \to D \to 0$$

(in the category of commutative algebras over κ) equipped with an isomorphism of D-modules $I \simeq \kappa$. Since D is smooth over κ every such extension is automatically split. □

Remark 3.2.3. Proposition 3.2.2 does not require any connectivity assumption on the relative cotangent complex $L_{X/Y}$. Consequently, it can be used to prove a generalization of Theorem 3.2.1 to the setting of Artin stacks. We will return to this point in a future work.

Our next goal is to show that in the situation of Proposition 3.2.4, we can modify the map $\mathrm{Spec}^{\mathrm{f}} B \to X$ to obtain a map which is formally étale, in the sense that the relative cotangent complex $L_{\mathrm{Spec}^{\mathrm{f}} B/X}$ vanishes.

Proposition 3.2.4. *Let R be an $\mathbb{E}_\infty$-ring, let $Y = \mathrm{Spec}^{\mathrm{f}} R$, and suppose we are given a natural transformation $q : X \to Y$ of functors $X, Y \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Assume that the following conditions are satisfied:*

(1) *The functor X is infinitesimally cohesive, nilcomplete, and integrable.*

(2) *The $\mathbb{E}_\infty$-ring R is Noetherian and $\pi_0 R$ is a Grothendieck commutative ring.*

(3) *The natural transformation q is locally almost of finite presentation.*

(4) *The map q admits a connective cotangent complex $L_{X/Y}$.*

Suppose we are given an a connective $\mathbb{E}_\infty$-ring A and a map $f : \operatorname{Spec}^{\mathrm{f}} A \to X$. Let $\mathfrak{p}$ be a prime ideal of $\operatorname{Spec}^{\mathrm{Z}} A$, and let $\kappa(\mathfrak{p})$ denote the residue field of A at the point p. Then there exists an étale A-algebra A' and a prime ideal $\mathfrak{p}'$ of A' lying over $\mathfrak{p}$ such that the induced map $\operatorname{Spec}^{\mathrm{f}} A' \to X$ factors as a composition

$$\operatorname{Spec}^{\mathrm{f}} A' \to \operatorname{Spec}^{\mathrm{f}} B \to X$$

where B is almost of finite presentation over R, and the relative cotangent complex $L_{\operatorname{Spec}^{\mathrm{f}} B/X}$ vanishes.

Proof. Assume that $q : X \to Y = \operatorname{Spec}^{\mathrm{f}} R$ satisfies hypotheses (1) through (4). Let $f : \operatorname{Spec}^{\mathrm{f}} A \to X$ be an arbitrary map which exhibits A as almost of finite presentation over R, and let $\mathfrak{p}$ be a prime ideal of $\pi_0 A$. We wish to show that, after passing to an étale neighborhood of $\mathfrak{p}$, the map f factors as a composition $\operatorname{Spec}^{\mathrm{f}} A \to \operatorname{Spec}^{\mathrm{f}} B \xrightarrow{g} X$, where g exhibits B as almost of finite presentation over R and $L_{\operatorname{Spec}^{\mathrm{f}} B/X} \simeq 0$. Note that, if these conditions are satisfied, then g induces a homotopy equivalence

$$(\operatorname{Spec}^{\mathrm{f}} B)(A) \to (\operatorname{Spec}^{\mathrm{f}} B)(\pi_0 A) \times_{X(\pi_0 A)} X(A).$$

We may therefore replace A by $\pi_0 A$, and thereby reduce to the case where A is discrete.

Write A as a filtered colimit of subalgebras A_α which are finitely generated as commutative rings over $\pi_0 R$. Since $X \to \operatorname{Spec}^{\mathrm{f}} R$ is locally almost of finite presentation, the map f factors through $\operatorname{Spec}^{\mathrm{f}} A_\alpha$ for some α. Replacing A by A_α and $\mathfrak{p}$ by $A_\alpha \cap \mathfrak{p}$, we may reduce to the case where A is finitely generated as a commutative ring over R.

Let κ denote the residue field of A at $\mathfrak{p}$, and consider the composite map

$$f_\kappa : \operatorname{Spec}^{\mathrm{f}} \kappa \to \operatorname{Spec}^{\mathrm{f}} A \xrightarrow{f} X.$$

Applying Proposition 3.2.2, we see that f_κ factors as a composition

$$\operatorname{Spec}^{\mathrm{f}} \kappa \xrightarrow{j} \operatorname{Spec}^{\mathrm{f}} B \xrightarrow{g} X,$$

where B is almost of finite presentation over R and the vector space $\pi_1(\kappa \otimes_B L_{\operatorname{Spec}^{\mathrm{f}} B/X})$ vanishes. Since the relative cotangent complex $L_{X/Y}$ is connective, the fiber sequence

$$g^* L_{X/Y} \to L_{B/R} \to L_{\operatorname{Spec}^{\mathrm{f}} B/X}$$

shows that the map $\pi_0 L_{B/R} \to \pi_0 L_{\operatorname{Spec}^{\mathrm{f}} B/X}$ is surjective. Since $\pi_0 L_{\operatorname{Spec}^{\mathrm{f}} B/X}$ is a finitely generated module over $\pi_0 B$ (Corollary 2.3.7), we can find a finite sequence of elements $b_1, \ldots, b_m \in \pi_0 B$ such that the images of $db_1, db_2, \ldots, db_m$ in $\pi_0 L_{\operatorname{Spec}^{\mathrm{f}} B/X}$ form a basis for the vector space $\pi_0(\kappa \otimes_B L_{\operatorname{Spec}^{\mathrm{f}} B/X})$. The choice of these elements determines a map of B-modules $\psi : B^m \to L_{\operatorname{Spec}^{\mathrm{f}} B/X}$. The κ-module $\kappa \otimes_B \operatorname{cofib}(\psi)$ is 2-connective. Replacing B by a localization if necessary, we may suppose that $\operatorname{cofib}(\psi)$ is 2-connective.

The map $\operatorname{Spec}^{\mathrm{f}} B \to X$ is locally almost of finite presentation (Remark 2.3.3). Set $X' = \operatorname{Spec}^{\mathrm{f}} \kappa \times_X \operatorname{Spec}^{\mathrm{f}} B$, so that the projection map $q' : X' \to \operatorname{Spec}^{\mathrm{f}} \kappa$ satisfies hypotheses (1), (2), and (3). The map f_κ determines a section s of q'. Applying Proposition 3.2.2 again, we deduce that s factors as a composition

$$\operatorname{Spec}^{\mathrm{f}} \kappa \xrightarrow{\nu} \operatorname{Spec}^{\mathrm{f}} C \xrightarrow{g'} X'$$

where C is almost of finite presentation over κ and $\pi_1(\kappa \otimes_C L_{\operatorname{Spec}^{\mathrm{f}} C/X'}) \simeq 0$. Using the fiber sequence

$$g'^* L_{X'/\kappa} \to L_{C/\kappa} \to L_{\operatorname{Spec}^{\mathrm{f}} C/X'},$$

we deduce that $\pi_1(\kappa \otimes L_{C/\kappa}) \simeq 0$. It follows that the ordinary scheme $\operatorname{Spec} \pi_0 C$ is smooth over κ at the point determined by ν. Replacing C by a localization if necessary, we may suppose that the ordinary scheme $\operatorname{Spec} \pi_0 C$ is smooth over κ.

For $1 \leq i \leq m$, let c_i denote the image of b_i in the commutative ring $\pi_0 C$. Since $\pi_1(\kappa \otimes_C L_{\operatorname{Spec}^{\mathrm{f}} C/X'}) = 0$, the map $\pi_0(\kappa \otimes_B L_{\operatorname{Spec}^{\mathrm{f}} B/X}) \to \pi_0(\kappa \otimes_C L_{\operatorname{Spec}^{\mathrm{f}} C/\kappa})$ is injective: in other words, as functions on the affine scheme $\operatorname{Spec}(\pi_0 C)$, the c_i have linearly independent derivatives and therefore induce a smooth map of ordinary schemes $h : \operatorname{Spec} \pi_0 C \to \mathbf{A}^m_\kappa$. The image of h is a nonempty open subscheme U of the affine space $\mathbf{A}^m_\kappa$. Let $\kappa_0 \subseteq \kappa$ denote the prime field of κ. Then there exists a finite Galois extension κ_0' of κ_0 such that U contains a point u which is rational over κ_0'. Let κ' be a separable extension of κ containing κ_0', so that u defines a map of schemes $\operatorname{Spec} \kappa' \to U$. Since h defines a smooth surjection $\operatorname{Spec} \pi_0 C \to U$, we may (after enlarging κ' if necessary) assume that u factors as a composition $\operatorname{Spec} \kappa' \to \operatorname{Spec} \pi_0 C \to U$. This determines a new map $j' : \operatorname{Spec}^{\mathrm{f}} \kappa' \to \operatorname{Spec}^{\mathrm{f}} B$, whose composition with g agrees with the composition

$$\operatorname{Spec}^{\mathrm{f}} \kappa' \to \operatorname{Spec}^{\mathrm{f}} \kappa \to \operatorname{Spec}^{\mathrm{f}} A \xrightarrow{f} X.$$

Since κ' is a finite separable extension of κ, we can write $\kappa' = \kappa[x]/(r(x))$ for some separable polynomial r. After localizing A, we can assume that r lifts to a separable polynomial $\overline{r}(x) \in (\pi_0 A)[x]$. Then $(\pi_0 A)[x]/(\overline{r}(x))$ is a finite étale extension of $\pi_0 A$. Using Theorem A.8.5.0.6, we can write $(\pi_0 A)[x]/(\overline{r}(x)) \simeq \pi_0 A'$, where A' is a finite étale A-algebra. Replacing A by A', κ by κ', and j by j', we can reduce to the case where j is given by a ring homomorphism $\pi_0 B \to \kappa$ which carries each b_i to an element $\lambda_i \in \kappa$ belongs to a subfield $\kappa_0' \subseteq \kappa$ which is algebraic over the prime field κ_0.

Choose an integer N and a finite étale $\mathbf{Z}[N^{-1}]$-algebra D such that $\kappa_0' \simeq \kappa_0 \otimes_{\mathbf{Z}} D$. Enlarging N if necessary, we may suppose that each λ_i can be lifted to an element $\overline{\lambda}_i \in D$. Replacing B by $B[N^{-1}]$ if necessary, we may suppose that N is invertible in $\pi_0 B$, so that $(\pi_0 B) \otimes_{\mathbf{Z}[N^{-1}]} D$ is a finite étale extension of $\pi_0 B$. Applying Theorem A.8.5.0.6, we can write $(\pi_0 B) \otimes_{\mathbf{Z}[N^{-1}]} D \simeq \pi_0 \overline{B}$ for some finite étale extension $\overline{B}$ of B. Moreover, the embedding $\kappa_0' \hookrightarrow \kappa'$ induces a map $\overline{B} \to \kappa$, which annihilates the elements $b_i - \overline{\lambda}_i \in \pi_0 \overline{B}$. Replacing B by $\overline{B}$ and the elements $b_i \in B$ by the differences $b_i - \overline{\lambda_i}$ (note that this does not change the differentials db_i), we may reduce to the case where the map $j : \operatorname{Spec}^{\mathrm{f}} \kappa \to \operatorname{Spec}^{\mathrm{f}} B$ annihilates each b_i.

Let $B_0 = B \otimes_{S\{b_1, \ldots, b_m\}} S$ denote the $\mathbb{E}_\infty$-algebra over B obtained by killing each b_i. Then we have a fiber sequence

$$B_0 \otimes_B L_{\operatorname{Spec}^{\mathrm{f}} B/X} \to L_{\operatorname{Spec}^{\mathrm{f}} B_0/X} \to L_{B_0/B}.$$

We note that $L_{B_0/B} \simeq \Sigma(B_0^m)$, and the boundary map $\Omega L_{B_0/B} \to B_0 \otimes_B L_{\operatorname{Spec}^{\mathrm{f}} B/X}$ is induced by the map $\psi : B^m \to L_{\operatorname{Spec}^{\mathrm{f}} B/X}$ given by the elements b_i. Since $\operatorname{cofib}(\psi)$ is 2-connective, we deduce that $L_{\operatorname{Spec}^{\mathrm{f}} B_0/X}$ is 2-connective. The map $j : \operatorname{Spec}^{\mathrm{f}} \kappa \to \operatorname{Spec}^{\mathrm{f}} B$ annihilates each b_i, and therefore factors through $\operatorname{Spec}^{\mathrm{f}} B_0$. We may therefore replace B by B_0, and thereby reduce to the case where $L_{\operatorname{Spec}^{\mathrm{f}} B/X}$ is 2-connective.

Proposition 3.1.4 implies that the map $g : \operatorname{Spec}^{\mathrm{f}} B \to X$ factors as a composition

$$\operatorname{Spec}^{\mathrm{f}} B \to \operatorname{Spec}^{\mathrm{f}} \overline{B} \to X,$$

where $\pi_0 \overline{B} \simeq \pi_0 B$ and $L_{\overline{B}/X} \simeq 0$. It follows from Corollary 2.3.7 that $L_{B/\overline{B}} \simeq L_{\operatorname{Spec}^{\mathrm{f}} B/X}$ is an almost perfect module over B. Since B is almost of finite presentation over R, $L_{B/R}$ is almost perfect. Using the fiber sequence

$$B \otimes_{\overline{B}} L_{\overline{B}/R} \to L_{B/R} \to L_{B/\overline{B}},$$

we deduce that $B \otimes_{\overline{B}} L_{\overline{B}/R}$ is almost perfect as a B-module. It follows from Lemma 2.4.11 that $L_{\overline{B}/R}$ is almost perfect as an $\overline{B}$-module. Since $\pi_0 \overline{B} \simeq \pi_0 B$ is finitely presented as an commutative ring over $\pi_0 R$, Theorem A.8.4.3.18 implies $\overline{B}$ is almost of finite presentation over R. We may therefore replace B by $\overline{B}$, and thereby reduce to the case where $L_{\operatorname{Spec}^{\mathrm{f}} B/X} \simeq 0$.

We will now complete the proof by showing that there exists a finite étale A-algebra A' with $\kappa \otimes_A A' \neq 0$, such that the induced map $\operatorname{Spec}^{\mathrm{f}} A' \to X$ factors through $\operatorname{Spec}^{\mathrm{f}} B$. Since X is nilcomplete and infinitesimally cohesive, the vanishing of the relative cotangent complex $L_{\operatorname{Spec}^{\mathrm{f}} B/X}$ implies that the diagram

$$\begin{array}{ccc} (\operatorname{Spec}^{\mathrm{f}} B)(A') & \longrightarrow & (\operatorname{Spec}^{\mathrm{f}} B)(\pi_0 A') \\ \downarrow & & \downarrow \\ X(A') & \longrightarrow & X(\pi_0 A') \end{array}$$

is a pullback square for every $\mathbb{E}_\infty$-ring A'. We may therefore replace A by $\pi_0 A$, and thereby reduce to the case where A is discrete.

Let $\widehat{A}$ denote the completion of the local ring $A_{\mathfrak{p}}$ at its maximal ideal, and let $\mathfrak{m}$ denote the maximal ideal of $\widehat{A}$, so that $\widehat{A}/\mathfrak{m} \simeq 0$. Each quotient $\widehat{A}/\mathfrak{m}^{n+1}$ is a square-zero extension of $\widehat{A}/\mathfrak{m}$. Since g is infinitesimally cohesive with $L_{\operatorname{Spec}^{\mathrm{f}} B/X} \simeq 0$, it follows that each of the diagrams

$$\begin{array}{ccc} (\operatorname{Spec}^{\mathrm{f}} B)(\widehat{A}/\mathfrak{m}^{n+1}) & \longrightarrow & (\operatorname{Spec}^{\mathrm{f}} B)(\widehat{A}/\mathfrak{m}) \\ \downarrow & & \downarrow \\ X(\widehat{A}/\mathfrak{m}^{n+1}) & \longrightarrow & X(\widehat{A}/\mathfrak{m}^{n}). \end{array}$$

is a pullback square It follows that the diagram

$$\begin{array}{ccc} \varprojlim (\operatorname{Spec}^{\mathrm{f}} B)(\widehat{A}/\mathfrak{m}^{n}) & \longrightarrow & (\operatorname{Spec}^{\mathrm{f}} B)(\kappa) \\ \downarrow & & \downarrow \\ \varprojlim X(\widehat{A}/\mathfrak{m}^{n}) & \longrightarrow & X(\kappa) \end{array}$$

is a pullback square. Since both X and $\operatorname{Spec}^{\mathrm{f}} B$ are integrable, we obtain a pullback square

$$\begin{array}{ccc} (\operatorname{Spec}^{\mathrm{f}} B)(\widehat{A}) & \longrightarrow & (\operatorname{Spec}^{\mathrm{f}} B)(\kappa) \\ \downarrow & & \downarrow \\ X(\widehat{A}) & \longrightarrow & X(\kappa). \end{array}$$

It follows that the map $j : \operatorname{Spec}^{\mathrm{f}} \kappa \to \operatorname{Spec}^{\mathrm{f}} B$ admits an essentially unique factorization as a composition

$$\operatorname{Spec}^{\mathrm{f}} \kappa \to \operatorname{Spec}^{\mathrm{f}} \widehat{A} \xrightarrow{\widehat{j}} \operatorname{Spec}^{\mathrm{f}} B,$$

where $\widehat{j}$ fits into a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}^{\mathrm{f}} \widehat{A} & \xrightarrow{\widehat{j}} & \operatorname{Spec}^{\mathrm{f}} B \\ \downarrow & & \downarrow g \\ \operatorname{Spec}^{\mathrm{f}} A & \xrightarrow{f} & X. \end{array}$$

By assumption, $\pi_0 R$ is a Grothendieck ring. Since A is finitely generated as an algebra over $\pi_0 R$, it is also a Grothendieck ring (Theorem 0.0.5), so that the map $A_{\mathfrak{p}} \to \widehat{A}$ is geometrically regular. Applying Theorem 0.0.6, we can write $\widehat{A}$ as a filtered colimit $\varinjlim A_\alpha$, where each A_α is smooth over $A_{\mathfrak{p}}$ (in the sense of classical commutative algebra). Since B is almost of finite presentation over R and the natural transformation $X \to \operatorname{Spec}^{\mathrm{f}} R$ is locally almost of finite presentation, the natural transformation g is locally almost of finite presentation. It follows that for some index α, there exists a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}^{\mathrm{f}} A_\alpha & \longrightarrow & \operatorname{Spec}^{\mathrm{f}} B \\ \downarrow & & \downarrow g \\ \operatorname{Spec}^{\mathrm{f}} A & \longrightarrow & X. \end{array}$$

Since A_α is smooth over $A_{\mathfrak{p}}$, we can choose a smooth A-algebra $\overline{A}$ over A such that $A_\alpha \simeq \overline{A} \otimes_A A_{\mathfrak{p}}$. Then A_α can be written as a filtered colimit of A-algebras of the form $\overline{A}[a^{-1}]$, where $a \in A - \mathfrak{p}$. Using the fact that g is locally almost of finite presentation again, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}^{\mathrm{f}}\, \overline{A}[a^{-1}] & \longrightarrow & \mathrm{Spec}^{\mathrm{f}}\, B \\ \downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{Spec}^{\mathrm{f}}\, A & \longrightarrow & X. \end{array}$$

for some $a \in A - \mathfrak{p}$. Note that $v : \mathrm{Spec}\, \overline{A}[a^{-1}] \to \mathrm{Spec}\, A$ is a smooth map of affine schemes whose image contains the prime ideal $\mathfrak{p}$. It follows that there exists an étale A-algebra A' such that $\kappa \otimes_A A' \neq 0$, and the map $\mathrm{Spec}\, A' \to \mathrm{Spec}\, A$ factors through v. Then the map $\mathrm{Spec}^{\mathrm{f}}\, A' \to \mathrm{Spec}^{\mathrm{f}}\, A \xrightarrow{f} X$ factors through the map $g : \mathrm{Spec}^{\mathrm{f}}\, B \to X$, as desired. □

We now turn to the proof of our main result.

Proof of Theorem 3.2.1. Suppose that X is representable by a spectral Deligne-Mumford n-stack. Conditions (1) and (2) are obviously satisfied. Condition (3) is satisfied by Proposition 2.1.7 and condition (4) by Proposition 1.3.17. If f is locally almost of finite presentation, then condition (5) follows from Proposition 2.3.9. This proves the necessity of conditions (1) through (5).

Now suppose that conditions (1) through (5) are satisfied; we wish to prove that X is representable by a spectral Deligne-Mumford stack $\mathfrak{X}$. Then $\mathfrak{X}$ is automatically a spectral Deligne-Mumford n-stack (by virtue of condition (1)) and locally almost of finite presentation over R (by condition (5) and Proposition 2.3.9). To prove the existence of $\mathfrak{X}$, we first note that hypothesis (1) can be restated as follows:

($1'_n$) For every discrete commutative ring A, the map $X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A)$ has n-truncated homotopy fibers.

Note that condition ($1'$) makes sense for all $n \geq -2$. We will show that for all $n \geq -2$, if $f : X \to \mathrm{Spec}^{\mathrm{f}}\, R$ satisfies conditions ($1'_n$), (2), (3), (4), and (5), then X is representable by a spectral Deligne-Mumford stack $\mathfrak{X}$.

Our proof now proceeds by induction on n. We begin by treating the case $n = -2$. In this case, condition ($1'$) asserts that the map $X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A)$ is a homotopy equivalence for every discrete commutative ring A. In this case, the existence of $\mathfrak{X}$ follows from Theorem 3.1.2.

Now suppose that $n \geq -2$. Let $\mathcal{S}\mathrm{hv}_{\text{ét}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors which are sheaves for the étale topology. Let S be a set of representatives for all equivalences classes of maps $\mathrm{Spec}^{\mathrm{f}}\, B_\alpha \to X$ for which $L_{\mathrm{Spec}^{\mathrm{f}}\, B_\alpha / X} = 0$ and exhibit B_α as almost of finite presentation over R, and let X_0 denote the coproduct $\coprod_{\alpha \in S} \mathrm{Spec}^{\mathrm{f}}\, B_\alpha$ formed in the ∞-category $\mathcal{S}\mathrm{hv}_{\text{ét}}$, and let $X_\bullet$ denote the simplicial object of $\mathcal{S}\mathrm{hv}_{\text{ét}}$ given by the Čech nerve of the map $X_0 \to X$. Note that each X_m is given as a coproduct (in the ∞-category $\mathcal{S}\mathrm{hv}_{\text{ét}}$)

$$\coprod_{(\alpha_1, \ldots, \alpha_m) \in S^m} X_{\alpha_1, \ldots, \alpha_m},$$

$$X_{\alpha_1, \ldots, \alpha_m} = (\mathrm{Spec}^{\mathrm{f}}\, B_{\alpha_1}) \times_X \cdots \times_X (\mathrm{Spec}^{\mathrm{f}}\, B_{\alpha_m}),$$

and therefore admits a map

$$X_{\alpha_1, \ldots, \alpha_m} \to \mathrm{Spec}^{\mathrm{f}}(B_{\alpha_1} \otimes_R \cdots \otimes_R B_{\alpha_m})$$

satisfying condition ($1'_{n-1}$). Applying the inductive hypothesis, we deduce that each $X_{\alpha_1, \ldots, \alpha_m}$ is representable by a spectral Deligne-Mumford stack, so that each X_m is representable by a spectral Deligne-Mumford stack $\mathfrak{X}_m$. Proposition 2.3.9 implies that each $\mathfrak{X}_m$ is locally almost of finite presentation over R, so that each of the transition maps $\mathfrak{X}_m \to \mathfrak{X}_{m'}$ is locally almost of finite presentation. By construction, we have $L_{X_0/X} \simeq 0$, which implies that each transition map $\mathfrak{X}_m \to \mathfrak{X}_{m'}$ has vanishing cotangent complex.

Applying Proposition 1.2.13, we deduce that each of the maps $\mathfrak{X}_m \to \mathfrak{X}_{m'}$ is étale. It follows that the simplicial object $\mathfrak{X}_\bullet$ admits a geometric realization $|\,\mathfrak{X}_\bullet\,|$ in the ∞-category of spectral Deligne-Mumford stacks (Proposition V.2.3.10). Lemma V.2.4.13 implies that $|\,\mathfrak{X}_\bullet\,|$ represents the functor $|X_\bullet|$, where the geometric realization is formed in the ∞-category $\mathcal{S}\mathrm{hv}_{\text{ét}}$. To complete the proof that X is representable, it will suffice to show that the canonical map $|X_\bullet| \to X$ is an equivalence. Since $\mathcal{S}\mathrm{hv}_{\text{ét}}$ is an ∞-topos, this is equivalent to the requirement that the map $X_0 \to X$ is an effective epimorphism of étale sheaves, which follows from Proposition 3.2.4. $\square$

3.3 Application: Existence of Weil Restrictions

Let X be an affine scheme defined over the complex numbers, and let $X(\mathbf{C})$ denote the collection of $\mathbf{C}$-points of X. Then $X(\mathbf{C})$ can be described as the set of $\mathbf{R}$-points of an affine $\mathbf{R}$-scheme Y. For example, if X is given as a closed subscheme of affine space $\mathbf{A}^n$ defined by a collection of polynomial equations $f_\alpha(z_1, \ldots, z_n) = 0$ with complex coefficients, then Y can be described as the closed subvariety of $\mathbf{A}^{2n}$ defined by the real polynomial equations

$$\Re(f_\alpha(x_1 + iy_1, \ldots, x_n + iy_n)) = 0 = \Im(f_\alpha(x_1 + iy_1, \ldots, x_n + iy_n)).$$

Here the scheme Y is called the *Weil restriction* of X along the morphism $\operatorname{Spec} \mathbf{C} \to \operatorname{Spec} \mathbf{R}$. It is characterized by the following universal property: for every $\mathbf{R}$-scheme S, there is a canonical bijection

$$\operatorname{Hom}(S, Y) \simeq \operatorname{Hom}(S \times_{\operatorname{Spec} \mathbf{R}} \operatorname{Spec} \mathbf{C}, X),$$

where the first Hom is computed in the category $\mathbf{R}$-schemes and the second Hom is computed in the category of $\mathbf{C}$-schemes.

In this section, we will study the operation of Weil restriction in the context of spectral algebraic geometry. Suppose that we are given a map of spectral Deligne-Mumford stacks $f : \mathfrak{Y} \to \mathfrak{Z}$, and another map $\mathfrak{X} \to \mathfrak{Y}$. A *Weil restriction* of $\mathfrak{X}$ along f is another spectral Deligne-Mumford stack $\mathfrak{W}$ equipped with a map $\mathfrak{W} \to \mathfrak{Z}$ and a commutative diagram

$$\begin{array}{ccc} \mathfrak{W} \times_{\mathfrak{Z}} \mathfrak{Y} & \xrightarrow{\rho} & \mathfrak{X} \\ & \searrow \quad \swarrow & \\ & \mathfrak{Y} & \end{array}$$

satisfying the following universal property: for every map of spectral Deligne-Mumford stacks $\mathfrak{U} \to \mathfrak{Z}$, composition with ρ induces a homotopy equivalence

$$\operatorname{Map}_{\mathrm{Stk}_{/\mathfrak{Z}}}(\mathfrak{U}, \mathfrak{W}) \to \operatorname{Map}_{\mathrm{Stk}_{/\mathfrak{Y}}}(\mathfrak{U} \times_{\mathfrak{Z}} \mathfrak{Y}, \mathfrak{X}).$$

In this case, the spectral Deligne-Mumford stack $\mathfrak{W}$ is determined up to canonical equivalence, and will be denoted by $\operatorname{Res}_{\mathfrak{Y}/\mathfrak{Z}}(\mathfrak{X})$. Our main result can be stated as follows:

Theorem 3.3.1. *Let $\phi : \mathfrak{Y} \to \mathfrak{Z}$ be a morphism of spectral Deligne-Mumford stacks which is strongly proper, flat, and locally almost of finite presentation, and let $\mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks which is geometric and locally almost of finite presentation. Then there exists a Weil restriction $\operatorname{Res}_{\mathfrak{Y}/\mathfrak{Z}}(\mathfrak{X})$. Moreover, the canonical map $\operatorname{Res}_{\mathfrak{Y}/\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z}$ is geometric and locally almost of finite presentation.*

We will give the proof of Theorem 3.3.1 at the end of this section. Let us begin by treating the affine case:

Proposition 3.3.2. *Let $\phi : \mathfrak{Y} \to \mathfrak{Z}$ be a morphism of spectral Deligne-Mumford stacks which is strongly proper, flat, and locally almost of finite presentation, and let $\mathfrak{X} \to \mathfrak{Y}$ be an affine morphism of spectral Deligne-Mumford stacks. Then there exists a Weil restriction $\operatorname{Res}_{\mathfrak{Y}/\mathfrak{Z}}(\mathfrak{X})$. Moreover, the canonical map $\operatorname{Res}_{\mathfrak{Y}/\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z}$ is affine.*

Proposition 3.3.2 is essentially equivalent to the following:

Proposition 3.3.3. *Let $\phi : \mathfrak{Y} \to \mathfrak{Z}$ be a morphism between quasi-compact, quasi-separated spectral algebraic spaces which is strongly proper, flat, and locally almost of finite presentation. Then the pullback functor $\phi^* : \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Z})) \to \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y}))$ admits a left adjoint $\phi_\dagger$. Moreover, the functor $\phi_\dagger$ carries connective objects of $\mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y}))$ to connective objects of $\mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Z}))$.*

Proof. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y}))$ spanned by those objects A for which the functor

$$B \mapsto \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y}))}(A, \phi^* B)$$

is corepresentable by an object $\phi_\dagger(A) \in \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Z}))$. To prove the existence of $\phi_\dagger$, it will suffice to show that $\mathcal{C} = \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y}))$. Note that $\mathcal{C}$ is closed under small colimits in $\mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y}))$. Let $\mathrm{Sym}^*_{\mathfrak{Y}} : \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y}))$ be a left adjoint to the forgetful functor, and define $\mathrm{Sym}^*_{\mathfrak{Z}}$ similarly. It follows from Proposition A.6.2.2.12 that $\mathcal{C}$ is generated under small colimits by the essential image of $\mathrm{Sym}^*_{\mathfrak{Y}}$. It will therefore suffice to show that $\mathcal{C}$ contains the essential image of $\mathrm{Sym}^*_{\mathfrak{Y}}$. In other words, it suffices to show that for each quasi-coherent sheaf M on $\mathfrak{Y}$, the functor

$$B \mapsto \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y}))}(\mathrm{Sym}^*_{\mathfrak{Y}}(M), \phi^* B) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{Y})}(M, \phi^* B)$$

is corepresentable.

According to Proposition XII.3.3.23, the pullback functor $\mathrm{QCoh}(\mathfrak{Z}) \to \mathrm{QCoh}(\mathfrak{Y})$ admits a left adjoint ϕ_+. We then have $\mathrm{Map}_{\mathrm{QCoh}(\mathfrak{Y})}(M, \phi^* B) \simeq \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{Z})}(\phi_+ M, B) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Z}))}(\mathrm{Sym}^*_{\mathfrak{Z}}(\phi_+(M)), B)$. It follows that $\mathrm{Sym}^*_{\mathfrak{Y}}(M)$ belongs to $\mathcal{C}$, and that we have a canonical equivalence $\phi_\dagger(\mathrm{Sym}^*_{\mathfrak{Y}}(M)) \simeq \mathrm{Sym}^*_{\mathfrak{Z}}(\phi_+ M)$. This completes the proof of the existence of $\phi_\dagger$.

Now suppose that $A \in \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y}))$ is connective; we wish to show that $\phi_\dagger(A)$ is connective. Let B be the connective cover of $\phi_\dagger(A)$, and let $u : B \to \phi_\dagger(A)$ be the canonical map. Since ϕ is flat, the induced map $\phi^*(u) : \phi^*(B) \to \phi^*\phi_\dagger(A)$ exhibits $\phi^*(B)$ as a connective cover of $\phi^*\phi_\dagger(A)$. Since A is connective, the unit map $A \to \phi^*\phi_\dagger(A)$ factors through $\phi^*(B)$. It follows that the map u admits a section, so that $\phi_\dagger(A)$ is a retract of B and therefore connective. ☐

Proof of Proposition 3.3.2. Let $\phi : \mathfrak{Y} \to \mathfrak{Z}$ be strongly proper, flat, and locally almost of finite presentation, and let $\mathfrak{X} \to \mathfrak{Y}$ be affine. We wish to prove that the Weil restriction $\mathrm{Res}_{\mathfrak{Y}/\mathfrak{Z}}(\mathfrak{X})$ exists and is affine over $\mathfrak{Z}$. Both assertions can be tested locally on $\mathfrak{Z}$ with respect to the étale topology. We may therefore suppose that $\mathfrak{Z} = \mathrm{Spec}\, R$ is affine, so that $\mathfrak{Y}$ is a spectral algebraic space which is proper, flat, and locally almost of finite presentation over R. Since the map $\mathfrak{X} \to \mathfrak{Y}$ is affine, it is classified by an object $A \in \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}})$. Let $\phi_\dagger(A) \in \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Z})^{\mathrm{cn}})$ be as in Proposition 3.3.3. Then $\phi_\dagger(A)$ determines an affine spectral Deligne-Mumford stack $\mathfrak{W}$ equipped with a map $\rho : \mathfrak{W} \times_{\mathfrak{Z}} \mathfrak{Y} \to \mathfrak{X}$ in $\mathrm{Stk}_{/\mathfrak{Y}}$. It is easy to see that ρ exhibits $\mathfrak{W}$ as a Weil restriction of $\mathfrak{X}$ along ϕ. ☐

To prove Theorem 3.3.1 in general, it will be convenient to work in the more general context of functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, where the existence of Weil restrictions is more or less tautological. We will then analyze the deformation theory of functors given by Weil restriction, ultimately allowing us to deduce Theorem 3.3.1 from Theorem 3.2.1.

Notation 3.3.4. Fix a functor $Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will regard $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$ as a symmetric monoidal ∞-category with respect to the operation of Cartesian product. Given objects $X, Y \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$, we let $\underline{\mathrm{Map}}_{/Z}(X, Y)$ denote a classifying object for morphisms from X to Y (if such an object exists). More precisely, $\underline{\mathrm{Map}}_{/Z}(X, Y)$ denotes an object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$ equipped with an evaluation map $e : X \times_Z \underline{\mathrm{Map}}_{/Z}(X, Y) \to Y$ with the following universal property: for every object $W \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$, composition with e induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}}(W, \underline{\mathrm{Map}}_{/Z}(X, Y)) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}}(W \times_Z X, Y).$$

Proposition 3.3.5. *Suppose we are given morphisms $X \to Z \leftarrow Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Assume that the functors Y and Z are sheaves with respect to the étale topology, and that the map $X \to Z$ is representable (by spectral Deligne-Mumford stacks). Then a morphism object $\underline{\mathrm{Map}}_{/Z}(X, Y)$ exists. Moreover, $\underline{\mathrm{Map}}_{/Z}(X, Y)$ is also a sheaf with respect to the étale topology.*

Proof. Let us regard $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}$ as a full subcategory of the ∞-category $\mathcal{C} = \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/Z}$. The ∞-category $\mathcal{C}$ can be regarded as an ∞-topos in a larger universe, so that the Cartesian monoidal structure on $\mathcal{C}$ is closed. In particular, the morphism object $\underline{\mathrm{Map}}_{/Z}(X, Y)$ exists as an object of $\mathcal{C}$. To prove the first assertion, it will suffice to show that for every connective $\mathbb{E}_\infty$-ring R, the space $\underline{\mathrm{Map}}_{/Z}(X, Y)(R)$ is essentially small. Since $Z(R)$ is small, it will suffice to show that for every point $\eta \in Z(R)$, the homotopy fiber $\underline{\mathrm{Map}}_{/Z}(X, Y)(R) \times_{Z(R)} \{\eta\}$ is essentially small. The point η determines a map of functors $\mathrm{Spec}^{\mathrm{f}} R \to Z$, and we can identify $\underline{\mathrm{Map}}_{/Z}(X, Y)(R) \times_{Z(R)} \{\eta\}$ with the mapping space $\mathrm{Map}_{\mathcal{C}}(\mathrm{Spec}^{\mathrm{f}} R, \underline{\mathrm{Map}}_{/Z}(X, Y)) \simeq \mathrm{Map}_{\mathcal{C}}(\mathrm{Spec}^{\mathrm{f}} R \times_Z X, Y)$. Since the morphism $X \to Z$ is representable, the functor $\mathrm{Spec}^{\mathrm{f}} R \times_Z X$ is representable by a spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For every object $U \in \mathcal{X}$, let X_U denote the functor represented by the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U)$. The construction $U \mapsto \mathrm{Map}_{\mathcal{C}}(X_U, Y)$ determines a functor $F : \mathcal{X}^{op} \to \widehat{\mathcal{S}}$. To complete the proof, it will suffice to show that for each $U \in \mathcal{X}$, the space $F(U)$ is essentially small. Since Y is a sheaf with respect to the étale topology, the functor F preserves small limits. It will therefore suffice to show that $F(U)$ is essentially small when $U \in \mathcal{X}$ is affine. In this case, we can write $X_U = \mathrm{Spec}^{\mathrm{f}} R'$, and $F(U)$ can be identified with a homotopy fiber of the map $Y(R') \to Z(R')$.

It remains to show that $\underline{\mathrm{Map}}_{/Z}(X, Y)$ is a sheaf with respect to the étale topology. Let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by those maps $W \to Z$ where $W : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is an étale sheaf, and let $L : \mathcal{C} \to \mathcal{C}_0$ be a left adjoint to the inclusion. We wish to show that $\underline{\mathrm{Map}}_{/Z}(X, Y)$ is L-local. Let $\alpha : W \to W'$ be a morphism in $\mathcal{C}$ such that $L(\alpha)$ is an equivalence; we wish to show that composition with α induces a homotopy equivalence $\theta : \mathrm{Map}_{\mathcal{C}}(W', \underline{\mathrm{Map}}_{/Z}(X, Y)) \to \mathrm{Map}_{\mathcal{C}}(W, \underline{\mathrm{Map}}_{/Z}(X, Y))$. Using the universal property of $\underline{\mathrm{Map}}_{/Z}(X, Y)$, we can identify θ with the canonical map $\mathrm{Map}_{\mathcal{C}}(W' \times_Z X, Y) \to \mathrm{Map}_{\mathcal{C}}(W \times_Z X, Y)$. Since $Y \in \mathcal{C}_0$, we are reduced to proving that $L(\beta)$ is an equivalence, where $\beta : W \times_Z X \to W' \times_Z X$ is the map induced by α. This follows from our assumption that $L(\alpha)$ is an equivalence, since the sheafification functor L is left exact. □

Proposition 3.3.6. *Suppose we are given morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. Assume that the functors Y and Z are sheaves with respect to the étale topology, and that the map $f : X \to Z$ is representable (by spectral Deligne-Mumford stacks). Then:*

(1) *If the map g is cohesive, then the induced map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is cohesive.*

(2) *If the map g is infinitesimally cohesive, then the induced map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is infinitesimally cohesive.*

(3) *If the map g is nilcomplete, then the induced map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is nilcomplete.*

(4) *Assume that f is representable by spectral algebraic spaces which are quasi-compact and quasi-separated. If g is locally of finite presentation, then the induced map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is locally of finite presentation.*

(5) *Assume that f is representable by spectral algebraic spaces which are quasi-compact and quasi-separated. If f is flat and g is locally of finite presentation to order n (locally almost of finite presentation), then the induced map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is locally of finite presentation to order n (locally almost of finite presentation).*

(6) *Assume that f is representable by spectral algebraic spaces which are proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude over Z. If g admits a relative cotangent complex, then the induced map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ admits a relative cotangent complex.*

Proof. Using Propositions 1.3.22, 2.2.7, and Remark 2.3.5, we can reduce to the case where $Z = \mathrm{Spec}^{\mathrm{f}} R$ is a corepresentable functor, so that X is representable by a spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For each object $U \in \mathcal{X}$, let X_U denote the functor represented by the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U)$. For the first three assertions, it will suffice to show that if Y is cohesive (infinitesimally cohesive, nilcomplete) then $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ has the same property, for each object $U \in \mathcal{X}$ (Remark 2.2.3). Since Y is a sheaf with respect to the étale topology, the construction $U \mapsto \underline{\mathrm{Map}}_{/Z}(X_U, Y)$ carries colimits in $\mathcal{X}$ to limits in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. It will therefore suffice to show that $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ is cohesive (infinitesimally cohesive, nilcomplete) in the special case where $U \in \mathcal{X}$ is affine, so that $X_U \simeq \mathrm{Spec}^{\mathrm{f}} R'$ for some connective $\mathbb{E}_\infty$-ring R'. Let $F : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be the functor corresponding to Y under the equivalence of ∞-categories $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z} \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})$. Unwinding the definitions, we see that $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ corresponds to the functor $F_U : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ given by the formula $F_U(A) = F(R' \otimes_R A)$. We now consider each case in turn:

(1) To prove that $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ is cohesive, we must show that for every pullback diagram τ :

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

in $\mathrm{CAlg}^{\mathrm{cn}}_R$ for which the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, the induced diagram σ :

$$\begin{array}{ccc} F_U(A) & \longrightarrow & F_U(A_0) \\ \downarrow & & \downarrow \\ F_U(A_1) & \longrightarrow & F_U(A_{01}) \end{array}$$

is a pullback square in $\mathcal{S}$. We can identify σ with the diagram

$$\begin{array}{ccc} F(R' \otimes_R A) & \longrightarrow & F(R' \otimes_R A_0) \\ \downarrow & & \downarrow \\ F(R' \otimes_R A_1) & \longrightarrow & F(R' \otimes_R A_{01}). \end{array}$$

This is a pullback square by virtue of our assumption that Y is cohesive, since the diagram of $\mathbb{E}_\infty$-rings τ' :

$$\begin{array}{ccc} R' \otimes_R A & \longrightarrow & R' \otimes_R A_0 \\ \downarrow & & \downarrow \\ R' \otimes_R A_1 & \longrightarrow & R' \otimes_R A_{01} \end{array}$$

is also a pullback square which induces surjections

$$\pi_0(R' \otimes_R A_0) \to \pi_0(R' \otimes_R A_{01}) \leftarrow (R' \otimes_R A_1).$$

(2) The argument is identical to that given in case (1), noting that if the diagram τ induces surjections $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ with nilpotent kernels, then τ' has the same property.

(3) Assume that Y is nilcomplete; we wish to show that $\underline{\mathrm{Map}}_{/Z}(X_U, Y)$ is nilcomplete. For this, it suffices to show that for every connective R-algebra A, the canonical map

$$F_U(A) \simeq F(R' \otimes_R A) \to \varprojlim F(R' \otimes_R \tau_{\leq n} A) \simeq \varprojlim F_U(\tau_{\leq n} A)$$

is an equivalence. This follows from Proposition 2.1.8.

We now prove (4). Assume that Y is locally of finite presentation over R, and that $\mathfrak{X}$ is a quasi-compact, quasi-separated spectral algebraic space. Let us say that an object $U \in \mathcal{X}$ is *good* if the functor $F_U : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ commutes with filtered colimits. It is easy to see that the collection of good objects of $\mathcal{X}$ is closed under finite colimits; we wish to prove that the final object of $\mathcal{X}$ is good. Using Proposition VIII.2.5.8 and Theorem XII.1.3.8, we are reduced to proving that every affine object $U \in \mathcal{X}$ is good. In this case, we can write $X_U = \mathrm{Spec}^{\mathrm{f}} R'$ as above, so that F_U is given by the formula $F_U(A) = F(A \otimes_R R')$ and therefore commutes with filtered colimits as desired.

The proof of (5) is similar. It will suffice to show that if $\mathfrak{X}$ is a quasi-compact, quasi-separated spectral algebraic space which is flat over R and g is locally of finite presentation to order n over R, then the map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is locally of finite presentation to order n. Let us say that a functor $G : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is *good* if it satisfies condition (c) of Remark 2.3.5, and let us say that an object $U \in \mathcal{X}$ is *good* if the functor F_U is good. The collection of good functors is closed under finite limits, so the collection of good objects of $\mathcal{X}$ is closed under finite colimits. We wish to prove that the final object of $\mathcal{X}$ is good. Invoking Proposition VIII.2.5.8 and Theorem XII.1.3.8 again, we are reduced to proving that every affine object $U \in \mathcal{X}$ is good. In this case, we can write $X_U = \mathrm{Spec}^{\mathrm{f}} R'$, where R' is flat over R. Then F_U is given by the formula $F_U(A) = F(A \otimes_R R')$. If $\{A_\alpha\}$ is a filtered diagram of m-truncated R-algebras having colimit A, then $\{A_\alpha \otimes_R R'\}$ is a filtered diagram of m-truncated R'-algebras having colimit $\{A \otimes_R R'\}$. If the functor F is good, we deduce that the map

$$\varinjlim F_U(A_\alpha) \simeq \varinjlim F(A_\alpha \otimes_R R') \to F(A \otimes_R R') \simeq F_U(A)$$

is $(m - n - 1)$-truncated, so that F_U is also good.

We now prove (6) by verifying conditions (a) and (b) of Remark 1.3.14. We first verify (a). Fix a connective $\mathbb{E}_\infty$-ring R' and a point $\eta \in \underline{\mathrm{Map}}_{/Z}(X, Y)(R')$, and consider the functor $G : \mathrm{Mod}_{R'}^{\mathrm{cn}} \to \mathcal{S}$ given by the formula

$$G(M) = \mathrm{fib}(\underline{\mathrm{Map}}_{/Z}(X, Y)(R' \oplus M) \to \underline{\mathrm{Map}}_{/Z}(X, Y)(R') \times_{Z(R')} Z(R' \oplus M)).$$

Let $\mathfrak{X}'$ be the spectral Deligne-Mumford stack representing the functor $\mathrm{Spec}^{\mathrm{f}} R' \times_Z X$, let $p : \mathfrak{X}' \to \mathrm{Spec}\, R'$ be the projection map, and let $q : \mathrm{Spec}^{\mathrm{f}} R' \times_Z X \to Y$ be the map determined by η. Then p and q determine pullback functors

$$p^* : \mathrm{Mod}_{R'} \simeq \mathrm{QCoh}(\mathrm{Spec}\, R') \to \mathrm{QCoh}(\mathfrak{X}') \qquad q^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} R' \times_Z X) \simeq \mathrm{QCoh}(\mathfrak{X}').$$

Unwinding the definitions, we see that G is given by the formula $G(M) = \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X}')}(q^* L_{Y/Z}, p^* M)$. Since $\mathfrak{X}'$ is a proper algebraic space which is locally almost of finite presentation and locally almost of finite Tor-amplitude, the functor p^* admits a left adjoint $p_+ : \mathrm{QCoh}(\mathfrak{X}') \to \mathrm{Mod}_{R'}$ (Proposition XII.3.3.23). It follows that the functor G is corepresented by the object $p_+ q^* L_{Y/X}$ (which is almost connective by virtue of Remark XII.3.3.24). This completes the verification of condition (a) of Remark 1.3.14. Condition (b) follows from the second part of Proposition XII.3.3.23. □

Remark 3.3.7. In the situation of part (4) of Proposition 3.3.6, suppose that the relative cotangent complex $L_{Y/Z}$ is perfect (almost perfect). Then the relative cotangent complex $L_{\underline{\mathrm{Map}}_{/Z}(X,Y)/Z}$ is perfect (almost perfect). This follows by combining the proof of Proposition 3.3.6 with Remark XII.3.3.25.

In good cases, we can use Theorem 3.2.1 to verify the representability of a functor $\underline{\mathrm{Map}}_{/Z}(X, Y)$.

Proposition 3.3.8. *Let R be a Noetherian $\mathbb{E}_\infty$-ring such that $\pi_0 R$ is a Grothendieck ring and let $Z = \operatorname{Spec}^{\mathrm{f}} R$. Suppose we are given natural transformations $X \to Z \leftarrow Y$ for some pair of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, where X is representable by a spectral algebraic space $\mathfrak{X}$ which is proper, flat, and locally almost of finite presentation over R, and Y is representable by a geometric spectral Deligne-Mumford stack $\mathfrak{Y}$ which is locally almost of finite presentation over R. Then the functor $\underline{\mathrm{Map}}_{/Z}(X, Y)$ is representable by a spectral Deligne-Mumford stack which is locally almost of finite presentation over R.*

Remark 3.3.9. In the situation of Proposition 3.3.8, the spectral Deligne-Mumford stack representing $\underline{\mathrm{Map}}_{/Z}(X, Y)$ is geometric. This can be deduced from Theorem 3.3.1.

Proof. Set $F = \underline{\mathrm{Map}}_{/Z}(X, Y)$, and let $F_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be given by the formula $F_0(A) = \mathrm{fib}(F(A) \to Z(A))$. We will show that F is representable by a spectral Deligne-Mumford 1-stack by verifying conditions (1) through (5) of Theorem 3.2.1:

(1) If A is a discrete commutative ring, then the space $F(A)$ is 1-truncated. To prove this, it will suffice to show that the fibers of the map $F(A) \to Z(A)$ are 1-truncated (since $Z(A) \simeq \mathrm{Map}_{\mathrm{CAlg}}(R, A)$) is discrete. That is, we must show that if A is a discrete $\mathbb{E}_\infty$-algebra over R, then $F_0(A)$ is 1-truncated. Unwinding the definitions, we have

$$F_0(A) = \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}}(\operatorname{Spec}^{\mathrm{f}} A \times_Z X, Y).$$

Since $\mathfrak{X}$ is flat over R, $\operatorname{Spec}^{\mathrm{f}} A \times_Z X$ is representable by a spectral algebraic space $\mathfrak{X} \times_{\operatorname{Spec}^{\mathrm{f}} R} \operatorname{Spec}^{\mathrm{f}} A$ which is flat over A. The desired result now follows from Lemma VIII.1.3.6 (by assumption $\mathfrak{Y}$ is geometric, and therefore a spectral Deligne-Mumford 1-stack).

(2) The functor F is a sheaf for the étale topology. This follows from Proposition 3.3.5.

(3) It follows from Proposition 3.3.6 that the forgetful functor $F \to Z$ is nilcomplete and infinitesimally cohesive (in fact, it is even cohesive). We claim that it is integrable. To prove this, suppose that A is a local Noetherian $\mathbb{E}_\infty$-algebra which is complete with respect its maximal ideal. We wish to show that the diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\operatorname{Spf} A, F) \\ \downarrow & & \downarrow \\ Z(A) & \longrightarrow & \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(\operatorname{Spf} A, Z) \end{array}$$

is a pullback square. Unwinding the definitions, we must show that for every map of $\mathbb{E}_\infty$-algebra $R \to A$, the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}}(\operatorname{Spec}^{\mathrm{f}} A \times_{\operatorname{Spec}^{\mathrm{f}} R} X, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/Z}}(\operatorname{Spf} A \times_{\operatorname{Spec}^{\mathrm{f}} R} X, Y)$$

is a homotopy equivalence. This follows immediately from Theorem XII.5.4.1.

(4) It follows from Proposition 3.3.6 that the natural transformation $F \to Z$ admits a relative cotangent complex $L_{F/Z}$. We must show that $L_{F/Z}$ is connective. To prove this, suppose we are given a point $\eta \in F(A)$, corresponding to an $\mathbb{E}_\infty$-ring morphism $R \to A$ and a map $u : \operatorname{Spec} A \times_{\operatorname{Spec} R} \mathfrak{X} \to \mathfrak{Y}$. Let $p : \operatorname{Spec} A \times_{\operatorname{Spec} R} \mathfrak{X} \to \operatorname{Spec} A$ be the projection map. The proof of Proposition 3.3.6 shows that we can identify $\eta^* L_{F/Z}$ with the A-module given by $p_+ u^* L_{\mathfrak{Y} / \operatorname{Spec} R}$. Since $u^* L_{\mathfrak{Y} / \operatorname{Spec} R}$ is connective, it suffices to show that the functor p_+ is right t-exact. This is clear, since p_+ is defined as the left adjoint to the pullback functor p^* (which is left t-exact by virtue of our assumption that $\mathfrak{X}$ is flat over R).

(5) The map $F \to Z$ is locally almost of finite presentation. This follows immediately from Proposition 3.3.6, since the map $Y \to Z$ is locally almost of finite presentation.

□

We next show that Proposition 3.3.8 is valid without Noetherian hypotheses:

Proposition 3.3.10. *Suppose we are given functors $X, Y, Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and natural transformations $f : X \to Z$, $g : Y \to Z$. Assume that f is representable by spectral algebraic spaces which are proper, flat, and locally almost of finite presentation, and that g is representable by geometric Deligne-Mumford stacks which are locally almost of finite presentation. Then the map $\underline{\mathrm{Map}}_{/Z}(X, Y) \to Z$ is representable by spectral Deligne-Mumford stacks.*

Proof. We may assume without loss of generality that $Z = \mathrm{Spec}^{\mathrm{f}} R$ for some connective $\mathbb{E}_\infty$-ring R, so that X is representable by a spectral algebraic space $\mathfrak{X}$ and Y is representable by a geometric spectral Deligne-Mumford stack $\mathfrak{Y}$. It follows from Proposition 3.3.6 that the functor $F = \underline{\mathrm{Map}}_{/Z}(X, Y)$ is nilcomplete, infinitesimally cohesive, and admits a cotangent complex. According to Theorem 3.1.2, to prove that F is representable by a spectral Deligne-Mumford stack, it will suffice to show that the restriction $F|\,\mathrm{CAlg}^0$ is representable by a spectral Deligne-Mumford stack, where CAlg^0 denotes the full subcategory of CAlg spanned by the discrete $\mathbb{E}_\infty$-rings. We may therefore replace R by $\pi_0 R$, and thereby reduce to the case where R is discrete. Write R as the union of finitely generated subrings R_α. Using Theorem XII.2.3.2, we can choose an index α and spectral Deligne-Mumford stacks $\mathfrak{X}_\alpha$ and $\mathfrak{Y}_\alpha$ which are finitely 0-presented over R_α, together with equivalences

$$\mathfrak{X} \simeq \tau_{\leq 0}(\mathrm{Spec}\, R \times_{\mathrm{Spec}\, R_\alpha} \mathfrak{X}_\alpha) \qquad \tau_{\leq 0}\, \mathfrak{Y} \simeq \tau_{\leq 0}(\mathrm{Spec}\, R \times_{\mathrm{Spec}\, R_\alpha} \mathfrak{Y}_\alpha).$$

Enlarging α if necessary, we can ensure that $\mathfrak{X}_\alpha$ is a spectral algebraic space which is proper and flat over R_α (Proposition XII.3.1.10 and Corollary 11.2.6.1 of [12]) and that $\mathfrak{Y}_\alpha$ is geometric (Proposition XII.2.5.1). Then $\mathfrak{X} \simeq \mathrm{Spec}\, R \times_{\mathrm{Spec}\, R_\alpha} \mathfrak{X}_\alpha$. Set $\mathfrak{Y}' = \mathrm{Spec}\, R \times_{\mathrm{Spec}\, R_\alpha} \mathfrak{X}_\alpha$, let Y' be the functor represented by $\mathfrak{Y}'$, and let $F' = \underline{\mathrm{Map}}_{/Z}(X, Y')$. Then $F|\,\mathrm{CAlg}^0 \simeq F'|\,\mathrm{CAlg}^0$. Consequently, we are free to replace Y by Y' and reduce the case where $\mathfrak{Y} = \mathrm{Spec}\, R \times_{\mathrm{Spec}\, R_\alpha} \mathfrak{Y}_\alpha$. We may therefore replace R by R_α, thereby reducing to the case where R is finitely generated as a commutative ring and therefore a Grothendieck ring (Theorem 0.0.5). In this case, the desired result follows from Proposition 3.3.8. □

Notation 3.3.11. Let $\mathcal{S}\mathrm{hv}_{\text{ét}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ spanned by those functors which are sheaves with respect to the étale topology. Suppose that $f : Y \to Z$ is a morphism in $\mathcal{S}\mathrm{hv}_{\text{ét}}$ which is representable by spectral Deligne-Mumford stacks. Then f determines a pullback functor $(\mathcal{S}\mathrm{hv}_{\text{ét}})_{/Z} \to (\mathcal{S}\mathrm{hv}_{\text{ét}})_{/Y}$, given by $X \mapsto X \times_Z Y$. Using Proposition 3.3.5, we deduce that this pullback functor admits a right adjoint $\mathrm{Res}_{Y/Z} : (\mathcal{S}\mathrm{hv}_{\text{ét}})_{/Y} \to (\mathcal{S}\mathrm{hv}_{\text{ét}})_{/Z}$, given by the formula $\mathrm{Res}_{Y/Z}(X) = \underline{\mathrm{Map}}_{/Z}(Y, X) \times_{\underline{\mathrm{Map}}_{/Z}(Y,Y)} Z$. We will refer to $\mathrm{Res}_{Y/Z}$ as the functor of *Weil restriction* along the map $Y \to Z$.

We are now in a position to prove our main result.

Proof of Theorem 3.3.1. Let $\phi : \mathfrak{Y} \to \mathfrak{Z}$ be a morphism of spectral Deligne-Mumford stacks which is strongly proper, flat, and locally almost of finite presentation, and let $\psi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks which is geometric and locally almost of finite presentation. Then ϕ and ψ determine natural transformations $X \to Y \to Z$ between functors $X, Y, Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. It follows from Proposition 3.3.10 that $\underline{\mathrm{Map}}_{/Z}(Y, X)$ and $\underline{\mathrm{Map}}_{/Z}(Y, Y)$ are representable by spectral Deligne-Mumford stacks which are locally almost of finite presentation over $\mathfrak{Z}$.

$$\mathrm{Res}_{Y/Z}(X) = \underline{\mathrm{Map}}_{/Z}(Y, X) \times_{\underline{\mathrm{Map}}_{/Z}(Y,Y)} Z$$

is also represented by a spectral Deligne-Mumford stack $\mathrm{Res}_{\mathfrak{Y}\,/\,\mathfrak{Z}}(\mathfrak{X})$ which is locally almost of finite presentation over $\mathfrak{Z}$. We now complete the proof by showing that the morphism $\mathrm{Res}_{\mathfrak{Y}\,/\,\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z}$ is geometric. Equivalently, we must show that the diagonal map

$$\theta : \mathrm{Res}_{Y/Z}(X) \to \mathrm{Res}_{Y/Z}(X) \times_Z \mathrm{Res}_{Y/Z}(X) \simeq \mathrm{Res}_{Y/Z}(X \times_Y X)$$

is representable and affine. To prove this, we set

$$Z' = \mathrm{Res}_{Y/Z}(X \times_Y X) \qquad Y' = Y \times_Z Z' \qquad X' = Y' \times_{X \times_Y X} X.$$

Since $X \to Y$ is geometric, the projection map $X' \to Y'$ is affine. Unwinding the definitions, we can identify θ with the canonical map $\mathrm{Res}_{Y'/Z'}(X') \to Z'$, which is affine by Proposition 3.3.2. □

We conclude this section with a few general remarks about the deformation theory of Weil restrictions.

Proposition 3.3.12. *Let $f : Y \to Z$ be a morphism in $\mathcal{S}\mathrm{hv}_{\acute{e}t}$ which is representable by spectral Deligne-Mumford stacks, and let $p : X \to Y$ be an arbitrary morphism in $\mathcal{S}\mathrm{hv}_{\acute{e}t}$. If p is cohesive (infinitesimally cohesive, nilcomplete), then the induced map $q : \mathrm{Res}_{Y/Z}(X) \to Z$ is cohesive (infinitesimally cohesive, nilcomplete).*

Proof. We will show that if p is cohesive, then q is cohesive; the proof in the other two cases is the same. We have a pullback diagram

$$\begin{array}{ccc} \mathrm{Res}(X) & \longrightarrow & \underline{\mathrm{Map}}_{/Z}(Y, X) \\ \downarrow q & & \downarrow q' \\ Z & \longrightarrow & \underline{\mathrm{Map}}_{/Z}(Y, Y). \end{array}$$

We may therefore reduce to proving that q' is cohesive. For this, it suffices to show that both of the projection maps $\underline{\mathrm{Map}}_{/Z}(Y, Y) \to Z \leftarrow \underline{\mathrm{Map}}_{/Z}(Y, X)$ are cohesive (Remark 2.2.3). Using Proposition 3.3.6, we are reduced to showing that f and $f \circ p$ are cohesive. In the case of f, this follows from Proposition 2.1.7. Because f and p are both cohesive, Remark 2.2.3 guarantees that $f \circ p$ is cohesive. □

Proposition 3.3.13. *Let $f : Y \to Z$ be a morphism in $\mathcal{S}\mathrm{hv}_{\acute{e}t}$ which is representable by spectral algebraic spaces, proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude. Let $p : X \to Y$ be an arbitrary morphism in $\mathcal{S}\mathrm{hv}_{\acute{e}t}$. If $f \circ p$ admits a cotangent complex, then the induced map $q : \mathrm{Res}(X) \to Z$ admits a cotangent complex. Moreover, if $L_{X/Z}$ is almost perfect, then $L_{\mathrm{Res}_{Y/Z}(X)/Z}$ is almost perfect.*

Proof. We have a pullback diagram

$$\begin{array}{ccc} \mathrm{Res}(X) & \longrightarrow & \underline{\mathrm{Map}}_{/Z}(Y, X) \\ \downarrow q & & \downarrow q' \\ Z & \longrightarrow & \underline{\mathrm{Map}}_{/Z}(Y, Y). \end{array}$$

It will therefore suffice to show that q' admits a cotangent complex, which is almost perfect if $L_{X/Z}$ is almost perfect. Using Proposition 1.3.18, we are reduced to proving that the maps $\underline{\mathrm{Map}}_{/Z}(Y, Y) \to Z \leftarrow \underline{\mathrm{Map}}_{/Z}(Y, X)$ admit cotangent complexes (which are almost perfect if $L_{X/Z}$ is almost perfect). Using Proposition 3.3.6 and Remark 3.3.7, we are reduced to proving that the maps f and $f \circ p$ admit cotangent complexes (which are almost perfect if $L_{X/Z}$ are almost perfect). This follows immediately from Proposition 1.3.18, since the relative cotangent complex $L_{Y/Z}$ is almost perfect by virtue of our assumption that Y is locally almost of finite presentation over Z. □

Remark 3.3.14. Let $f : Y \to Z$ and $p : X \to Y$ be as in Proposition 3.3.13. Fix a connective $\mathbb{E}_\infty$-ring R and a point $\eta \in Z(R)$, and let X_η and Y_η denote the fiber products $\mathrm{Spec}^{\mathrm{f}} R \times_Z X$ and $\mathrm{Spec}^{\mathrm{f}} R \times_Z Y$. Let $\overline{\eta} \in \mathrm{Res}(X)(R)$ be a point lifting η, corresponding to a section s of the canonical map $p_\eta : X_\eta \to Y_\eta$. Combining the proofs of Propositions 3.3.13, 3.3.6, and 1.3.18, we deduce that there is a canonical equivalence of R-modules $\overline{\eta}^* L_{\mathrm{Res}_{Y/Z}(X)/Z} \simeq f'_+ s^* L_{X_\eta / Y_\eta}$, where f'_+ denotes a left adjoint to the pullback functor $\mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} R) \to \mathrm{QCoh}(Y_\eta)$.

3.4 Example: The Picard Functor

Let X be a projective algebraic variety over a field k. The *Picard group* of X is defined to be the group of isomorphism classes of line bundles on X. In good cases, one can show that the Picard group of X itself has the structure of an algebraic variety. More precisely, there exists a group scheme E over k whose group of k-rational points is canonically isomorphic to the Picard group of X. In this section, we will apply Theorem 3.2.1 to prove an analogous result in the setting of spectral algebraic geometry. First, we need to introduce a bit of terminology.

Definition 3.4.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that an object $\mathcal{F} \in \mathrm{QCoh}(X)$ is a *line bundle* on X if is an invertible object of the symmetric monoidal ∞-category $\mathrm{QCoh}(X)^{\mathrm{cn}}$. We let $\mathrm{Pic}(X)$ denote the subcategory of $\mathrm{QCoh}(X)$ whose objects are line bundles on X, and whose morphisms are equivalences.

Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack representing a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that an object $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is a *line bundle* on $\mathfrak{X}$ if its image under the equivalence $\mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{QCoh}(X)$ is a line bundle on X. We let $\mathrm{Pic}(\mathfrak{X})$ denote the subcategory of $\mathrm{QCoh}(\mathfrak{X})$ whose objects are line bundles on X and whose morphisms are equivalences of line bundles on X.

Remark 3.4.2. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. Then $\mathrm{Pic}(\mathfrak{X}) \subseteq \mathrm{QCoh}(\mathfrak{X})$ is evidently a Kan complex. Moreover, it is essentially small. To prove this, choose a regular cardinal κ for which the global sections functor $\mathcal{F} \mapsto \Gamma(\mathfrak{X}; \mathcal{F})$ commutes with κ-filtered colimits. If $\mathcal{L}$ is an invertible object of $\mathrm{QCoh}(\mathfrak{X})$, then the canonical equivalence

$$\mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{L}, \mathcal{F}) \simeq \Gamma(\mathfrak{X}; \mathcal{L}^{-1} \otimes \mathcal{F})$$

shows that $\mathcal{L}$ is a κ-compact object of $\mathrm{QCoh}(\mathfrak{X})$.

Construction 3.4.3. Let R be a connective $\mathbb{E}_\infty$-ring and let $f : \mathfrak{X} \to \operatorname{Spec} R$ be a map of spectral Deligne-Mumford stacks. We define a functor $\mathrm{Pic}_{\mathfrak{X}/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ by the formula $\mathrm{Pic}_{\mathfrak{X}/R}(R') = \mathrm{Pic}(\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathfrak{X})$.

In general, we cannot expect the functor $\mathrm{Pic}_{\mathfrak{X}/R}$ to be representable by a spectral Deligne-Mumford stack over R, because it does not have unramified diagonal: line bundles on $\mathfrak{X}$ generally admit continuous families of automorphisms. To address this issue, we introduce a rigidification of the functor $\mathrm{Pic}_{\mathfrak{X}/R}$:

Definition 3.4.4. Let $f : \mathfrak{X} \to \operatorname{Spec} R$ be a map of spectral Deligne-Mumford stacks, and suppose that f admits a section $x : \operatorname{Spec} R \to \mathfrak{X}$. Then pullback along x determines a natural transformation of functors

$$\mathrm{Pic}_{\mathfrak{X}/R} \to (\mathrm{Pic} \,|\, \mathrm{CAlg}^{\mathrm{cn}}_R).$$

We will denote the fiber of this map by $\mathrm{Pic}^x_{\mathfrak{X}/R} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$.

Remark 3.4.5. In the situation of Definition 3.4.4, the fiber sequence

$$\mathrm{Pic}^x_{\mathfrak{X}/R} \to \mathrm{Pic}_{\mathfrak{X}/R} \to \mathrm{Pic} \,|\, \mathrm{CAlg}^{\mathrm{cn}}_R$$

splits canonically, with the splitting given by the pullback functor $\mathcal{L} \mapsto f^* \mathcal{L}$. It follows that we have an equivalence of functors

$$\mathrm{Pic}^x_{\mathfrak{X}/R} \times (\mathrm{Pic} \,|\, \mathrm{CAlg}^{\mathrm{cn}}_R) \simeq \mathrm{Pic}_{\mathfrak{X}/R},$$

given informally the formula $(\mathcal{L}, \mathcal{L}') \mapsto \mathcal{L} \otimes f^* \mathcal{L}'$.

More informally, $\mathrm{Pic}^x_{\mathfrak{X}/R}$ is the functor which assigns to every connective R-algebra R' a classifying space for pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a line bundle on $\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathfrak{X}$ and α is an equivalence of R'-modules $R' \to x'^* \mathcal{L}$, where $x' : \operatorname{Spec} R' \to \operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathfrak{X}$ is the map determined by x.

We are now ready to state our main result.

Theorem 3.4.6. *Let $f : \mathfrak{X} \to \operatorname{Spec} R$ be a map of spectral algebraic spaces which is flat, proper, and locally almost of finite presentation. Let $u : R \to f_* \mathcal{O}_{\mathfrak{X}}$ be the evident map, and suppose that* $\operatorname{cofib}(u)$ *is an R-module of* Tor*-amplitude ≤ -1. Let $x : \operatorname{Spec} R \to \mathfrak{X}$ be a section of f. Then the functor $\operatorname{Pic}^x_{\mathfrak{X}/R}$ is representable by a spectral algebraic space which is quasi-separated and locally of finite presentation over R.*

The analogue of Theorem 3.4.6 in classical algebraic geometry was proven by Artin as an application of his representability criterion. It is possible to deduce Theorem 3.4.6 from its classical analogue, using Theorem 3.1.2. We will give a slightly different argument at the end of this section, which appeals instead to Theorem 3.2.1. The main point is to show that the functor $\operatorname{Pic}^x_{\mathfrak{X}/R}$ has a well-behaved deformation theory.

We begin with a general analysis of the deformation theory of quasi-coherent sheaves.

Proposition 3.4.7. *Let $F : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the functor given by $R \mapsto \mathrm{Mod}_R^{\mathrm{acn}}$, where $\mathrm{Mod}_R^{\mathrm{acn}}$ denotes the full subcategory of Mod_R spanned by those objects which are almost connective (that is, n-connective for some integer n). Then the functor F is cohesive and nilcomplete.*

Proof. The assertion that F is cohesive follows from Theorem IX.7.2. To prove that F is nilcomplete, we let R denote an arbitrary $\mathbb{E}_\infty$-ring; we will show that the canonical functor $F : \mathrm{Mod}_R^{\mathrm{acn}} \to \varprojlim_n \mathrm{Mod}^{\mathrm{acn}}_{\tau_{\leq n}}$ is an equivalence of ∞-categories. We can identify an object of $\varprojlim_n \mathrm{Mod}^{\mathrm{acn}}_{\tau_{\leq n} R}$ with a sequences of objects $\{M_n \in \mathrm{Mod}^{\mathrm{acn}}_{\tau_{\leq n} R}\}$ together with equivalences $\alpha_n : M_n \simeq (\tau_{\leq n} R) \otimes_{\tau_{\leq n+1} R} M_{n+1}$. Using the assumption that M_{n+1} is almost connective, we deduce that M_n is nonzero if and only if M_{n+1} is nonzero. Suppose that M_0 is nonzero, so that M_n is nonzero for every integer n. Then there is some smallest integer $k(n)$ for which $\pi_{k(n)} M_n \neq 0$. Using the isomorphisms α_n, we deduce that all of the integers $k(n)$ are the same; let us denote this common value by k. For each integer n, the fiber of the canonical map $M_{n+1} \to M_n$ is given by the tensor product

$$\operatorname{fib}(\tau_{\leq n+1} R \to \tau_{\leq n} R) \otimes_{\tau_{\leq n+1} R} M_{n+1},$$

and is therefore $(k+n+1)$-connective. It follows that each of the towers of abelian groups $\{\pi_j M_n\}_{n \geq 0}$ are constant for $n \geq j-k$. Let $M = \varprojlim M_n$, so that the Milnor exact sequences

$$0 \to \lim^1 \pi_{j+1} M_n \to \pi_j M \to \lim^0 \pi_j M_n \to 0$$

specialize to give isomorphisms $\pi_j M \to \pi_j M_n$ for $n \geq j-k$. In particular, we deduce that M is k-connective. We conclude that the functor F admits a right adjoint $G : \varprojlim_n \mathrm{Mod}^{\mathrm{acn}}_{\tau_{\leq n} R} \to \mathrm{Mod}_R^{\mathrm{acn}}$, given by $\{M_n\}_{n \geq 0} \mapsto \varprojlim M_n$.

We next show that the unit map $\mathrm{id} \to G \circ F$ is an equivalence of functors from $\mathrm{Mod}_R^{\mathrm{acn}}$ to itself. Let M be a k-connective R-module; we wish to show that the canonical map $M \to \varprojlim (\tau_{\leq n} R) \otimes_R M$ is an equivalence. Fix an integer j, and consider the composition

$$\pi_j M \xrightarrow{\phi} \pi_j \varprojlim (\tau_{\leq n} R) \otimes_R M \xrightarrow{\psi} \pi_j (\tau_{\leq n} R) \otimes_R M.$$

Using the analysis above, we see that ψ is an isomorphism for $n \geq j-k$. It will therefore suffice to show that $\psi \circ \phi$ is an isomorphism for $n \geq j-k$. This follows from the existence of an exact sequence of abelian groups

$$\pi_j(\tau_{\geq n+1} R \otimes_R M) \to \pi_j M \to \pi_j(\tau_{\leq n} R \otimes_R M) \to \pi_{j-1}(\tau_{\geq n+1} R \otimes_R M),$$

since the k-connectivity of M implies that the abelian groups $\pi_j(\tau_{\geq n+1} R \otimes_R M)$ and $\pi_{j-1}(\tau_{\geq n+1} R \otimes_R M)$ are trivial.

To complete the proof, it will suffice to show that the functor G is conservative Since G is an exact functor between stable ∞-categories, we are reduced to proving that if $\{M_n\}_{n \geq 0}$ is an object of $\varprojlim_n \mathrm{Mod}^{\mathrm{acn}}_{\tau_{\leq n} R}$ such that $\varprojlim M_n \simeq 0$, then each M_n vanishes. Assume otherwise, and let k be defined as above. Then $\pi_k \varprojlim M_n \simeq \pi_k M_0 \neq 0$, and we obtain a contradiction. □

Corollary 3.4.8. *Let $F' : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ denote the functor given by $R \mapsto \mathrm{Mod}_R^{\mathrm{perf}}$. Then the functor F' is cohesive, nilcomplete, and commutes with filtered colimits.*

Proof. Let $F : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Proposition 3.4.7, so there is an evident natural transformation $\alpha : F' \to F$. The natural transformation α is cohesive (Proposition IX.7.7) and nilcomplete (Lemma 2.4.11). Since F is cohesive and nilcomplete, we conclude that F' is cohesive and nilcomplete.

It remains to show that F' commutes with filtered colimits. Choose a diagram of connective $\mathbb{E}_\infty$-rings $\{R_\alpha\}$ indexed by a filtered partially ordered set, having colimit R. We wish to prove that the canonical map

$$\theta : \varinjlim \mathrm{Mod}_{R_\alpha}^{\mathrm{perf}} \to \mathrm{Mod}_R$$

is an equivalence of ∞-categories. We first prove that θ is fully faithful. Choose any pair of objects of $\varinjlim \mathrm{Mod}_{R_\alpha}^{\mathrm{perf}}$, which we can represent using a pair of perfect R_α-modules M and N for some index α. Unwinding the definitions, we are reduced to proving that the canonical map

$$\phi : \varinjlim_{\beta \geq \alpha} \mathrm{Map}_{\mathrm{Mod}_{R_\beta}}(R_\beta \otimes_{R_\alpha} M, R_\beta \otimes_{R_\alpha} N) \to \mathrm{Map}_{\mathrm{Mod}_R}(R \otimes_{R_\alpha} M, R \otimes_{R_\alpha} N)$$

is an equivalence. We can identify ϕ with the canonical map

$$\varinjlim_{\beta \geq \alpha} \mathrm{Map}_{\mathrm{Mod}_{R_\alpha}}(M \otimes_{R_\alpha} N^\vee, R_\beta) \to \mathrm{Map}_{\mathrm{Mod}_{R_\alpha}}(M \otimes_{R_\alpha} N^\vee, R).$$

This map is a homotopy equivalence, since $M \otimes_{R_\alpha} N^\vee$ is a compact object of Mod_{R_α}.

It remains to prove that the functor θ is essentially surjective. Since θ is exact and fully faithful, its essential image is a stable subcategory $\mathcal{C} \subseteq \mathrm{Mod}_R^{\mathrm{perf}}$. Let M be a perfect R-module; we wish to show that $M \in \mathcal{C}$. Replacing M by a shift, we may assume without loss of generality that M is connective. We proceed by induction on the Tor-amplitude of M. If M has Tor-amplitude $n > 0$, then we can choose a fiber sequence

$$M' \xrightarrow{u} R^a \to M$$

where M' is a connective perfect R-module of Tor-amplitude $< n$. The inductive hypothesis then implies that $M' \in \mathcal{C}$, so that $M = \mathrm{cofib}(u) \in \mathcal{C}$. We are therefore reduced to the case where $n = 0$: that is, where M is a projective R-module of finite rank. Then $\pi_0 M$ is a summand of $(\pi_0 R)^m$ for some integer m, and can therefore be described as the image of an idempotent m-by-m matrix T over the commutative ring $\pi_0 R$. Since $\pi_0 R \simeq \varprojlim \pi_0 R_\alpha$, we may assume that T is the image of an m-by-m matrix T_α over $\pi_0 R_\alpha$ for some index α. Enlarging α if necessary, we may assume that T_α is idempotent, and therefore determines a projective module Q_0 of finite rank over $\pi_0 R_\alpha$. Using Corollary A.8.2.2.19, we can lift Q_0 to a projective module Q over R_α. Then $R \otimes_{R_\alpha} Q$ is a projective R-module Q' with $\pi_0 R \otimes_R Q' \simeq \pi_0 R \otimes_R M$. Using Corollary A.8.2.2.19 again, we deduce that $M \simeq Q'$ belongs to the essential image of θ. □

Notation 3.4.9. For every simplicial set K, we let $\mathrm{Perf}_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by the formula $\mathrm{Perf}_K(R) = \mathrm{Fun}(K, \mathrm{Mod}_R^{\mathrm{perf}})^{\simeq}$. If $K = \Delta^0$, we will denote the functor Perf_K simply by Perf.

Proposition 3.4.10. *The functor* $\mathrm{Perf} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *is cohesive, nilcomplete, locally of finite presentation, and admits a perfect cotangent complex.*

Lemma 3.4.11. *Let R be a connective $\mathbb{E}_\infty$-ring, let M and N be R-modules, and define a functor $F : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ by the formula $F(R') = \mathrm{Map}_{\mathrm{Mod}_{R'}}(R' \otimes_R M, R' \otimes_R N) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M, R' \otimes_R N)$. Let $\overline{F}$ denote the image of F under the equivalence of ∞-categories* $\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec}^{\mathrm{f}} R}$. *If M is almost connective and N is perfect, then the map $\alpha : \overline{F} \to \operatorname{Spec}^{\mathrm{f}} R$ admits a cotangent complex. Moreover, there is a canonical equivalaence*

$$L_{\overline{F}/\operatorname{Spec}^{\mathrm{f}} R} \simeq \alpha^*(M \otimes N^\vee),$$

where $N^\vee$ denotes the dual of M in the symmetric monoidal ∞-category $\mathrm{Mod}_R \simeq \mathrm{QCoh}(\operatorname{Spec}^{\mathrm{f}} R)$.

Proof. For every connective R-algebra R' and every connective R'-module Q, the fiber of the canonical map $F(R' \oplus Q) \to F(R)$ is given by

$$\mathrm{Map}_{\mathrm{Mod}_R}(M, Q \otimes_R N) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M \otimes N^\vee, Q).$$

□

Proof of Proposition 3.4.10. The first three assertions follow from Corollary 3.4.8. To prove that Perf admits a cotangent complex we will verify the conditions of Example 1.3.15:

(*a*) Let R be a connective $\mathbb{E}_\infty$-ring and let $\eta \in \mathrm{Perf}(R)$, corresponding to a perfect R-module N. Let $F : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ be the functor defined by the formula $F(M) = \mathrm{Perf}(R \oplus M) \times_{\mathrm{Perf}(R)} \{\eta\}$. We wish to show that F is almost corepresentable. Since the functor Perf is cohesive (Corollary 3.4.8), the pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R \oplus M & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \oplus \Sigma M \end{array}$$

gives rise to a pullback diagram of spaces

$$\begin{array}{ccc} \mathrm{Perf}(R \oplus M) & \longrightarrow & \mathrm{Perf}(R) \\ \downarrow & & \downarrow \\ \mathrm{Perf}(R) & \longrightarrow & \mathrm{Perf}(R \oplus \Sigma M). \end{array}$$

For every R-algebra R', let $G(R')$ denote the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_{R'}}(R' \otimes_R N, R' \otimes_R N).$$

We can identify $F(M)$ with the summand of the fiber $\mathrm{fib}(G(R \oplus \Sigma M) \to G(R))$ consisting of the equivalences of $(R \oplus \Sigma M) \otimes_R N$ with itself. Note that if α is an endomorphism of $(R \oplus \Sigma M) \otimes_R N$ which belongs to $\mathrm{fib}(G(R \oplus \Sigma M) \to G(R))$, then $\mathrm{cofib}(\alpha)$ is a perfect module over $R \oplus \Sigma M$ whose image in Mod_R is trivial. It follows that $\mathrm{cofib}(\alpha) \simeq 0$ and therefore that α is an equivalence, so that the inclusion $F(M) \hookrightarrow \mathrm{fib}(G(R \oplus \Sigma M) \to G(R))$ is a homotopy equivalence. Using Lemma 3.4.11, we obtain a canonical homotopy equivalence

$$F(M) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(N \otimes_R N^\vee, \Sigma M),$$

so that F is almost corepresentable by the R-module $\Sigma^{-1}(N \otimes_R N^\vee)$.

(*b*) For every map of connective $\mathbb{E}_\infty$-rings $R \to R'$ and every connective R'-module M, we must show that the diagram of spaces

$$\begin{array}{ccc} \mathrm{Perf}(R \oplus M) & \xrightarrow{\theta} & \mathrm{Perf}(R' \oplus M) \\ \downarrow & & \downarrow \\ \mathrm{Perf}(R) & \longrightarrow & \mathrm{Perf}(R') \end{array}$$

is a pullback square. Choose a point $\eta \in \mathrm{Perf}(R)$ corresponding to a perfect R-module N, and let $\eta' \in \mathrm{Perf}(R')$ be its image (corresponding to the perfect R'-module $R' \otimes_R N$). We will prove that θ induces a homotopy equivalence after passing to the homotopy fibers over the points η and η', respectively. Using the proof of (*a*), we are reduced to showing that the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_R}(N \otimes_R N^\vee, \Sigma M) \to \mathrm{Map}_{\mathrm{Mod}_{R'}}(N' \otimes_{R'} {N'}^\vee, \Sigma M)$$

is a homotopy equivalence, which is clear.

□

Remark 3.4.12. The proof of Proposition 3.4.10 supplies an explicit description of the cotangent complex of Perf: given a point $\eta \in \mathrm{Perf}(R)$ corresponding to a perfect R-module N, we have $\eta^* L_{\mathrm{Perf}} \simeq \Sigma^{-1}(N \otimes_R N^\vee)$.

Proposition 3.4.13. *Let R be a connective $\mathbb{E}_\infty$-ring and $f : \mathfrak{X} \to \operatorname{Spec} R$ be a map of spectral Deligne-Mumford stacks. Define a functor $\mathrm{Perf}_{\mathfrak{X}/R} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ by the formula*

$$\mathrm{Perf}_{\mathfrak{X}/R}(R') = \mathrm{QCoh}(\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathfrak{X})^{\mathrm{perf},\simeq}.$$

Let F denote the image of $\mathrm{Perf}_{\mathfrak{X}/R}$ under the equivalence of ∞-categories

$$\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\operatorname{Spec}^{\mathrm{f}} R}.$$

Then:

(1) *The functor F is nilcomplete and cohesive.*

(2) *Assume that $\mathfrak{X}$ is a quasi-compact, quasi-separated spectral algebraic space. Then the natural transformation $F \to \operatorname{Spec}^{\mathrm{f}} R$ is locally of finite presentation.*

(3) *Assume that $\mathfrak{X}$ is a spectral algebraic space which is proper and locally almost of finite presentation over R. Then the functor F is integrable.*

(4) *Assume that $\mathfrak{X}$ is a spectral algebraic space which is proper, locally almost of finite presentation, and locally of finite* Tor*-amplitude over R. Then the natural transformation $u : F \to \operatorname{Spec}^{\mathrm{f}} R$ admits a perfect cotangent complex.*

Proof. Assertions (1), (2) and (4) follow from Proposition 3.3.6, Remark 3.3.7, and Proposition 3.4.10. To prove (3), suppose that $\mathfrak{X}$ is a spectral algebraic space which is proper and locally almost of finite presentation over R; we wish to show that F is integrable. For this, it suffices to show that if A is a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal and $R \to A$ is a map of $\mathbb{E}_\infty$-rings, then the restriction functor

$$\theta : \mathrm{QCoh}(\operatorname{Spec} A \times_{\operatorname{Spec} R} \mathfrak{X})^{\mathrm{perf}} \to \mathrm{QCoh}(\operatorname{Spf} A \times_{\operatorname{Spec} R} \mathfrak{X})^{\mathrm{perf}}$$

is an equivalence of ∞-categories. According to Theorem XII.5.3.2, the restriction functor

$$\overline{\theta} : \mathrm{QCoh}(\operatorname{Spec} A \times_{\operatorname{Spec} R} \mathfrak{X})^{\mathrm{aperf}} \to \mathrm{QCoh}(\operatorname{Spf} A \times_{\operatorname{Spec} R} \mathfrak{X})^{\mathrm{aperf}},$$

is an equivalence of symmetric monoidal ∞-categories. We now observe that θ is obtained from $\overline{\theta}$ by restricting to the dualizable objects of $\mathrm{QCoh}(\operatorname{Spec} A \times_{\operatorname{Spec} R} \mathfrak{X})^{\mathrm{aperf}}$ and $\mathrm{QCoh}(\operatorname{Spf} A \times_{\operatorname{Spec} R} \mathfrak{X})^{\mathrm{aperf}}$, respectively. □

Remark 3.4.14. In the situation of Proposition 3.4.13, it is possible to describe the relative cotangent complex $L_{F/\operatorname{Spec}^{\mathrm{f}} R}$ explicitly. Suppose we are given a point $\eta \in F(A)$, corresponding to a map of connective $\mathbb{E}_\infty$-rings $R \to A$ and a perfect object $\mathcal{F} \in \mathrm{QCoh}(\operatorname{Spec} A \times_{\operatorname{Spec} R} \mathfrak{X})$. Let $f' : \operatorname{Spec} A \times_{\operatorname{Spec} R} \mathfrak{X} \to \operatorname{Spec} A$ denote the projection onto the first factor. Then $\eta^* L_{F/\operatorname{Spec}^{\mathrm{f}} R} \in \mathrm{QCoh}(\operatorname{Spec}^{\mathrm{f}} A) \simeq \mathrm{Mod}_A$ can be identified with

$$\Sigma^{-1} f'_+(\mathcal{F} \otimes \mathcal{F}^\vee) \simeq \Sigma^{-1} f'_*(\mathcal{F} \otimes \mathcal{F}^\vee)^\vee,$$

where f'_+ denotes the functor described in Proposition XII.3.3.23.

We next specialize to the study of algebraic vector bundles.

Definition 3.4.15. Let R be a connective $\mathbb{E}_\infty$-ring, let M be an R-module, and let $n \geq 0$ be an integer. We will say that M is *locally free of rank n* if the following conditions are satisfied:

(a) The module M is locally free of finite rank (equivalently, M is flat and almost perfect as an R-module: see Proposition A.8.2.5.20).

(b) For every field k and every map of $\mathbb{E}_\infty$-rings $R \to k$, the vector space $\pi_0(k \otimes_R M)$ has dimension n over k.

Remark 3.4.16. To verify condition (b) of Definition 3.4.15, we are free to pass to any field extension of k. We may therefore assume without loss of generality that k is algebraically closed.

The terminology of Definition 3.4.15 is motivated by the following observation:

Proposition 3.4.17. *Let R be a connective $\mathbb{E}_\infty$-ring and let M be an R-module which is locally free of finite rank. Then there exists a sequence of elements $x_1, \ldots, x_m \in \pi_0 R$ which generate the unit ideal, such that each of the modules $M[\frac{1}{x_i}] = R[\frac{1}{x_i}] \otimes_R M$ is free of rank n_i over $R[\frac{1}{x_i}]$. If M is locally free of rank n, then we can assume that $n_i = n$ for every integer i.*

Proof. Let us say that an element $x \in \pi_0 R$ is *good* if $M[\frac{1}{x}]$ is a free module of finite rank over $R[\frac{1}{x}]$ (which is of rank n in the case where M is locally free of rank n). To complete the proof, it will suffice to show that the collection of good elements of $\pi_0 R$ generate the unit ideal in $\pi_0 R$. Assume otherwise; then there exists a maximal ideal $\mathfrak{m}$ of $\pi_0 R$ which contains every good element of R. Let $k = (\pi_0 R)/\mathfrak{m}$ denote the residue field of $\pi_0 R$ at $\mathfrak{m}$. Then $\pi_0(k \otimes_R M)$ is a finite dimensional vector space over k (which is of dimension n in the case M is locally free of rank n). Let n' be the dimension of this vector space, and choose elements $y_1, \ldots, y_{n'} \in \pi_0 M$ whose images form a basis for $\pi_0(k \otimes_R M)$. Since $\pi_0 M$ is finitely generated over $\pi_0 R$, Nakayama's lemma implies that the images of the elements y_i generate the localization $(\pi_0 M)_{\mathfrak{m}}$. We may therefore choose an element $x \in (\pi_0 R) - \mathfrak{m}$ such that the elements y_i generate the module $(\pi_0 M)[\frac{1}{x}]$. It follows that there is a map a map $\phi : R[\frac{1}{x}]^{n'} \to M$ which induces a surjection $(\pi_0 R[\frac{1}{x}])^{n'} \to \pi_0 M[\frac{1}{x}]$. Since M is projective, the map ϕ admits a right homotopy inverse $\psi : M[\frac{1}{x}] \to R[\frac{1}{x}]^{n'}$. The composite map $\psi \circ \phi : R[\frac{1}{x}]^{n'} \to R[\frac{1}{x}]^{n'}$ determines an n'-by-n' matrix A_{ij} with values in $\pi_0 R[\frac{1}{x}]$. Let D denote the determinant of this matrix, and choose an element $x' \in \pi_0 R$ with $x^a D = x'$. Since ϕ induces an isomorphism of vector spaces $k^{n'} \simeq \pi_0(k \otimes_R M)$, the element x' does not belong to $\mathfrak{m}$. We note that the image of D in $\pi_0 R[\frac{1}{xx'}]$ is invertible, so that ϕ induces an equivalence $R[\frac{1}{xx'}]^{n'} \to M[\frac{1}{xx'}]$. It follows that $xx' \in \pi_0 R$ is good. Since the product xx' does not belong to $\mathfrak{m}$, we obtain a contradiction. □

Proposition 3.4.18. *The condition that an R-module M be locally free of rank n is stable under base change and local with respect to the fpqc topology (see Definitions VIII.2.6.14 and VIII.2.7.19).*

Proof. According to Proposition VIII.2.7.31, the condition of being locally free of finite rank is stable under base change and local with respect to the fpqc topology. It will therefore suffice to prove the following:

(∗) Let R be a connective $\mathbb{E}_\infty$-ring, M a locally free R-module of finite rank, and $R \to \prod_{1 \le i \le m} R_i$ a faithfully flat map of $\mathbb{E}_\infty$-rings. If each tensor product $R_i \otimes_R M$ satisfies condition (b) of Definition 3.4.15, then so does M.

To prove (∗), let us suppose we are given a field k and a map $R \to k$; we wish to prove that $\pi_0(k \otimes_R M)$ is a k-vector space of dimension n. Since $R \to \prod_{1 \le i \le m} R_i$ is faithfully flat, there exists an index i such that $k \otimes_R R_i \neq 0$. Let k' be a residue field of the commutative ring $k \otimes_R R_i$. Then k' is a field extension of k, so it will suffice to show that $\pi_0(k' \otimes_R M)$ has dimension n over k' (Remark 3.4.16). This follows from the existence of an isomorphism

$$\pi_0(k' \otimes_R M) \simeq \pi_0(k' \otimes_{R_i} (R_i \otimes_R M)),$$

since $R_i \otimes_R M$ is locally free of rank n over R_i. □

Notation 3.4.19. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. We say that a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is *locally free of rank n* if, for every étale morphism $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{QCoh}(\operatorname{Spec} A) \simeq \mathrm{Mod}_A$ is locally free of rank n when regarded as an A-module (see Definition VIII.2.6.17). If this condition is satsified, then $f^* \mathcal{F} \in \mathrm{Mod}_A$ is locally free of rank n for every map $f : \operatorname{Spec} A \to \mathfrak{X}$.

More generally, if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is a functor and $\mathcal{F} \in \mathrm{QCoh}(X)$, we say that $\mathcal{F}$ is *locally free of rank n* if $\eta^* \mathcal{F} \in \mathrm{Mod}_A$ is locally free of rank n for every point $\eta \in X(A)$ (see Definition VIII.2.7.21).

Lemma 3.4.20. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack, and let $\mathcal{F}$ be an almost perfect quasi-coherent sheaf on $\mathfrak{X}$. For every integer n, there exists quasi-compact open immersion $i : \mathfrak{U} \to \mathfrak{X}$ with the following property: a morphism of spectral Deligne-Mumford stacks $f : \mathfrak{X}' \to \mathfrak{X}$ factors through $\mathfrak{U}$ if and only if $f^* \mathcal{F}$ is n-connective.*

Proof. The assertion is local on $\mathfrak{X}$. We may therefore assume without loss of generality that $\mathfrak{X}$ is quasi-compact, so that $\mathcal{F}$ is m-connective for some integer m. We proceed by induction on the difference $n - m$. If $n - m \leq 0$, then we can take $\mathfrak{U} = \mathfrak{X}$. Assume that $m < n$. Using the inductive hypothesis, we can choose a quasi-compact open immersion $j : \mathfrak{V} \to \mathfrak{X}$ such that a map $\mathfrak{X}' \to \mathfrak{X}$ factors through j if and only if $f^* \mathcal{F}$ is $(n-1)$-connective. Replacing $\mathfrak{X}$ by $\mathfrak{V}$, we may assume that $\mathcal{F}$ is $(n-1)$-connective. Since the assertion is local on $\mathfrak{X}$, we may assume without loss of generality that $\mathfrak{X} = \operatorname{Spec} R$ is affine, so that $\mathcal{F}$ corresponds to an $(n-1)$-connective R-module M. Since $\mathcal{F}$ is almost perfect, $\pi_{n-1} M$ is finitely presented over $\pi_0 R$. We may therefore choose a presentation

$$(\pi_0 R)^{m'} \xrightarrow{T} (\pi_0 R)^m \to \pi_{n-1} M \simeq 0.$$

Let $I \subseteq \pi_0 R$ be the ideal generated by all m-by-m minors of the matrix representing the map T. Let $U = \{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{Z}} R : I \nsubseteq \mathfrak{p}\} \subseteq \operatorname{Spec}^{\mathrm{Z}} R$ and let $\mathfrak{U}$ be the corresponding open substack of $\mathfrak{X}$. We claim that $\mathfrak{U}$ has the desired properties. To prove this, it suffices to observe that a map $\operatorname{Spec} R' \to \mathfrak{X}$ factors through $\mathfrak{U}$ if and only if the abelian group $\operatorname{Tor}_0^{\pi_0 R}(\pi_0 R', \pi_{n-1} M) \simeq \pi_{n-1}(R' \otimes_R M)$ vanishes. □

Proposition 3.4.21. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack, and let $\mathcal{F}$ be a perfect quasi-coherent sheaf on $\mathcal{F}$. Then there exists a quasi-compact open immersion $i : \mathfrak{U} \to \mathfrak{X}$ with the following property: a morphism of spectral Deligne-Mumford stacks $f : \mathfrak{X}' \to \mathfrak{X}$ factors through $\mathfrak{U}$ if and only if $f^* \mathcal{F}$ is locally free of finite rank.*

Proof. Since $\mathcal{F}$ is perfect, it is a dualizable object of $\operatorname{QCoh}(\mathfrak{X})$ (Proposition VIII.2.7.28); let us denote its dual by $\mathcal{F}^\vee$. Note that $\mathcal{F}$ is locally free of finite rank if and only if both $\mathcal{F}$ and $\mathcal{F}^\vee$ are connective (Proposition A.8.2.5.20). The desired result now follows from Lemma 3.4.20. □

Proposition 3.4.22. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})$ be locally free of finite rank. Then:*

(1) *For every integer n, there exists a largest open substack $i_n : \mathfrak{X}_n \hookrightarrow \mathfrak{X}$ such that $i_n^* \mathcal{F}$ is locally free of rank n.*

(2) *The canonical map $\theta : \coprod_n \mathfrak{X}_n \to \mathfrak{X}$ is an equivalence of spectral Deligne-Mumford stacks. In particular, each i_n is a clopen immersion.*

Proof. The existence of the open immersion i_n follows from Proposition 3.4.18. The map θ is evidently étale , and is surjective by virtue of Proposition 3.4.17. To prove that θ is an equivalence, it will suffice to show that the diagonal map

$$\coprod_n \mathfrak{X}_n \to (\coprod_n \mathfrak{X}_n) \times_{\mathfrak{X}} (\coprod_n \mathfrak{X}_n) \simeq \coprod_{m,n} (\mathfrak{X}_m \times_{\mathfrak{X}} \mathfrak{X}_n)$$

is an equivalence. Since each i_n is an open immersion, each of the maps $\mathfrak{X}_n \to \mathfrak{X}_n \times_{\mathfrak{X}} \mathfrak{X}_n$ is an equivalence. It will therefore suffice to show that $\mathfrak{X}_m \times_{\mathfrak{X}} \mathfrak{X}_n$ is trivial for $m \neq n$. Equivalently, we must show that if R is a connective $\mathbb{E}_\infty$-ring and M is a locally free R-module of rank m which is also of rank $n \neq m$, then $R \simeq 0$. Assume otherwise, and let k be a residue field of $\pi_0 R$. We obtain an immediate contradiction, since $\pi_0(k \otimes_R M)$ is a vector space over k which is dimension m and also of dimension $n \neq m$. □

We now introduce some terminology to place Propositions 3.4.21 and 3.4.22 in context.

Notation 3.4.23. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \operatorname{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We say that f is an *open immersion* if, for every map $\operatorname{Spec}^{\mathrm{f}} R \to Y$, the fiber product $X \times_Y \operatorname{Spec}^{\mathrm{f}} R$ is representable by a spectral Deligne-Mumford stack $\mathfrak{X}_R$, and the projection $\mathfrak{X} \to \operatorname{Spec} R$ is an open immersion (see Example VIII.3.1.27). We note that f is an open immersion if and only the following conditions are satisfied:

(*a*) For every connective $\mathbb{E}_\infty$-ring R, the map $X(R) \to Y(R)$ induces a homotopy equivalence from $X(R)$ to a summand $Y_0(R) \subseteq Y(R)$.

(*b*) For every point $\eta \in Y(R)$, there exists an open subset $U \subseteq \operatorname{Spec}^{\mathrm{Z}} R$ with the following property: if $R \to R'$ is a map of connective $\mathbb{E}_\infty$-rings, then the image of η in $Y(R')$ belong to $Y_0(R')$ if and only if the map of topological spaces $\operatorname{Spec}^{\mathrm{Z}} R' \to \operatorname{Spec}^{\mathrm{Z}} R$ factors through U.

Remark 3.4.24. Let $f : X \to Y$ be an open immersion of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Then f is cohesive, nilcomplete, integrable, and admits a cotangent complex (Corollary 2.2.8). Moreover, the relative cotangent complex $L_{X/Y}$ is a zero object of $\mathrm{QCoh}(X)$. Using Proposition 2.3.9, we deduce that f is locally of finite presentation.

Proposition 3.4.25. *Suppose we are given a commutative diagram*

$$\begin{array}{ccc} Xh & \xrightarrow{f} & Y \\ & \searrow & \swarrow_{g} \\ & Z & \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. *Assume that f is an open immersion. If g is cohesive (infinitesimally cohesive, nilcomplete, integrable, locally of finite presentation to order n, locally almost of finite presentation, locally of finite presentation), then h has the same property. If g admits a cotangent complex, then so does h; moreover, we have a canonical equivalence $L_{X/Z} \simeq f^* L_{Y/Z}$ in* $\mathrm{QCoh}(X)$.

Proof. The first assertions follow from Remark 3.4.24. The existence of a cotangent complex $L_{X/Z}$ follows from the existence of $L_{Y/Z}$ by virtue of the criterion supplied by Remark 1.3.14. □

Definition 3.4.26. Let $\mathrm{Perf} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor defined in Proposition 3.4.10. For every connective $\mathbb{E}_\infty$-ring R, we let $\mathrm{Perf}^{\mathrm{lf}}(R)$ denote the summand of $\mathrm{Perf}(R)$ spanned by those perfect R-modules M which are locally free of finite rank over R, and $\mathrm{Perf}\,\mathrm{lf}_n(R)$ the summand of $\mathrm{Perf}(R)$ spanned by those perfect R-modules which are locally free of rank n over R.

The following assertions follow immediately from Propositions 3.4.21 and 3.4.22:

Proposition 3.4.27. *For every integer $n \geq 0$, the inclusions*

$$\mathrm{Perf}^{\mathrm{lf}}_n \hookrightarrow \mathrm{Perf}^{\mathrm{lf}} \hookrightarrow \mathrm{Perf}$$

are open immersions.

Corollary 3.4.28. *The functor* $\mathrm{Perf}^{\mathrm{lf}} : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *is cohesive, nilcomplete, locally of finite presentation, and admits a perfect cotangent complex. Moreover, if $\eta \in \mathrm{Perf}^{\mathrm{lf}}(R)$ classifies a locally free R-module M of finite rank, then $\eta^* L_{\mathrm{Perf}^{\mathrm{lf}}} \in \mathrm{Mod}_R$ can be identified with the R-module $\Sigma^{-1}(M \otimes_R M^\vee)$.*

For every integer $n \geq 0$, the functor $\mathrm{Perf}^{\mathrm{lf}}_n : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *is also cohesive, nilcomplete, locally of finite presentation and admits a perfect cotangent complex, given by the image of $L_{\mathrm{Perf}^{\mathrm{lf}}}$ in* $\mathrm{QCoh}(\mathrm{Perf}^{\mathrm{lf}}_n)$.

Proof. Combine Proposition 3.4.10, Remark 3.4.12, Proposition 3.4.27, and Proposition 3.4.25. □

We now specialize to the study of locally free sheaves of rank 1.

Proposition 3.4.29. *Let* $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *be a functor and let* $\mathcal{F} \in \mathrm{QCoh}(X)$. *The following conditions are equivalent:*

(1) *The quasi-coherent sheaf $\mathcal{F}$ is locally free of rank* 1.

(2) *The quasi-coherent sheaf $\mathcal{F}$ is a line bundle: that is, it is an invertible object of* $\mathrm{QCoh}(X)^{\mathrm{cn}}$.

Proof. Suppose first that (1) is satisfied. Then $\mathcal{F}$ is locally free of finite rank, and therefore a dualizable object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ (Proposition VIII.2.7.32). Let us denote its dual by $\mathcal{F}^\vee$, and let $e : \mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}$ be the evaluation map (where $\mathcal{O}$ denotes the structure sheaf of X). We claim that e is an equivalence. To prove this, it suffices to show that e induces an equivalence of R-modules $\eta^*(\mathcal{F} \otimes \mathcal{F}^\vee) \to R$ for every point $\eta \in X(R)$. The assertion is local on $\mathrm{Spec}^{\mathrm{Z}} R$, so we can apply Proposition 3.4.17 to reduce further to the case where $\eta^* \mathcal{F} \simeq R$, in which case the result is obvious.

We now prove (2). Assume that $\mathcal{F}$ is an invertible object in $\mathrm{QCoh}(X)^{\mathrm{cn}}$. We wish to show that for every point $\eta \in X(R)$, the pullback $M = \eta^* \mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} R) \simeq \mathrm{Mod}_R$ is locally free of rank 1. Note that M is a dualizable object of $\mathrm{Mod}_R^{\mathrm{cn}}$ and therefore locally free of finite rank (Proposition VIII.2.7.32). In particular, for every map from R to a field k, the tensor product $k \otimes_R M$ can be identified with a finite dimensional vector space over k, which is invertible as an object of $\mathrm{Mod}_k^{\mathrm{cn}}$. It follows easily that $\pi_0(k \otimes_R M)$ has dimension 1 over k, so that M is locally free of rank 1. □

Proposition 3.4.30. *Let R be a connective $\mathbb{E}_\infty$-ring and $f : \mathfrak{X} \to \mathrm{Spec}\, R$ be a map of spectral Deligne-Mumford stacks. Let F denote the image of $\mathrm{Pic}_{\mathfrak{X}/R}$ under the equivalence of ∞-categories $\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\,\mathrm{Spec}^{\mathrm{f}} R}$. Then:*

1. *The functor F is nilcomplete and cohesive.*
2. *Assume that $\mathfrak{X}$ is a quasi-compact, quasi-separated spectral algebraic space. Then the natural transformation $F \to \mathrm{Spec}^{\mathrm{f}} R$ is locally of finite presentation.*
3. *Assume that $\mathfrak{X}$ is a spectral algebraic space which is proper and locally almost of finite presentation over R. Then F is integrable.*
4. *Assume that $\mathfrak{X}$ is a proper algebraic space which is locally almost of finite presentation and locally of finite* Tor*-amplitude over R. Then the natural transformation $F \to \mathrm{Spec}^{\mathrm{f}} R$ admits a perfect cotangent complex.*

Proof. Assertions (1), (2), and (4) follow from Proposition 3.3.6, Remark 3.3.7, and Corollary 3.4.28. To prove (3), it will suffice to show that if A is a local Noetherian $\mathbb{E}_\infty$-ring which is complete with respect to its maximal ideal and we are given a map of $\mathbb{E}_\infty$-rings $R \to A$, then the restriction map $\theta : \mathrm{Pic}(\mathrm{Spec}\, A \times_{\mathrm{Spec}\, R} \mathfrak{X}) \to \mathrm{Pic}(\mathrm{Spf}\, A \times_{\mathrm{Spec}\, R} \mathfrak{X})$ is a homotopy equivalence. To prove this, we observe that Proposition 3.4.29 implies that θ is obtained from the symmetric monoidal forgetful functor

$$\overline{\theta} : \mathrm{QCoh}(\mathrm{Spec}\, A \times_{\mathrm{Spec}\, R} \mathfrak{X})^{\mathrm{aperf,cn}} \to \mathrm{QCoh}(\mathrm{Spf}\, A \times_{\mathrm{Spec}\, R} \mathfrak{X})^{\mathrm{aperf,cn}}$$

by restricting to the subcategories spanned by invertible objects and equivalences between them. It now suffices to observe that $\overline{\theta}$ is an equivalence of symmetric monoidal ∞-categories, by Theorem XII.5.3.2. □

Remark 3.4.31. If $f : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spec}\, R$ is proper, the evident inclusion $\mathrm{Pic}_{\mathfrak{X}/R} \hookrightarrow \mathrm{Perf}_{\mathfrak{X}/R}$ is an open immersion of functors. In this case, we can deduce Proposition 3.4.30 from Propositions 3.4.13 and 3.4.25. Moreover, Remark 3.4.14 implies that the cotangent complex of the map $q : F \to \mathrm{Spec}^{\mathrm{f}} R$ is given by the formula

$$L_{F/\,\mathrm{Spec}^{\mathrm{f}} R} \simeq \Sigma^{-1} q^*(f_* \mathcal{O}_{\mathcal{X}})^\vee.$$

In particular, the relative cotangent complex of F over $\mathrm{Spec}^{\mathrm{f}} R$ is constant along the fibers of F. This is a reflection of the fact that the functor $\mathrm{Pic}_{\mathfrak{X}/R}$ admits a group structure, given pointwise by the formation of tensor products of line bundles on $\mathfrak{X}$.

Proof of Theorem 3.4.6. Let $f : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spec}\, R$ be a morphism of spectral algebraic spaces which is proper, flat, and locally almost of finite presentation, and suppose that the cofiber of the unit map $u : R \to f_* \mathcal{O}_{\mathcal{X}}$ has Tor-amplitude ≤ -1 as an R-module. Let $x : \mathrm{Spec}\, R \to \mathfrak{X}$ be a section of f; we wish to show that the functor $\mathrm{Pic}^x_{\mathfrak{X}/R}$ is representable by a spectral algebraic space which is quasi-separated and locally almost of finite presentation over R. Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor corresponding to $\mathrm{Pic}^{x}_{\mathfrak{X}/R}$ under the equivalence of ∞-categories

$$\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\,\mathrm{Spec}^{\mathrm{f}} R}.$$

Let $Y' = Y \times_{\mathrm{Spec}^{\mathrm{f}} R} \mathrm{Spec}^{\mathrm{f}}(\pi_0 R)$. We will prove that Y' is representable by a quasi-separated spectral algebraic space which is locally almost of finite presentation over $\pi_0 R$. Note that $Y|\,\mathrm{CAlg}^0 \simeq Y'|\,\mathrm{CAlg}^0$. Since the functor Y is nilcomplete, infinitesimally cohesive, and admits a cotangent complex (Proposition 3.4.30), it follows that Y is representable by a spectral Deligne-Mumford stack $\mathfrak{Y}$ (Theorem 3.1.2). Note that that $\tau_{\leq 0}\,\mathfrak{Y} \simeq \tau_{\leq 0}\,\mathfrak{Y}'$. Since $\mathfrak{Y}'$ is a quasi-separated spectral algebraic space, it follows immediately that $\mathfrak{Y}$ is also a quasi-separated spectral algebraic space. Since $L_{Y/\,\mathrm{Spec}^{\mathrm{f}} R}$ is perfect (Proposition 3.4.30), Proposition 2.3.8 shows that $\mathfrak{Y}$ is locally of finite presentation over R.

Replacing R by $\pi_0 R$, we can assume that R is discrete (so that $Y = Y'$). Write R as the union of finitely generated subrings R_α. Using Theorem XII.2.3.2, we can choose an index α, a spectral Deligne-Mumford stack $\mathfrak{X}_\alpha$ which is finitely 0-presented over R_α, and an equivalence $\mathfrak{X} \simeq \tau_{\leq 0}(\mathrm{Spec}\, R \times_{\mathrm{Spec}\, R_\alpha} \mathfrak{X}_\alpha)$. Enlarging α if necessary, we may suppose that $\mathfrak{X}_\alpha$ is a spectral algebraic space which is proper and flat over R_α (Proposition XII.3.1.10 and Corollary 11.2.6.1 of [12]). Let $f_\alpha : \mathfrak{X}_\alpha \to \mathrm{Spec}\, R_\alpha$ denote the projection map, let $\mathcal{O}_\alpha$ denote the structure sheaf of $\mathfrak{X}_\alpha$, and let M denote the cofiber of the unit map $R_\alpha \to f_{\alpha *}\,\mathcal{O}_\alpha$. Then M is a perfect R_α-module (Proposition XII.3.3.20) and $R \otimes_{R_\alpha} M$ has Tor-amplitude ≤ -1. Let $M^\vee$ be the R_α-linear dual of M, so that $R \otimes_{R_\alpha} M^\vee$ is 1-connective. Enlarging α if necessary, we may suppose that $M^\vee$ is 1-connective, so that M has Tor-amplitude ≤ 1. We may therefore replace R by R_α and $\mathfrak{X}$ by $\mathfrak{X}_\alpha$, and thereby reduce to the case where R is finitely generated as a commutative ring. In particular, we may assume that R is a Grothendieck ring.

We next prove that Y is representable by a spectral algebraic space which is locally almost of finite presentation over R by verifying the hypotheses of Theorem 3.2.1. Hypothesis (2) is obvious, and (3) and (5) follow immediately from Proposition 3.4.30. Let us check the remaining hypotheses:

(1) For every discrete commutative ring A, the space $Y(A)$ is discrete. Equivalently, we must show that if A is a discrete R-algebra, then the space $\mathrm{Pic}^{x}_{\mathfrak{X}/R}(A)$ is discrete. Let A be a discrete R-algebra and let $\mathcal{L}$ be a line bundle on $\mathfrak{X}_A = \mathrm{Spec}\, A \times_{\mathrm{Spec}\, R} \mathfrak{X}$. Then the mapping space $\mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X}_A)}(\mathcal{L}, \mathcal{L})$ is given by

$$\begin{aligned} \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X}_A)}(\mathcal{O}_{\mathfrak{X}}, \mathcal{L} \otimes \mathcal{L}^\vee) &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X}_A)}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_A}(A, A \otimes_R f_*\,\mathcal{O}_{\mathfrak{X}}) \\ &\simeq \Omega^\infty A \oplus \Omega^\infty(A \otimes_R \mathrm{cofib}(u)). \end{aligned}$$

Our assumption on the Tor-amplitude of $\mathrm{cofib}(u)$ guarantees that $\mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X}_A)}(\mathcal{L}, \mathcal{L})$ is homotopy equivalent to the discrete commutative ring $\pi_0 A \simeq \Omega^\infty A$. In particular, if we let $x' : \mathrm{Spec}\, A \to \mathfrak{X}_A$ denote the map induced by x, then pullback along x' induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{QCoh}(\mathfrak{X}_A)}(\mathcal{L}, \mathcal{L}) \to \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} A)}(x'^*\,\mathcal{L}, x'^*\,\mathcal{L}).$$

It follows that the space $\mathrm{Pic}^{x}_{\mathfrak{X}/R}(A)$ is discrete.

(4) The natural transformation $Y \to \mathrm{Spec}^{\mathrm{f}} R$ admits a connective cotangent complex $L_{Y/\,\mathrm{Spec}^{\mathrm{f}} R}$. Let $Z : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the image of the functor $\mathrm{Pic}_{\mathfrak{X}/R}$ under the equivalence of ∞-categories

$$\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})_{/\,\mathrm{Spec}^{\mathrm{f}} R},$$

and let Z_0 be the image of $\mathrm{Pic}_{\mathrm{Spec}\, R/R}$ under the same equivalence, so Y can be identified with the fiber of the map $Z \to Z_0$ determined by the section x. Using Remark 3.4.31, we deduce that the relative cotangent complexes $L_{Z/\,\mathrm{Spec}^{\mathrm{f}} R}$ and $L_{Z_0/\,\mathrm{Spec}^{\mathrm{f}} R}$ are given by the pullbacks of the R-modules

$\Sigma^{-1}(f_* \mathcal{O}_{\mathfrak{X}})^\vee$ and $\Sigma^{-1}R$, respectively. It follows that the relative cotangent complex $L_{Y/\operatorname{Spec}^{\mathrm{f}} R}$ exists, and is given by the pullback of the R-linear dual of

$$\Sigma^1 \operatorname{fib}(f_* \mathcal{O}_{\mathfrak{X}} \xrightarrow{x^*} R) \simeq \Sigma^1 \operatorname{cofib}(u).$$

By assumption, $\operatorname{cofib}(u)$ has Tor-amplitude ≤ -1, so that $\Sigma^{-1} \operatorname{cofib}(u)^\vee$ is connective and therefore $L_{Y/\operatorname{Spec}^{\mathrm{f}} R}$ is connective.

This completes the proof that the functor Y is representable by a spectral algebraic space $\mathfrak{Y}$ which is locally almost of finite presentation over R. It remains to verify that $\mathfrak{Y}$ is quasi-separated. Suppose we are given a pair of connective $\mathbb{E}_\infty$-rings A and B and maps $\operatorname{Spec} A \xrightarrow{\phi} \mathfrak{Y} \xleftarrow{\phi'} \operatorname{Spec}^{\text{ét}} B$; we wish to prove that the fiber product $\operatorname{Spec} A \times_{\mathfrak{Y}} \operatorname{Spec} B$ is quasi-compact. Replacing R by $A \otimes_R B$, we may reduce to the case where $A = B = R$. Then ϕ and ϕ' determine line bundles $\mathcal{L}$ and $\mathcal{L}'$ on $\mathfrak{X}$ equipped with trivializations of $x^* \mathcal{L}$ and $x^* \mathcal{L}'$. For every object $R' \in \mathrm{CAlg}^{\mathrm{cn}}_R$, let $\mathcal{L}_{R'}$ and $\mathcal{L}'_{R'}$ denote the pullbacks of $\mathcal{L}$ and $\mathcal{L}'$ to $\mathfrak{X}_{R'}$. Define functors $F, F' : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ by the formulas

$$F(R') = \operatorname{Map}_{\mathrm{QCoh}(\mathfrak{X}_{R'})}(\mathcal{L}_{R'}, \mathcal{L}'_{R'}) \qquad F'(R') = \operatorname{Map}_{\mathrm{Mod}_{R'}}(R', R').$$

Since $f_*(\mathcal{L}' \otimes \mathcal{L}^\vee)$ is perfect, we can identify $F(R')$ with $\operatorname{Map}_{\mathrm{Mod}_R}((f_* \mathcal{L}' \otimes \mathcal{L}^\vee)^\vee, R')$. Note that $\mathcal{L}' \otimes \mathcal{L}^\vee$ is a line bundle on $\mathfrak{X}$. Since f is flat, the pushforward $f_*(\mathcal{L}' \otimes \mathcal{L}^\vee)$ has Tor-amplitude ≤ 0, so that $(f_* \mathcal{L}' \otimes \mathcal{L}^\vee)^\vee$ is connective. It follows that the functor F is representable by the affine spectral Deligne-Mumford stack $\mathfrak{Z} = \operatorname{Spec} \operatorname{Sym}^*_R(f_*(\mathcal{L}' \otimes \mathcal{L}^\vee)^\vee)$. Similarly, the functor F' is representable by the affine spectral Deligne-Mumford stack $\operatorname{Spec} R\{x\}$.

Let $g : \mathfrak{Z} \times_{\operatorname{Spec} R} \mathfrak{X} \to \mathfrak{X}$ be the projection onto the second factor. By construction, we have a canonical map of line bundles $\alpha : g^* \mathcal{L} \to g^* \mathcal{L}'$. Lemma 3.4.20 implies that there is a quasi-compact open immersion $\mathfrak{U} \hookrightarrow \mathfrak{Z} \times_{\operatorname{Spec} R} \mathfrak{X}$ such that a map $h : \operatorname{Spec} C \to \mathfrak{Z} \times_{\operatorname{Spec} R} \mathfrak{X}$ factors through $\mathfrak{U}$ if and only if $h^* \operatorname{cofib}(\alpha)$ is 1-connective. Then $\mathfrak{U}$ determines a constructible closed subset $K \subseteq |\mathfrak{Z} \times_{\operatorname{Spec} R} \mathfrak{X}|$. Since f is proper, the image of K is a constructible closed subset of $\mathfrak{Z}$, which determines a quasi-compact open immersion $\mathfrak{Z}_0 \hookrightarrow \mathfrak{Z}$. Unwinding the definitions, we see that $\mathfrak{Z}_0$ represents the subfunctor F_0 of F which carries an object $R' \in \mathrm{CAlg}^{\mathrm{cn}}_R$ to the summand of $F(R') = \operatorname{Map}_{\mathrm{QCoh}(\mathfrak{X}_{R'})}(\mathcal{L}_{R'}, \mathcal{L}'_{R'})$ consisting of equivalences of $\mathcal{L}_{R'}$ with $\mathcal{L}'_{R'}$.

Unwinding the definitions, we obtain a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} A \times_{\mathfrak{Y}} \operatorname{Spec} B & \longrightarrow & \mathfrak{Z}_0 \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \mathfrak{Z}'. \end{array}$$

In particular, we deduce that $\operatorname{Spec} A \times_{\mathfrak{Y}} \operatorname{Spec} B$ is quasi-affine (and therefore quasi-compact). □

4 Tangent Complexes and Dualizing Modules

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Theorem 3.1.2 supplies necessary and sufficient conditions for X to be representable by a spectral Deligne-Mumford stack. Among these conditions is the requirement that X admits a cotangent complex, in the sense of Definition 1.3.13. Consequently, in order to apply Theorem 3.1.2 to prove the representability of X, we first need to verify a "linearized" version of the same representability result. Our goal in this section is to give a reformulation of this condition, which does not explicitly mention the representability of any functor.

Let $f : X \to Y$ be an infinitesimally cohesive natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Suppose that we wish to show that there exists a relative cotangent complex $L_{X/Y}$, in the sense of Definition

1.3.13. Our starting point is that the *dual* of $L_{X/Y}$ can be defined under very mild hypotheses. In §4.1, we will explain how to associate to each point $\eta \in X(A)$ an A-module $T_{X/Y}(\eta)$, which we will refer to as the *relative tangent complex of X over Y at the point η*. When f admits a cotangent complex $L_{X/Y}$, $T_{X/Y}(\eta)$ can be identified with the A-linear dual of $\eta^* L_{X/Y}$ (Example 4.1.2). In §4.5, we will show that if the tangent complexes $T_{X/Y}(\eta)$ satisfy some reasonable finiteness conditions, then f admits a cotangent complex (Theorem 4.5.1). The proof makes use of characterization of those functors $\mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ which are corepresentable by almost perfect R-modules (Theorem 4.4.2), which we will establish in §4.4. For this, we will need a generalization of Grothendieck's theory of dualizing complex to the setting of $\mathbb{E}_\infty$-rings, which we review in §4.2 and 4.3.

4.1 The Tangent Complex

In §XII.6, we studied the tangent complex associated to a formal moduli problem X over a field k. In this section, we will consider the global analogue of this notion, obtained by replacing X by an arbitrary infinitesimally cohesive morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$.

Construction 4.1.1. Let $f : X \to Y$ be an infinitesimally cohesive natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. For every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, let $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by the formula

$$F_\eta(M) = \mathrm{fib}(X(A \oplus M) \to Y(A \oplus M) \times_{Y(A)} X(A)),$$

where the fiber is taken over the point determined by η. Since f is infinitesimally cohesive, the canonical map $F_\eta(M) \to \Omega F_\eta(\Sigma M)$ is an equivalence for each $M \in \mathrm{Mod}_A^{\mathrm{cn}}$, so that F_η is a reduced excisive functor (Proposition A.1.4.2.13). Applying Lemma 1.3.2, we see that F_η admits an essentially unique extension to a left exact functor $F_\eta^+ : \mathrm{Mod}_A^{\mathrm{acn}} \to \mathcal{S}$. We can identify the restriction $F_\eta^+ | \mathrm{Mod}_A^{\mathrm{perf}}$ with an object of $\mathrm{Ind}((\mathrm{Mod}_A^{\mathrm{perf}})^{op}) \simeq \mathrm{Ind}(\mathrm{Mod}_A^{\mathrm{perf}}) \simeq \mathrm{Mod}_A$ (see §A.8.2.5). We will denote the corresponding A-module by $T_{X/Y}(\eta)$, and refer to it as the *relative tangent complex to f at the point η*. It is characterized by existence of a canonical homotopy equivalence $F_\eta(M) \simeq \Omega^\infty(T_{X/Y}(\eta) \otimes_A M)$ whenever M is a perfect, connective A-module. In particular, we have a homotopy equivalence $\Omega^{\infty - n} T_{X/Y}(\eta) \simeq F_\eta(\Sigma^n A)$ for each $n \geq 0$.

Example 4.1.2. Let $f : X \to Y$ be an infinitesimally cohesive morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, and suppose that f admits a relative cotangent complex $L_{X/Y}$. For every point $\eta \in X(A)$, the functor F_η of Construction 4.1.1 is given by $F_\eta(M) = \mathrm{Map}_{\mathrm{Mod}_A}(\eta^* L_{X/Y}, M)$. It follows that the relative tangent complex $T_{X/Y}(\eta)$ can be identified with the A-linear dual $\underline{\mathrm{Map}}_A(\eta^* L_{X/Y}, A)$ of $\eta^* L_{X/Y}$.

Variant 4.1.3. In the situation of Construction 4.1.1, suppose we are given an almost connective A-module N. The functor $M \mapsto F_\eta^+(M \otimes_A N)$ is left exact, and therefore its restriction to perfect A-modules determines an A-module $T_{X/Y}(\eta; N)$ equipped with canonical equivalences $\Omega^{\infty - n} T_{X/Y}(\eta; N) \simeq F_\eta(\Sigma^n N)$. Note that we have $T_{X/Y}(\eta) \simeq T_{X/Y}(\eta; A)$. Moreover, the construction $N \mapsto T_{X/Y}(\eta; N)$ is an exact functor from Mod_A to itself.

Remark 4.1.4. Let $f : X \to Y$ be an infinitesimally cohesive morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, let $\phi : A \to A'$ be a map of connective $\mathbb{E}_\infty$-rings, let $\eta \in X(A)$ and let η' denote its image in $X(A')$. Define

$$F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S} \qquad F_{\eta'} : \mathrm{Mod}_{A'}^{\mathrm{cn}} \to \mathcal{S}$$

as in Construction 4.1.1, and let $U : \mathrm{Mod}_{A'}^{\mathrm{acn}} \to \mathrm{Mod}_A^{\mathrm{acn}}$ denote the forgetful functor. Then ϕ induces a natural transformation of reduced excisive functors $F_\eta \circ (U | \mathrm{Mod}_{A'}^{\mathrm{cn}}) \to F_{\eta'}$, which extends to a natural transformation of left exact functors $F_\eta^+ \circ U \to F_{\eta'}^+$. For every A'-module N, we obtain a map of A-modules $T_{X/Y}(\eta; N) \to T_{X/Y}(\eta', N)$, which is an equivalence when f is cohesive and ϕ induces a surjection $\pi_0 A \to \pi_0 A'$.

In particular, by taking $N = A'$, we obtain a map of A-modules

$$T_{X/Y}(\eta) = T_{X/Y}(\eta; A) \to T_{X/Y}(\eta; A') \to T_{X/Y}(\eta', A') = T_{X/Y}(\eta'),$$

which is adjoint to a map of A'-modules $A' \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$.

Remark 4.1.5. Suppose we are given a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, where the vertical maps are infinitesimally cohesive. Let A be a connective $\mathbb{E}_\infty$-ring, let $\eta \in X(A)$, and let η' denote the image of η in $X'(A)$. These is a canonical equivalence of A-modules $T_{X/Y}(\eta) \simeq T_{X'/Y'}(\eta')$.

Our goal in this section is to prove the following result:

Proposition 4.1.6. *Let $f : X \to Y$ be a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, where Y is corepresentable by a Noetherian $\mathbb{E}_\infty$-ring R, and let $X_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ denote the functor given by $X_0(A) = \mathrm{fib}(X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A))$. Assume that f is cohesive, nilcomplete, and locally almost of finite presentation. The following conditions are equivalent:*

(1) *For every morphism $\phi : A \to B$ in $\mathrm{CAlg}^{\mathrm{cn}}_R$ and every connective B-module M, the diagram*

$$\begin{array}{ccc} X_0(A \oplus M) & \longrightarrow & X_0(B \oplus M) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

is a pullback square.

(2) *For every point $\eta \in X(A)$ and every flat morphism $\phi : A \to B$ carrying η to a point $\eta' \in X(B)$, the map $B \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$ of Remark 4.1.4 is an equivalence, where η' denotes the image of η in $X(B)$.*

(3) *For every discrete integral domain A and every point $\eta \in X(A)$ which exhibits A as a finitely generated algebra over $\pi_0 R$, the map $A[x] \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$ of Remark 4.1.4 is an equivalence, where η' denotes the image of $\eta \in X(A[x])$.*

(4) *For every discrete integral domain A with fraction field K, every point $\eta \in X(A)$ which exhibits A as a finitely generated algebra over $\pi_0 R$, and every extension field L of K, the canonical map*

$$L \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$$

is an equivalence, where η' denotes the image of η in $X(L)$.

Proof. We first show that (1) $\Rightarrow$ (2). Fix a point $\eta \in X(A)$, let B be a flat $\mathbb{E}_\infty$-algebra over A, and let $\eta' \in X(B)$ denote the image of η. Define $F_\eta : \mathrm{Mod}^{\mathrm{cn}}_A \to \mathcal{S}$ and $F_{\eta'} : \mathrm{Mod}^{\mathrm{cn}}_B \to \mathcal{S}$ as in Construction 4.1.1. To prove that the canonical map $B \otimes_A T_{X/Y}(\eta) \to T_{X/Y}(\eta')$ is an equivalence, it will suffice to show that for each $n \geq 0$ the map

$$\theta : \Omega^{\infty - n}(B \otimes_A T_{X/Y}(\eta)) \to \Omega^{\infty - n} T_{X/Y}(\eta')$$

is a homotopy equivalence of spaces. Since B is flat over A, we can write B as a filtered colimit $\varinjlim P_\alpha$, where each P_α is a free A-module of finite rank (Theorem A.8.2.2.15). We can then identify θ with the composite map

$$\begin{array}{rcl} \Omega^{\infty-n}(B \otimes_A T_{X/Y}(\eta)) & \simeq & \varinjlim \Omega^{\infty-n}(P_\alpha \otimes_A T_{X/Y}(\eta)) \\ & \simeq & \varinjlim F_\eta(\Sigma^n P_\alpha) \\ & \xrightarrow{\theta'} & F_\eta(B) \\ & \xrightarrow{\theta''} & F_{\eta'}(B). \end{array}$$

The map θ' is a homotopy equivalence by virtue of our assumption that f is locally almost of finite presentation, and the map θ'' is a homotopy equivalence by virtue of assumption (1).

The implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are obvious. We next show that (3) $\Rightarrow$ (1). Choose a connective $\mathbb{E}_\infty$-ring R and a point $\eta \in Y(R)$, and let $X_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be as in Notation 2.2.4. We wish to show that for every morphism $\phi : A \to B$ in $\mathrm{CAlg}^{\mathrm{cn}}_R$ and every connective B-module M, the diagram σ_M :

$$\begin{array}{ccc} X_0(A \oplus M) & \longrightarrow & X_0(B \oplus M) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

is a pullback square. Since X is nilcomplete, σ_M is the limit of the diagrams $\sigma_{\tau_{\leq n} M}$. It will therefore suffice to show that each $\sigma_{\tau_{\leq n} M}$ is a pullback diagram. We proceed by induction on n, the case $n = 0$ being trivial. If $n > 0$, we have a fiber sequence of B-modules

$$\tau_{\leq n} M \to \tau_{\leq n-1} M \to \Sigma^{n+1} N$$

where $N \simeq \pi_n M$ is a discrete B-module. The square $\sigma_{\tau_{\leq n} M}$ fits into a larger diagram

$$\begin{array}{ccccc} X_0(A \oplus \tau_{\leq n} M) & \longrightarrow & X_0(B \oplus \tau_{\leq n} M) & \longrightarrow & X_0(B \oplus \tau_{\leq n-1} M) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \end{array}$$

The right square in this diagram is a pullback since X is cohesive. It will therefore suffice to show that the outer rectangle is a pullback diagram. That is, we must show that the outer rectangle in the diagram

$$\begin{array}{ccccc} X_0(A \oplus \tau_{\leq n} M) & \longrightarrow & X_0(A \oplus \tau_{\leq n-1} M) & \longrightarrow & X_0(B \oplus \tau_{\leq n-1} M) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A \oplus K) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \end{array}$$

is a pullback square. Since the left square is a pullback by virtue of our assumption that X is cohesive, it suffices to show that the right square is also a pullback. This right square fits into a commutative diagram

$$\begin{array}{ccc} X_0(A \oplus \tau_{\leq n-1} M) & \longrightarrow & X_0(B \oplus \tau_{\leq n-1} M) \\ \downarrow & & \downarrow \\ X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

where the outer rectangle is a pullback square by the inductive hypothesis. It will therefore suffice to show that $\sigma_{\Sigma^{n+1} N}$ is a pullback diagram.

Since N is a module over $\pi_0 B$, $\sigma_{\Sigma^{n+1} N}$ fits into a commutative diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(\pi_0 B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) & \longrightarrow & X_0(\pi_0 B) \end{array}$$

The right square is a pullback diagram by virtue of our assumption that X is cohesive. It will therefore suffice to show that the outer rectangle is a pullback. Equivalently, we must show that the outer rectangle in the diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(\pi_0 A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(\pi_0 B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(\pi_0 A) & \longrightarrow & X_0(\pi_0 B) \end{array}$$

is a pullback square. Since the left square is a pullback (because X is cohesive), we are reduced to proving that the right square is a pullback. In other words, we may replace A by $\pi_0 A$ and B by $\pi_0 B$, and thereby reduce to the case where A and B are discrete.

Write A as a filtered colimit of subalgebras A_α which are finitely generated over $\pi_0 R$. Since f is locally almost of finite presentation, the functor X_0 commutes with filtered colimits when restricted to $(n+1)$-connective R-algebras. It will therefore suffice to show that each of the diagrams

$$\begin{array}{ccc} X_0(A_\alpha \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow \\ X_0(A_\alpha) & \longrightarrow & X_0(B). \end{array}$$

We may therefore replace A by A_α and thereby reduce to the case where A is finitely generated as an algebra over $\pi_0 R$. In particular, A is a Noetherian ring. Choose a surjection of commutative A-algebras $P \rightarrow B$, where P is a polynomial ring over A. We then have a commutative diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(P \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(B \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(P) & \longrightarrow & X_0(B). \end{array}$$

The right square is a pullback since X_0 is cohesive. It will therefore suffice to show that the left square is a homotopy pullback. Write $P \simeq \varinjlim P_\beta$, where each P_β is a polynomial ring over A on finitely many generators. It will therefore suffice to show that each of the diagrams

$$\begin{array}{ccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(P_\beta \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(P_\beta). \end{array}$$

is a pullback square. Write $P_\beta = A[x_1, \ldots, x_k]$. Working by induction on k, we can reduce to the case where $k = 1$: that is, we are given a discrete $A[x]$-module N, and we wish to show that the diagram τ :

$$\begin{array}{ccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+1} N) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]) \end{array}$$

is a pullback square.

Let N_0 denote the underlying A-module of N, and regard $N_0[x]$ as an $A[x]$-module. We have a short exact sequence of discrete $A[x]$-modules

$$0 \rightarrow N_0[x] \rightarrow N_0[x] \rightarrow N \rightarrow 0,$$

hence a fiber sequence

$$\Sigma^{n+1} N \to \Sigma^{n+2} N_0[x] \to \Sigma^{n+2} N_0[x].$$

It follows that τ fits into a commutative diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]). \end{array}$$

Our assumption that X is cohesive guarantees that the right square is a pullback. It will therefore suffice to show that the outer rectangle is also a pullback. Equivalently, we must show that the outer rectangle in the diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+1} N) & \longrightarrow & X_0(A \oplus \Sigma^{n+2} N_0[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A \oplus \Sigma^{n+2} N_0[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \end{array}$$

is a pullback square. Since the left square is a pullback (because X is cohesive), it will suffice to show that the right square is also a pullback. This square fits into a larger diagram

$$\begin{array}{ccc} X_0(A \oplus \Sigma^{n+2} N_0[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \\ \downarrow & & \downarrow \\ X_0(A \oplus \Sigma^{n+2} N_0[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} N_0[x]) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]). \end{array}$$

It will therefore suffice to show that the lower square and the outer rectangle in this diagram are pullback squares. For this, it suffices to verify the following general assertion: for every discrete A-module T, the diagram τ_T :

$$\begin{array}{ccc} X_0(A \oplus \Sigma^{n+2} T[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T[x]) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]) \end{array}$$

is a pullback square.

Since f is locally almost of finite presentation, the construction $T \mapsto \tau_T$ commutes with filtered colimits. Writing T as a filtered colimit of its finitely generated submodules, we are reduced to proving that τ_T is an equivalence when T is finitely generated over A. Since A is Noetherian, T is also Noetherian. Working by Noetherian induction, we can assume that for every nonzero submodule $T' \subseteq T$, the diagram $\tau_{T/T'}$ is a pullback square. If $T = 0$, there is nothing to prove. Otherwise, T has an associated prime: that is, we can choose a nonzero element $x \in T$ whose annihilator is a prime ideal $\mathfrak{p} \subseteq A$. Let T' denote the submodule of T generated by x. The diagram τ_T fits into a commutative square

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+2} T[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T/T'[x]) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+3} T'[x]). \end{array}$$

Since X is cohesive, the right square is a pullback. It will therefore suffice to show that the outer rectangle is a pullback. This is equivalent to the outer rectangle in the diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+2} T[x]) & \longrightarrow & X_0(A \oplus \Sigma^{n+2} T/T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T/T'[x]) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+3} T'[x]). \end{array}$$

Since the left square in this diagram is a pullback (by virtue of the assumption that X is cohesive), we are reduced to proving that the right square is also a pullback. To prove this, we consider the rectangular diagram

$$\begin{array}{ccc} X_0(A \oplus \Sigma^{n+2} T/T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+2} T/T'[x]) \\ \downarrow & & \downarrow \\ X_0(A \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+3} T'[x]) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]). \end{array}$$

The inductive hypothesis implies that the outer rectangle is a pullback diagram. It will therefore suffice to show that the lower square is also a pullback diagram.

Write $T' = A/\mathfrak{p}$, and consider the diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A[x] \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A[x]/\mathfrak{p} \oplus \Sigma^{n+3} T'[x]) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A[x]) & \longrightarrow & X_0(A[x]/\mathfrak{p}). \end{array}$$

Since X is cohesive, the left square is a pullback. It will therefore suffice to show that the outer rectangle is a pullback. Equivalently, we must show that the outer rectangle in the diagram

$$\begin{array}{ccccc} X_0(A \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A/\mathfrak{p} \oplus \Sigma^{n+3} T'[x]) & \longrightarrow & X_0(A[x]/\mathfrak{p} \oplus \Sigma^{n+3} T'[x]) \\ \downarrow & & \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(A/\mathfrak{p}) & \longrightarrow & X_0(A[x]/\mathfrak{p}) \end{array}$$

is a pullback. Here the left square is a pullback by virtue of our assumption that X is cohesive. We are therefore reduced to proving that the right square is a pullback diagram, which follows assumption (3).

We now complete the proof by showing that (4) $\Rightarrow$ (3). Suppose that (4) is satisfied. We will prove the following more general version of (3):

($*$) Let A be a commutative ring, let $\eta \in X(A)$ exhibit A as a finitely generated algebra over $\pi_0 R$, and let M be a finitely generated (discrete) R-module. Let η' denote the image of η in $X(A[x])$. Then the canonical map

$$\psi_M : A[x] \otimes_A T_{X/Y}(\eta; M) \simeq T_{X/Y}(\eta; M[x]) \to T_{X/Y}(\eta', M[x])$$

is an equivalence.

To prove ($*$), we first note that A is Noetherian, so that M is a Noetherian A-module. Working by Noetherian induction, we may suppose that $\psi_{M/M'}$ is an equivalence for every nonzero submodule $M' \subseteq M$.

If $M = 0$ there is nothing to prove. Otherwise, M has an associated prime ideal: that is, there is an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where $M' \simeq A/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq A$. Since $\psi_{M''}$ is an equivalence by virtue of the inductive hypothesis, we are reduced to proving that $\psi_{M'}$ is an equivalence. Since f is cohesive, we may replace A by $A/\mathfrak{p}$, and thereby reduce to the special case where $A$is an integral domain and $M = A$.

For each nonzero element $a \in A$, we have an exact sequence

$$0 \to aM \to M \to M/aM \to 0.$$

The inductive hypothesis implies that $\psi_{M/aM}$ is an equivalence. It follows that multiplication by a induces an equivalence from $\mathrm{cofib}(\psi_M)$ to itself. Let K denote the fraction field of A, so that $K \otimes_A \mathrm{cofib}(\psi_M)$ is equivalent to $\mathrm{cofib}(\psi_M)$. We are therefore reduced to proving that ψ_M induces an equivalence $K[x] \otimes_A T_{X/Y}(\eta) \to K[x] \otimes_{A[x]} T_{X/Y}(\eta')$.

Let $h(x) \in A[x]$ be a polynomial whose image $K[x]$ is irreducible. Let $B = A[x]/(h(x))$, and let $L = K[x]/(h(x))$ be the fraction field of B. Let η'_B denote the image of η in $X(B)$ and define η'_L similarly. Since X is infinitesimally cohesive, can identify $T_{X/Y}(\eta'_B)$ with the cofiber of the map $h(x) : T_{X/Y}(\eta') \to T_{X/Y}(\eta')$. Using condition (4), we can identify $T_{X/Y}(\eta'_L)$ with the cofiber of $h(x)$ on $K[x] \otimes_{A[x]} T_{X/Y}(\eta')$. We therefore have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} K[x] \otimes_A T_{X/Y}(\eta) & \xrightarrow{h(x)} & K[x] \otimes_A T_{X/Y}(\eta) & \longrightarrow & L \otimes_A T_{X/Y}(\eta) \\ \downarrow & & \downarrow & & \downarrow \\ K[x] \otimes_{A[x]} T_{X/Y}(\eta') & \xrightarrow{h(x)} & K[x] \otimes_{A[x]} T_{X/Y}(\eta') & \longrightarrow & T_{X/Y}(\eta'_L) \end{array}$$

where condition (4) implies that the right vertical map is an equivalence. It follows that multiplication by $h(x)$ acts invertibly on $\mathrm{cofib}(\psi_M)$.

Let K' denote the fraction field of the integral domain $A[x]$. The reasoning above shows that $\mathrm{cofib}(\psi_M) \simeq K' \otimes_{A[x]} \mathrm{cofib}(\psi_M)$. Consequently, to show that $\mathrm{cofib}(\psi_M) \simeq 0$, it will suffice to show that the horizontal map in the diagram

$$\begin{array}{ccc} K' \otimes_A T_{X/Y}(\eta) & \longrightarrow & K' \otimes_{A[x]} T_{X/Y}(\eta') \\ & \searrow \quad \swarrow & \\ & T_{X/Y}(\eta_{K'}) & \end{array}$$

is an equivalence, where $\eta_{K'}$ denotes the image of η in $X(K')$. It now suffices to observe that condition (4) implies that both of the vertical maps are equivalences. □

4.2 Dualizing Modules

Let R be a Noetherian commutative ring. In his treatment of the duality theory of coherent sheaves, Grothendieck introduced the notion of a *dualizing complex* of R-modules. In this section, we discuss a generalization of the theory of dualizing complexes to the setting of modules over an arbitrary Noetherian $\mathbb{E}_\infty$-ring A. In accordance with our usual convention of referring to A-module spectra simply as A-modules, we will refer to a dualizing object of Mod_A as a *dualizing module* rather than a dualizing complex.

The main results of this section can be stated as follows:

(a) Let A be a Noetherian $\mathbb{E}_\infty$-ring. If A admits a dualizing module K, then there is a contravariant equivalence from the ∞-category of coherent A-modules to itself, given by $M \mapsto \underline{\mathrm{Map}}_A(M, K)$ (Theorem 4.2.7).

(b) If A is a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module K, then K is essentially unique: any other dualizing module has the form $K \otimes_A L$, where L is an invertible A-module (Proposition 4.2.9).

We begin by introducing some terminology.

Notation 4.2.1. Let A be an $\mathbb{E}_\infty$-ring, and let M and N be A-modules. We let $\underline{\mathrm{Map}}_A(M, N)$ denote a classifying object for morphisms from M to N in the ∞-category Mod_A. That is, $\underline{\mathrm{Map}}_A(M, N)$ is an A-module with the following universal property: there exists a map

$$e : \underline{\mathrm{Map}}_A(M, N) \otimes_A M \to N$$

such that, for every A-module M', composition with e induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_A}(M', \underline{\mathrm{Map}}_A(M, N)) \to \mathrm{Map}_{\mathrm{Mod}_A}(M' \otimes_A M, N).$$

Definition 4.2.2. Let A be a connective $\mathbb{E}_\infty$-ring and let K be an A-module. We will say that K has *injective dimension* $\leq n$ if, for every discrete A-module M, the abelian groups $\mathrm{Ext}^i_A(M, K)$ vanish for $i > n$. We say that K has *finite injective dimension* if it has injective dimension $\leq n$ for some integer n.

Remark 4.2.3. Let A be a connective $\mathbb{E}_\infty$-ring and let K be an A-module. Then K has injective dimension $\leq n$ if and only if the following condition is satisfied:

($*$) For every m-truncated A-module M, the groups $\mathrm{Ext}^i_A(M, N)$ vanish for $i > n + m$.

The "if" direction is obvious. Conversely, suppose that K has injective dimension $\leq n$ and that M is m-truncated; we wish to show that $\underline{\mathrm{Map}}_A(M, K)$ is $(-m-n)$-connective. The A-module $\underline{\mathrm{Map}}_A(M, K)$ is the limit of a tower of A-modules $\{\underline{\mathrm{Map}}_A(\tau_{\geq m-k} M, K)\}_{k \geq 0}$, where $\underline{\mathrm{Map}}_A(\tau_{\geq m+1} M, K)$ vanishes. It will therefore suffice to show that for $k \geq 0$, the map

$$\underline{\mathrm{Map}}_A(\tau_{\geq m-k} M, K) \to \underline{\mathrm{Map}}_A(\tau_{\geq m-k+1} M, K)$$

is $(-m-n)$-connective. The fiber of this map is given by $\underline{\mathrm{Map}}_A(\pi_{m-k} M[m-k], K) \simeq \underline{\mathrm{Map}}_A(\pi_{m-k} M, K)[k-m]$. We are therefore reduced to proving that $\mathrm{Ext}^i_A(\pi_{m-k} M, K)$ vanishes for $i \geq n-k$, which follows from our assumption on K.

Remark 4.2.4. Let A be a connective $\mathbb{E}_\infty$-ring and let K be an A-module such that $\pi_i K \simeq 0$ for $i > 0$. Then K has injective dimension ≤ 0 if and only if K is an injective object of Mod_A, in the sense of Definition A.8.2.3.2 (see Proposition A.8.2.3.1). More generally, choose an injective object $I \in \mathrm{Mod}_A$ and a map $\alpha : K \to I$ which induces an injection $\pi_0 K \to \pi_0 I$. For $n > 0$, the object K has injective dimension $\leq n$ if and only if the cofiber $\mathrm{cofib}(\alpha)$ has injective dimension $\leq n-1$. Consequently, if K has injective dimension $\leq n$, then it can be written as a successive extension of A-modules $\{I_m[-m]\}_{0 \leq m \leq n}$, where each I_m is injective.

Definition 4.2.5. Let A be a Noetherian $\mathbb{E}_\infty$-ring and let K be an A-module. We will say that K is a *dualizing module* if it has the following properties:

(1) Each homotopy group $\pi_n K$ is a finitely generated module over $\pi_0 A$, and $\pi_n K$ vanishes for $n \gg 0$.

(2) The canonical map $A \to \underline{\mathrm{Map}}_A(K, K)$ is an equivalence. In other words, for every A-module M, the canonical map
$$\mathrm{Map}_{\mathrm{Mod}_A}(M, A) \to \mathrm{Map}_{\mathrm{Mod}_A}(M \otimes_A K, K)$$
is a homotopy equivalence.

(3) The module K has finite injective dimension.

Example 4.2.6. Let A be a Noetherian commutative ring. If A is Gorenstein, then $K = A$ is a dualizing module for A.

Theorem 4.2.7. *Let A be a Noetherian $\mathbb{E}_\infty$-ring and let K be a dualizing module for A. For every A-module M, we let $\mathbb{D}(M)$ denote the A-module $\underline{\mathrm{Map}}_A(M, K)$. Then:*

(1) *Let $\mathcal{C}$ denote the full subcategory of Mod_A spanned by those A-modules M such that $\pi_n M$ is finitely generated over $\pi_0 A$ for every integer n. Then the construction $M \mapsto \mathbb{D}(M)$ induces a contravariant equivalence of $\mathcal{C}$ with itself.*

(2) *Let $M \in \mathcal{C}$. Then the canonical map $M \to \mathbb{D}(\mathbb{D}(M))$ is an equivalence.*

(3) *Let $M \in \mathcal{C}$. Then M is almost perfect if and only if $\mathbb{D}(M)$ is truncated.*

Proof. We first show that if $M \in \mathcal{C}$, then $\mathbb{D}(M) \in \mathcal{C}$. We must show that each homotopy group $\pi_i \mathbb{D}(M)$ is finitely generated over $\pi_0 R$. Replacing M by a shift if necessary, we may suppose that $i = 0$. Choose an integer n such that K has injective dimension $\leq n$, and choose a fiber sequence

$$M' \to M \to M''$$

where M' is $(-n)$-connective and M'' is $(-n-1)$-truncated. We have an exact sequence

$$\pi_0 \mathbb{D}(M'') \to \pi_0 \mathbb{D}(M) \to \pi_0 \mathbb{D}(M')$$

where $\pi_0 \mathbb{D}(M'')$ vanishes by Remark 4.2.3. Since $\pi_0 R$ is Noetherian, it will suffice to show that $\pi_0 \mathbb{D}(M')$ is finitely generated over $\pi_0 R$. Replacing M by M', we may reduce to the case where there exists an integer k such that M is k-connective. We now proceed by descending induction on k. If $k \gg 0$, then $\pi_0 \mathbb{D}(M) \simeq 0$ (since K is truncated). To carry out the inductive step, choose a map $\alpha : \Sigma^k R^a \to M$ which induces a surjection $(\pi_0 R)^a \to \pi_k M$, and let $N = \mathrm{cofib}(\alpha)$. Then N is $(k+1)$-connective, so that $\pi_0 \mathbb{D}(N)$ is finitely generated by the inductive hypothesis. Using the exact sequence

$$\pi_0 \mathbb{D}(N) \to \pi_0 \mathbb{D}(M) \to (\pi_k K)^a,$$

we deduce that $\pi_0 \mathbb{D}(M)$ is finitely generated.

To complete the proof of (1), it will suffice to prove (2) (so that the duality functor $M \mapsto \mathbb{D}(M)$ is homotopy inverse to itself). Let $M \in \mathcal{C}$; we wish to show that the canonical map $u_M : M \to \mathbb{D}(\mathbb{D}(M))$ is an equivalence. For this, it suffices to show that u_M induces an isomorphism $\pi_i M \to \pi_i \mathbb{D}(\mathbb{D}(M))$ for every integer i. Replacing M by a shift if necessary, we may suppose that $i = 0$.

Choose an integer m such that K is m-truncated. Then for every k-truncated R-module N, the dual $\mathbb{D}(N)$ is $(-n-k)$-connective (Remark 4.2.3), so that the double dual $\mathbb{D}(\mathbb{D}(N))$ is $(m+n+k)$-truncated. Choose a negative integer k such that $m+n+k$ is also negative. Then the fiber sequence

$$\tau_{\geq k+1} M \to M \to \tau_{\leq k} M$$

gives rise to a fiber sequence

$$\mathbb{D}(\mathbb{D}(\tau_{\geq k+1} M)) \to \mathbb{D}(\mathbb{D}(M))\, \mathbb{D}(\mathbb{D}(\tau_{\leq k} M)),$$

which induces an isomorphism $\pi_0 \mathbb{D}(\mathbb{D}(\tau_{\geq k+1} M)) \to \pi_0 \mathbb{D}(\mathbb{D}(M))$. Consequently, to prove that $\pi_0 M \to \pi_0 \mathbb{D}(\mathbb{D}(M))$ is an isomorphism, we may replace M by $\tau_{\geq k+1} M$. It will therefore suffice to prove that u_M is an equivalence whenever $M \in \mathcal{C}$ is almost connective. Replacing M by a shift if necessary, we are reduced to proving the following:

$(*)$ Let $M \in \mathcal{C}$ be connective. Then the map $u_M : M \to \mathbb{D}(\mathbb{D}(M))$ is an equivalence.

We prove by induction on p that u_M is p-connective whenever $M \in \mathcal{C}$ is connective. If M is connective, then $\mathbb{D}(M)$ is m-truncated, so that $\mathbb{D}(\mathbb{D}(M))$ is $(-n-m)$-connective. Our claim therefore follows automatically if $p < 0, -n-m$. We proceed in general using induction on p. Since $\pi_0 M$ is finitely generated as a $\pi_0 R$-module, we can choose a fiber sequence

$$N \to R^a \to M$$

where $N \in \mathcal{C}$ is connective. We therefore have a fiber sequence

$$\operatorname{fib}(u_N) \to \operatorname{fib}(u_R)^a \to \operatorname{fib}(u_M).$$

The definition of a dualizing complex guarantees that u_R is an equivalence, so we obtain an equivalence $\operatorname{fib}(u_M) \simeq \Sigma \operatorname{fib}(u_N)$. The inductive hypothesis implies that $\Sigma \operatorname{fib}(u_N)$ is $(p-1)$-connective, so that $\operatorname{fib}(u_M)$ is p-connective as desired.

We now prove (3). If M is almost perfect, then it is a k-connective for some integer k. Since K is m-truncated, we deduce that $\mathbb{D}(M)$ is $(m-k)$-truncated. Conversely, suppose that $\mathbb{D}(M)$ is k'-truncated for some integer k'. Then $M \simeq \mathbb{D}(\mathbb{D}(M))$ is $(-n-k')$-connective (Remark 4.2.3), and therefore almost perfect (since $M \in \mathcal{C}$). □

Lemma 4.2.8. *Let R be a Noetherian $\mathbb{E}_\infty$-ring, let K be a dualizing module for R, and let Q be an R-module of finite injective dimension. For every almost perfect R-module M, the canonical map*

$$f_M : M \otimes_R \underline{\operatorname{Map}}_R(K, Q) \simeq \underline{\operatorname{Map}}_R(\mathbb{D}(M), K) \otimes_R \underline{\operatorname{Map}}_R(K, Q) \to \underline{\operatorname{Map}}_R(\mathbb{D}(M), Q)$$

is an equivalence.

Proof. Replacing M by a shift, we may assume without loss of generality that M is connective. Let K be m-truncated and let Q have injective dimension $\leq n$. Remark 4.2.3 implies that $\underline{\operatorname{Map}}_R(K, Q)$ is $(-n-m)$-connective $M \otimes_R \underline{\operatorname{Map}}_R(K, Q)$ is $(-n-m)$-connective. Similarly, the connectivity of M implies that $\mathbb{D}(M)$ is m-truncated, so that $\underline{\operatorname{Map}}_R(\mathbb{D}(M), Q)$ is $(-n-m)$-connective. It follows that f_M is $(-n-m-1)$-connective. We prove that f_M is k-connective for every integer k, using induction on k. Since $\pi_0 M$ is a finitely generated module over $\pi_0 R$, we can choose a fiber sequence

$$M' \to R^a \to M,$$

where M' is connective. We therefore obtain a fiber sequence

$$\operatorname{fib}(f_{M'}) \to \operatorname{fib}(f_R)^a \to \operatorname{fib}(f_M).$$

It follows immediately from the definitions that f_R is an equivalence, so that $\operatorname{fib}(f_M) \simeq \Sigma \operatorname{fib}(f_{M'})$. The inductive hypothesis implies that $\operatorname{fib}(f_{M'})$ is $(k-1)$-conncetive, so that $\operatorname{fib}(f_M)$ is connective as desired. □

Proposition 4.2.9. *Let R be a Noetherian $\mathbb{E}_\infty$-ring, and let K be a dualizing module for R. Then an arbitrary R-module K' is a dualizing module if and only if there is an equivalence $K' \simeq K \otimes_R P$, where P is an invertible R-module.*

Proof. The "if" direction is obvious. To prove the converse, suppose that K' is a dualizing module for R. Then K' has finite injective dimension. Let $P = \underline{\operatorname{Map}}_R(K, K')$. It follows from Lemma 4.2.8 that for every almost perfect R-module M, the canonical map $M \otimes_R P \to \underline{\operatorname{Map}}_R(\mathbb{D}(M), K')$ is an equivalence. Taking $M = \underline{\operatorname{Map}}_R(K', K)$ (which is almost perfect by Theorem 4.2.7), we deduce that $M \otimes_R P \simeq \underline{\operatorname{Map}}_R(K', K') \simeq R$, so that P is invertible. To complete the proof, it will suffice to show that the canonical map $K \otimes_R P \to K'$ is an equivalence. This is a consequence of the following more general assertion (applied in the case $N = K$):

($*$) Let N be an R-module such that each homotopy group $\pi_i N$ is finitely generated over $\pi_0 R$. Then the canonical map $u_N : N \otimes_R P \to \underline{\operatorname{Map}}_R(\mathbb{D}(N), K')$ (appearing in Lemma 4.2.8) is an equivalence.

To prove $(*)$, it will suffice to show that u_N induces an isomorphism

$$\pi_i(N \otimes_R P) \to \pi_i \underline{\mathrm{Map}}_R(\mathbb{D}(N), K')$$

for every integer i. Replacing N by a shift, we may suppose that $i = 0$. Let $M = \underline{\mathrm{Map}}_R(K', K)$ be as above, so we can rewrite the domain of u_N as $\underline{\mathrm{Map}}_R(M, N)$. Choose an integer a such that M is a-connective. For every integer k, the spectrum $\underline{\mathrm{Map}}_A(M, \tau_{\leq k} N)$ is $(a+k)$-truncated. Let K have injective dimension $\leq n$, so that $\mathbb{D}(\tau_{\leq k} N)$ is $(-n-k)$-connective (Remark 4.2.3). Choose m such that K' is m-truncated, so that $\underline{\mathrm{Map}}_R(\mathbb{D}(\tau_{\leq k} N), K')$ is $(k+m+n)$-truncated. It follows that if $k \leq -a, -m-n$, then the maps

$$\pi_0(\tau_{\geq k+1} N \otimes_R P) \to \pi_0(N \otimes_R P)$$

$$\pi_0 \underline{\mathrm{Map}}_R(\mathbb{D}(\tau_{\geq k+1} N), K') \to \pi_0 \underline{\mathrm{Map}}_R(\mathbb{D}(N), K')$$

are isomorphisms. We may therefore replace N by $\tau_{\geq k+1} N$, in which case the desired result follows from Lemma 4.2.8. □

Remark 4.2.10. In the situation of Proposition 4.2.9, the invertible module P is unique up to equivalence. In fact, we have the following more precise assertion: if R is a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module K, then the functor $P \mapsto P \otimes_R K$ induces an equivalence from the full subcategory of Mod_R spanned by the invertible R-modules to the full subcategory of Mod_R spanned by the dualizing modules. To prove this, it suffices to observe that if P and Q are invertible, then the canonical map

$$\underline{\mathrm{Map}}_R(P, Q) \to \underline{\mathrm{Map}}_R(P \otimes_R K, Q \otimes_R K)$$

is an equivalence. This reduces easily to assumption (2) of Definition 4.2.5.

4.3 Existence of Dualizing Modules

In §4.2, we introduced the notion of a *dualizing module* for a Noetherian $\mathbb{E}_\infty$-ring A. Moreover, we proved that if A admits a dualizing module K, then K is essentially unique (up to tensor product with an invertible A-module; see Proposition 4.2.9). In this section, we will discuss the existence problem for dualizing modules. Our results will show that dualizing modules almost always exist in practical situations. Our main results can be formulated more precisely as follows:

- (a) If A is a Noetherian $\mathbb{E}_\infty$-ring, then A admits a dualizing module if and only if the ordinary commutative ring $\pi_0 A$ admits a dualizing module (Theorem 4.3.5).
- (b) If R is a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module and A is an $\mathbb{E}_\infty$-ring which is almost of finite presentation over R, then A also admits a dualizing module (Theorem 4.3.14).

Remark 4.3.1. The commutative ring $\mathbf{Z}$ is Gorenstein, and is therefore a dualizing module for itself (Example 4.2.6). It follows from (a) that the sphere spectrum S admits a dualizing module (see Example 4.3.9). Using (b), we deduce that every $\mathbb{E}_\infty$-ring which is almost of finite presentation over S admits a dualizing module. In particular, every connective $\mathbb{E}_\infty$-ring R can be written as a filtered colimit of Noetherian $\mathbb{E}_\infty$-rings which admit dualizing modules. This can be quite useful in combination with the Noetherian approximation techniques of §XII.2.

Our first step is to prove the following special case of (b): if $\phi : R \to A$ is a map of Noetherian $\mathbb{E}_\infty$-rings which exhibits A as an almost perfect R-module and R admits a dualizing module, then A admits a dualizing module. In fact, the dualizing module of A admits a very explicit description.

Notation 4.3.2. Let $f : A \to B$ be a map of $\mathbb{E}_\infty$-rings. Then the forgetful functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$ preserves small colimits, and therefore admits a right adjoint $f^\dagger : \mathrm{Mod}_A \to \mathrm{Mod}_B$ (Corollary T.5.5.2.9). For any A-modules M and N, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_A}(N, f^\dagger M) \simeq \mathrm{Map}_{\mathrm{Mod}_B}(B \otimes_A N, f^\dagger M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(B \otimes_A N, M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(N, \underline{\mathrm{Map}}_A(B, M)).$$

In other words, the composite functor

$$\mathrm{Mod}_A \xrightarrow{f^\dagger} \mathrm{Mod}_B \to \mathrm{Mod}_A$$

is given by $M \mapsto \underline{\mathrm{Map}}_A(B, M)$. We will generally abuse notation by identifying the functor $f^\dagger$ with $M \mapsto \underline{\mathrm{Map}}_A(B, M)$. We can informally summarize the situation as follows: for every A-module M, the object $\underline{\mathrm{Map}}_A(B, M)$ admits the structure of a B-module which is *universal* among those B-modules P which admit an A-module map $P \to M$.

Proposition 4.3.3. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module K. Let $\phi : R \to A$ be a map of connective $\mathbb{E}_\infty$-rings which exhibits A as an almost perfect R-module. Then $K' = \underline{\mathrm{Map}}_R(A, K)$ is a dualizing module for A. In particular, A admits a dualizing module.*

Proof. It is clear that K' is truncated, and Theorem 4.2.7 implies that the homotopy groups of K' are finitely generated as modules over $\pi_0 R$ (and therefore also as modules over $\pi_0 A$). The canonical map

$$A \to \underline{\mathrm{Map}}_A(K', K') \simeq \underline{\mathrm{Map}}_A(K', \underline{\mathrm{Map}}_R(A, K)) \simeq \underline{\mathrm{Map}}_R(K', K)$$

can be identified with the double duality map $A \to \mathbb{D}(\mathbb{D}(A))$, which is an equivalence by Theorem 4.2.7. To complete the proof, it will suffice to show that K' has finite injective dimension over A. Let M be a discrete A-module. Then $\mathrm{Ext}^i_A(M, K') \simeq \mathrm{Ext}^i_R(M, K)$ vanishes for $i > n$, where n is the injective dimension of K over R. $\square$

Corollary 4.3.4. *Let R be a Noetherian $\mathbb{E}_\infty$-ring. If R admits a dualizing module, then the commutative ring $\pi_0 R$ admits a dualizing module.*

We next prove the converse of Corollary 4.3.4:

Theorem 4.3.5. *Let R be a Noetherian $\mathbb{E}_\infty$-ring. If $\pi_0 R$ admits a dualizing module, then R admits a dualizing module.*

The proof of Theorem 4.3.5 will require some preliminaries.

Lemma 4.3.6. *Let A be a Noetherian $\mathbb{E}_\infty$-ring, let M be an almost perfect A-module, and let N be a 0-truncated A-module whose homotopy groups are finitely generated over $\pi_0 A$. Then the groups $\mathrm{Ext}^i_A(M, N)$ are finitely generated modules over $\pi_0 A$.*

Proof. Shifting M if necessary, we are reduced to proving that $\mathrm{Ext}^0_A(M, N)$ is finitely generated over $\pi_0 A$. Choose an integer m such that M is m-connective. We proceed by descending induction on m. Note that if $m > 0$, then $\mathrm{Ext}^0_A(M, N) \simeq 0$ and there is nothing to prove. Otherwise, the assumption that M is almost perfect guarantees the existence of a fiber sequence

$$A^k[m] \to M \to M'$$

where M' is $(m+1)$-connective. We have an exact sequence of $\pi_0 A$-modules

$$\mathrm{Ext}^0_A(M', N) \to \mathrm{Ext}^0_A(M, N) \to \mathrm{Ext}^0_A(A^k[m], N).$$

The inductive hypothesis implies that $\mathrm{Ext}^0_A(M', N)$ is finitely generated over $\pi_0 A$, and we have a canonical isomorphism $\mathrm{Ext}^0_A(A^k[m], N) \simeq (\pi_m N)^k$. It follows that $\mathrm{Ext}^0_A(M, N)$ is finitely generated over $\pi_0 A$. $\square$

Lemma 4.3.7. *Let $f : A \to B$ be a map of Noetherian $\mathbb{E}_\infty$-rings, Suppose that the induced map $\pi_0 A \to \pi_0 B$ is a surjection of commutative rings whose kernel $I \subseteq \pi_0 A$ is nilpotent. Let K be a truncated A-module, and suppose that the homotopy groups $\pi_i \underline{\mathrm{Map}}_A(B, K)$ are finitely generated modules over $\pi_0 B$. Then the homotopy groups $\pi_i K$ are finitely generated modules over $\pi_0 A$.*

Proof. We may assume without loss of generality that K is 0-truncated. We prove that the homotopy groups $\pi_{-n} K$ are finitely generated over $\pi_0 A$ using induction on n, the case $n < 0$ being trivial. For each integer $k \geq 1$, let $M(k)$ denote the submodule of $\pi_0 K$ consisting of elements which are annihilated by I^k. Since $\pi_0 A$ is Noetherian, there exists a finite set of generators $x_1, \ldots, x_n$ for the ideal I. Multiplication by the elements x_i determines a map $M(k) \to M(k-1)^n$, which fits into an exact sequence

$$0 \to M(1) \to M(k) \to M(k-1)^n$$

Note that $M(1) \simeq \pi_0 \underline{\mathrm{Map}}_A(B, K)$ is finitely generated over $\pi_0 A$. It follows by induction on k that each $M(k)$ is finitely generated over $\pi_0 A$. Since the ideal I is nilpotent, we have $M(k) \simeq \pi_0 K$ for $k \gg 0$, so that $\pi_0 K$ is finitely generated over $\pi_0 A$. This completes the proof when $n = 0$. If $n > 0$, we apply Lemma 4.3.6 to deduce that the homotopy groups of $\underline{\mathrm{Map}}_A(B, \pi_0 K)$ are finitely generated over $\pi_0 A$, and therefore finitely generated over $\pi_0 B$. Let $K' = \Sigma(\tau_{\leq -1} K)$, so that we have a fiber sequence of A-modules

$$\pi_0 K \to K \to \Sigma^{-1} K'.$$

It follows that the homotopy groups of $\pi_0 \underline{\mathrm{Map}}_A(B, K')$ are finitely generated over $\pi_0 B$. Applying the inductive hypothesis, we deduce that $\pi_{-n} K \simeq \pi_{1-n} K'$ is finitely generated over $\pi_0 A$, as desired. □

Proposition 4.3.8. *Let $f : A \to B$ be a map of Noetherian $\mathbb{E}_\infty$-rings. Suppose that the induced map $\pi_0 A \to \pi_0 B$ is a surjection of commutative rings whose kernel $I \subseteq \pi_0 A$ is nilpotent. Let K be a truncated A-module, and suppose that $\underline{\mathrm{Map}}_A(B, K)$ is a dualizing module for B. Then K is a dualizing module for A.*

Example 4.3.9. Let S denote the sphere spectrum. Since $\pi_0 S \simeq \mathbf{Z}$ admits a dualizing module, S also admits a dualizing module. In fact, we can describe this dualizing module explicitly. Let $I \in \mathrm{Sp}$ be the Brown-Comenetz dual of the sphere spectrum (see Example A.8.2.3.9). Then I is an injective object of Sp, which is characterized up to equivalence by the formula

$$\pi_n \underline{\mathrm{Map}}_S(M, I) \simeq \mathrm{Hom}(\pi_{-n} M, \mathbf{Q}/\mathbf{Z})$$

for every integer n and every spectrum M. In particular, we have

$$\pi_n I \simeq \begin{cases} 0 & \text{if } n > 0 \\ \mathbf{Q}/\mathbf{Z} & \text{if } n = 0 \\ \mathrm{Hom}(\pi_{-n} S, \mathbf{Q}/\mathbf{Z}) & \text{if } n < 0. \end{cases}$$

Let $\mathbf{Q}$ denote the field of rational numbers, which we regard as a discrete spectrum. The map of abelian groups $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ induces a map of spectra $\alpha : \mathbf{Q} \to I$. We let K denote the fiber of α. Then

$$\underline{\mathrm{Map}}_S(\mathbf{Z}, K) \simeq \mathrm{fib}(\underline{\mathrm{Map}}_S(\mathbf{Z}, \mathbf{Q}) \to \underline{\mathrm{Map}}_S(\mathbf{Z}, I)) \simeq \mathrm{fib}(\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}) \simeq \mathbf{Z}$$

is a dualizing module for $\mathbf{Z}$. It follows from Proposition 4.3.8 that K is a dualizing module for S. The spectrum K is often called the *Anderson dual* of the sphere spectrum. Its homotopy groups are given by

$$\pi_n K \simeq \begin{cases} 0 & \text{if } n > 0 \\ \mathbf{Z} & \text{if } n = 0 \\ 0 & \text{if } n = -1 \\ \mathrm{Hom}(\pi_k S, \mathbf{Q}/\mathbf{Z}) & \text{if } n = -k-1, k > 0. \end{cases}$$

Proof of Proposition 4.3.8. It follows from Lemma 4.3.7 that the homotopy groups of $\pi_i K$ are finitely generated over $\pi_0 A$. Suppose that $\underline{\mathrm{Map}}_A(B, K)$ has injective dimension $\leq n$ as a B-module. We claim that K has injective dimension $\leq n$ as an A-module. Let M be a discrete A-module; we wish to prove that the groups $\mathrm{Ext}^i_A(M, K)$ vanish for $i > n$. Since I is nilpotent, the module M is annihilated by I^k for some integer $k \geq 1$. We proceed by induction on k. We have an exact sequence

$$0 \to IM \to M \to M/IM \to 0$$

of discrete $\pi_0 A$-modules, giving rise to short exact sequences

$$\mathrm{Ext}^i_A(M/IM, K) \to \mathrm{Ext}^i_A(M, K) \to \mathrm{Ext}^i_A(IM, K).$$

The groups $\mathrm{Ext}^i_A(IM, K)$ vanish for $i > n$ by the inductive hypothesis. The quotient M/IM has the structure of a module over $\pi_0 B$, so that $\mathrm{Ext}^i_A(IM, K) \simeq \mathrm{Ext}^i_B(IM, \underline{\mathrm{Map}}_A(B, K))$ vanishes since $\underline{\mathrm{Map}}_A(B, K)$ has injective dimension $\leq n$. It follows that $\mathrm{Ext}^i_A(M, K)$ vanishes, as desired.

To complete the proof, it will suffice to show that the canonical map $A \to \underline{\mathrm{Map}}_A(K, K)$ is an equivalence. We will prove more generally that for every almost perfect A-module M, the canonical map $u_M : M \to \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K)$ is an equivalence. For this, it suffices to show thatu_M induces an isomorphism $\pi_i M \to \pi_i \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K)$ for every integer i. Replacing M by a shift, we can assume that $i = 0$. Choose an integer m such that K is m-truncated. For every integer k, the module $\underline{\mathrm{Map}}_A(\tau_{\geq k} M, K)$ is $(m-k)$-truncated, so that $\underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(\tau_{\geq k} M, K), K)$ is $(k - n - m)$-connective (Remark 4.2.3). If $k > n + m$, it follows that the canonical map

$$\pi_0 \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K) \to \pi_0 \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(\tau_{\leq k-1} M, K), K)$$

is an isomorphism. Assuming also that k is positive (so that $\pi_0 M \simeq \pi_0 \tau_{\leq k-1} M$), we may replace M by $\tau_{\leq k-1} M$ and thereby reduce to the case where M is truncated. It will therefore suffice to show that u_M is an equivalence whenever M is truncated and almost perfect. In this case, M is a successive extension of A-modules which are concentrated in a single degree. It will therefore suffice to show that u_M is an equivalence when M is discrete A-module which is finitely generated over $\pi_0 A$. Since I is nilpotent, we can write M as a successive extension of discrete A-modules which are annihilated by I. We may therefore assume that M is annihilated by I, and therefore admits the structure of a B-module. In this case, we have

$$\underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_A(M, K), K) \simeq \underline{\mathrm{Map}}_A(\underline{\mathrm{Map}}_B(M, K'), K) \simeq \underline{\mathrm{Map}}_B(\underline{\mathrm{Map}}_B(M, K'), K'),$$

where $K' = \underline{\mathrm{Map}}_A(B, K)$. The assertion that u_M is an equivalence now follows from Theorem 4.2.7. □

Notation 4.3.10. Let A be a connective $\mathbb{E}_\infty$-ring. We let $(\mathrm{Mod}_A)_{<\infty}$ denote the full subcategory of Mod_A spanned by the truncated A-modules (that is, $(\mathrm{Mod}_A)_{<\infty} = \bigcup_n (\mathrm{Mod}_A)_{\leq n}$). Note that if $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings, then the functor $f^\dagger : \mathrm{Mod}_A \to \mathrm{Mod}_B$ of Remark 4.3.2 carries $(\mathrm{Mod}_A)_{<\infty}$ into $(\mathrm{Mod}_B)_{<\infty}$.

Lemma 4.3.11. *Suppose we are given a pullback diagram of connective $\mathbb{E}_\infty$-rings τ :*

$$\begin{array}{ccc} A' & \xrightarrow{f'} & A \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow{f} & B, \end{array}$$

where f and g induce surjections $\pi_0 A \to \pi_0 B$, $\pi_0 B' \to \pi_0 B$. Then the induced diagram σ :

$$\begin{array}{ccc} (\mathrm{Mod}_{A'})_{<\infty} & \xrightarrow{f'^\dagger} & (\mathrm{Mod}_A)_{<\infty} \\ \downarrow{\scriptstyle g'^\dagger} & & \downarrow{\scriptstyle g^\dagger} \\ (\mathrm{Mod}_{B'})_{<\infty} & \xrightarrow{f^\dagger} & (\mathrm{Mod}_B)_{<\infty} \end{array}$$

is a pullback square of ∞-categories.

Proof. Let $\mathcal{C}$ denote the fiber product $(\mathrm{Mod}_A)_{<\infty} \times_{(\mathrm{Mod}_B)_{<\infty}} (\mathrm{Mod}_{B'})_{<\infty}$. Unwinding the definitions, we can identify the objects of $\mathcal{C}$ with triples (M, N, α), where M is a truncated A-module, N is a truncated B'-module, and $\alpha : \underline{\mathrm{Map}}_A(B, M) \to \underline{\mathrm{Map}}_{B'}(B, N)$ is an equivalence of B'-modules. The diagram σ determines a functor $G : (\mathrm{Mod}_{A'})_{<\infty} \to \mathcal{C}$; we wish to prove that G is an equivalence. We note that G has a left adjoint F, given on objects by the formula $F(M, N, \alpha) = M \coprod_{\underline{\mathrm{Map}}_A(B,M)} N$. We first prove that the counit map $v : F \circ G \to \mathrm{id}$ is an equivalence from $(\mathrm{Mod}_{A'})_{<\infty}$ to itself. Unwinding the definitions, we must show that if M is a truncated A'-module, then the diagram

$$\begin{array}{ccc} \underline{\mathrm{Map}}_{A'}(B, M) & \longrightarrow & \underline{\mathrm{Map}}_{A'}(A, M) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Map}}_{A'}(B', M) & \longrightarrow & \underline{\mathrm{Map}}_{A'}(A', M) \end{array}$$

is a pushout square of A'-modules. This follows from our assumption that τ is a pullback square.

Since v is an equivalence, we deduce that the functor G is fully faithful. To complete the proof, it will suffice to show that F is conservative. Since F is an exact functor between stable ∞-categories, it will suffice to show that if (M, N, α) is an object of $\mathcal{C}$ which is annihilated by F, then M and N are both trivial. Suppose otherwise. Then there exists a smallest integer n such that $\pi_i M \simeq \pi_i N \simeq 0$ for $i > n$. Then $\pi_n M$ and $\pi_n N$ cannot both vanish; without loss of generality, we may assume that $\pi_n N \neq 0$. We have an exact sequence

$$0 \to \pi_n \underline{\mathrm{Map}}_A(B, M) \to \pi_n M \oplus \pi_n N \to \pi_n F(M, N, \alpha).$$

Since $\pi_n F(M, N, \alpha)$ vanishes, we deduce that the map $\pi_n \underline{\mathrm{Map}}_A(B, M) \to \pi_n M \oplus \pi_n N$ is an isomorphism. This contradicts the nontriviality of $\pi_n N$, since the map $\pi_n \underline{\mathrm{Map}}_A(B, M) \simeq \pi_n M$ is injective (because the homotopy groups $\pi_i M$ vanish for $i > n$). ☐

Lemma 4.3.12. *Let B be a Noetherian $\mathbb{E}_\infty$-ring, and let A be a square-zero extension of B by a connective, almost perfect B-module M. Suppose that B admits a dualizing module K. Then there exists a dualizing module K' for A and an equivalence $K \simeq \underline{\mathrm{Map}}_A(B, K')$.*

Proof. We will show that there exists a truncated A-module K' and an equivalence $K \simeq \underline{\mathrm{Map}}_A(B, K')$. Then K' is automatically a dualizing module for A, by Proposition 4.3.8. We have a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \eta \\ B & \xrightarrow{\eta_0} & B \oplus \Sigma M. \end{array}$$

By virtue of Lemma 4.3.11, it will suffice to show that $\eta^\dagger K$ and $\eta_0^\dagger K$ are equivalent as modules over $B \oplus \Sigma M$. Both $\eta^\dagger K$ and $\eta_0^\dagger K$ are dualizing modules for $B \oplus \Sigma M$ (Proposition 4.3.3). It follows that there exists an invertible module P for $B \oplus \Sigma M$ and an equivalence $\eta_0^\dagger K \simeq \eta^\dagger K \otimes_{B \oplus \Sigma M} P$. To complete the proof, it will suffice to show that P is trivial. Let $p : B \oplus \Sigma M \to B$ denote the projection map. Then

$$K \simeq p^\dagger \eta_0^\dagger K \simeq p^\dagger(\eta^\dagger K \otimes_{B \oplus \Sigma M} P) \simeq (p^\dagger \eta^\dagger K) \otimes_{B \oplus \Sigma M} P \simeq K \otimes_{B \oplus \Sigma M} P.$$

Invoking Remark 4.2.10, we deduce that $P \otimes_{B \oplus \Sigma M} B$ is equivalent to B (as an B-module). In particular, there exists an isomorphism

$$\pi_0 P \simeq \pi_0(P \otimes_{B \oplus \Sigma M} B) \simeq \pi_0 B.$$

Lifting the unit element of $\pi_0 B$ under such an isomorphism, we obtain an element $e \in \pi_0 P$, which determines a map $\gamma : B \oplus \Sigma M \to P$ of $(B \oplus \Sigma M)$-modules. Then $\mathrm{fib}(\gamma) \otimes_{B \oplus \Sigma M} B$ vanishes, so that $\mathrm{fib}(\gamma) \otimes_{B \oplus \Sigma M} N \simeq 0$

whenever N admits the structure of a B-module. SInce $B \oplus \Sigma M$ can be obtained as an extension of two $(B \oplus \Sigma M)$-modules which admit B-module structures, we deduce htat

$$\mathrm{fib}(\gamma) \simeq \mathrm{fib}(\gamma) \otimes_{B \oplus \Sigma M} (B \oplus \Sigma M)$$

vanishes, so that γ is an equivalence and $P \simeq B \oplus \Sigma M$ as desired. □

Proof of Theorem 4.3.5. Let R be a Noetherian $\mathbb{E}_\infty$-ring, and let $K(0)$ be a dualizing module for $\pi_0 R$. Without loss of generality, we may assume that $K(0)$ is 0-truncated. We will show that there exists a 0-truncated R-module K and an equivalence $K(0) \simeq \underline{\mathrm{Map}}_R(\pi_0 R, K)$. It will then follow from Proposition 4.3.8 that K is a dualizing module for R.

Since each truncation $\tau_{\leq n+1} R$ is a square-zero extension of $\tau_{\leq n} R$, Lemma 4.3.12 allows us to choose a sequence $K(n)$ of dualizing modules for $\tau_{\leq n} R$, together with equivalences

$$K(n) \simeq \underline{\mathrm{Map}}_{\tau_{\leq n+1} R}(\tau_{\leq n} R, K(n+1)).$$

It then follows by induction on n that each $K(n)$ is 0-truncated. Moreover, we have canonical fiber sequences

$$K(n-1) \xrightarrow{\beta_n} K(n) \to \underline{\mathrm{Map}}_{\tau_{\leq n+1} R}(\Sigma^n (\pi_n R), K(n)),$$

so that $\mathrm{cofib}(\beta_n)$ is $(-n)$-truncated for every integer n. Let $K = \varinjlim_n K(n)$, where the colimit is taken in the ∞-category Mod_R. Then K is a 0-truncated R-module, and we have a canonical map of $\pi_0 R$-modules $\alpha : K(0) \to K$. We will complete the proof by showing that α induces an equivalence $e : K(0) \to \underline{\mathrm{Map}}_R(\pi_0 R, K)$. Fix an integer $n \geq 0$, so that e is given by the composition

$$K(0) \simeq \underline{\mathrm{Map}}_{\tau_{\leq n} R}(\pi_0 R, K(n)) \xrightarrow{e'} \underline{\mathrm{Map}}_{\tau_{\leq n} R}(\pi_0 R, \underline{\mathrm{Map}}_R(\tau_{\leq n} R, K)) \simeq \underline{\mathrm{Map}}_R(\pi_0 R, K).$$

Here e' is induced by the map $f : K(n) \to \underline{\mathrm{Map}}_R(\tau_{\leq n} R, K)$. Let $f' : \underline{\mathrm{Map}}_R(\tau_{\leq n} R, K) \to K$ be the canonical map, so that $\mathrm{cofib}(f') \simeq \underline{\mathrm{Map}}_R(\tau_{\geq n+1} R, K)$ is $(-n-1)$-truncated. Since each of the maps β_m has $(-m)$-truncated cofiber, we deduce that $\mathrm{cofib}(f' \circ f)$ is $(-n-1)$-truncated. It follows that $\mathrm{cofib}(f)$ is $(-n-1)$-truncated, so that $\mathrm{cofib}(e') \simeq \mathrm{cofib}(e)$ is $(-n-1)$-truncated. Since we can choose n to be arbitrarily large, we deduce that e is an equivalence. □

Remark 4.3.13. Let R be a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module. Then $\pi_0 R$ admits a dualizing module (Corollary 4.3.4). In more classical language, this dualizing module is a *dualizing complex* in the sense of Grothendieck. The existence of such a dualizing complex implies that the commutative ring $\pi_0 R$ has finite Krull dimension (see [31]). We shall say that R is *of finite Krull dimension* if the commutative ring $\pi_0 R$ is of finite Krull dimension, so that any Noetherian $\mathbb{E}_\infty$-ring R which admits a dualizing module is necessarily of finite Krull dimension.

We now turn to the main result of this section.

Theorem 4.3.14. *Let R be a Noetherian $\mathbb{E}_\infty$-ring, and let $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ be almost of finite presentation over R. If R admits a dualizing module, then A admits a dualizing module.*

Remark 4.3.15. Using Theorem 4.3.5 and Remark 4.3.13, we see that Theorem 4.3.14 reduces to the following statement in commutative algebra: if R is a Noetherian ring which admits a dualizing complex, then any finitely generated R-algebra admits a dualizing complex. This statement is classical; we include a proof below for completeness.

Lemma 4.3.16. *Let R be a Noetherian $\mathbb{E}_\infty$-ring and let K be a truncated R-module. Then K has injective dimension $\leq n$ if and only if, for every finitely generated discrete R-module M and every integer $i > n$, the abelian group $\mathrm{Ext}^i_R(M, K)$ vanishes.*

Proof. Replacing K by $\Sigma^n K$, we may suppose that $n = 0$. Replacing R by $\pi_0 R$ and K by $\underline{\mathrm{Map}}_R(\pi_0 R, K)$, we may suppose that R is discrete. Then we can identify $(\mathrm{Mod}_R)_{<\infty}$ with the derived ∞-category of the abelian category $\mathrm{Mod}_R^{\heartsuit}$ of discrete R-modules (Proposition A.8.1.1.15). Since K is truncated, it can be represented by a chain complex

$$0 \to I_m \to I_{m-1} \to \dots$$

of discrete, injective R-modules. Let I denote the R-module represented by the complex

$$0 \to I_m \to I_{m-1} \to \dots \to I_1 \to 0 \to \dots.$$

Then I is evidently of injective dimension ≤ -1 and there is a triangle $I' \to K \to I$, where I' is 0-truncated. To complete the proof, it suffices to show that I' is discrete and is an injective object in the abelian category $\mathrm{Mod}_R^{\heartsuit}$.

For any discrete R-module M, we obtain a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(M, I') \to \mathrm{Ext}^i_R(M, K) \to \mathrm{Ext}^i_R(M, I) \to \mathrm{Ext}^{i+1}_R(M, I') \to \cdots,$$

which gives an isomorphism $\mathrm{Ext}^i_R(M, K) \simeq \mathrm{Ext}^i_R(M, I')$ for $i \geq 0$. It follows that $\mathrm{Ext}^i_R(M, I')$ vanishes when $i > 0$ and M is finitely generated. Taking $M = R$, we deduce that I' is connective. Since I' is 0-truncated, we conclude that it is a discrete R-module.

If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of finitely generated discrete R-modules, then the vanishing of $\mathrm{Ext}^1_R(M', R)$ implies that the induced sequence

$$0 \to \mathrm{Ext}^0_R(M'', I') \to \pi_0 \mathrm{Ext}^0_R(M, I') \to \mathrm{Ext}^0_R(M', I') \to 0$$

is exact. Since R is Noetherian, it follows that the induced map $I' \to \mathrm{Hom}_{\mathrm{Mod}_R}(J, I')$ is surjective for any ideal $J \subseteq I$. Using Zorn's lemma, we deduce that I' is injective. $\square$

Proof of Theorem 4.3.14. It follows from Theorem A.8.2.5.31 that A is Noetherian. Using Corollary 4.3.4 and Theorem 4.3.5, we can replace R and A by $\pi_0 R$ and $\pi_0 A$, and thereby reduce to the case where A is a commutative ring which is finitely presented as an algebra over a Noetherian commutative ring R. Choose a surjection $R[x_1, \dots, x_n] \to A$. To prove that A admits a dualizing module, it will suffice to show that $R[x_1, \dots, x_n]$ admits a dualizing module (Proposition 4.3.8). Proceeding by induction on n, we are reduced to proving the following:

$(*)$ Let R be a Noetherian commutative ring. If R admits a dualizing module K, then $R[x]$ admits a dualizing module.

To prove $(*)$, it will suffice to show that $K[x] = R[x] \otimes_R K$ is a dualizing module for K. It is clear that $K[x]$ is truncated and that its homotopy groups are finitely generated as discrete modules over $R[x]$. We next claim that the formation of $K[x]$ is compatible with "finite" base change in R. Namely, suppose that $R \to R'$ expresses R' as an almost perfect R-module. Then $K' = \underline{\mathrm{Map}}_R(R', K)$ is a dualizing module for R' (Proposition 4.3.8). We claim that the natural map $K'[x] \to \underline{\mathrm{Map}}_{R[x]}(R'[x], K[x])$ is an equivalence. We may rewrite the target as $\underline{\mathrm{Map}}_R(R', K[x]) = \underline{\mathrm{Map}}_R(R', \oplus_{n \geq 0} K)$. Since R' is almost perfect as an R-module, $\mathrm{Hom}_{\mathrm{Mod}_R}(R', \bullet)$ commutes with infinite direct sums when restricted to m-truncated modules, for every integer m. Since K is truncated, the claim follows.

We now prove that $K[x]$ has finite injective dimension as an $R[x]$-module. Suppose that K is of injective dimension $\leq n$. The existence of K implies that R has finite Krull dimension (Remark 4.3.13). It follows that $R[x]$ has Krull dimension $\leq m$ for some m. We will show that $K[x]$ has injective dimension $\leq n+m+1$. By virtue of Lemma 4.3.16, it will suffice to show that if M is a finitely generated discrete $R[x]$-module, then the groups $\mathrm{Ext}^i_{R[x]}(M, K[x])$ vanish for $i > n + m + 1$. We will show, more generally, that if the support of

M has Krull dimension $\leq j$, then $\mathrm{Ext}^i_{R[x]}(M, K[x])$ vanishes for $i > n + j + 1$. We prove this by induction on j. Filtering M and working by induction, we may suppose that $M \simeq R[x]/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of $R[x]$. Let $\mathfrak{q} = \mathfrak{p} \cap R$. Replacing R by $R/\mathfrak{q}$, we may reduce to the case where R is an integral domain and $\mathfrak{q} = 0$. If $\mathfrak{p} = 0$, then $\mathrm{Ext}^i_{R[x]}(M, K[x]) \simeq \pi_{-i} K[x]$, which vanishes for $i > n$ by virtue of our assumption that K has injective dimension $\leq n$.

If $\mathfrak{p} \neq 0$, then we may choose an element $y \in \mathfrak{p}$ which generates $\mathfrak{p}$ after tensoring with fraction field of R. Then there is an exact sequence

$$0 \to N \to R[x]/(y) \to M \to 0.$$

To show that $\mathrm{Ext}^i_{R[x]}(M, K[x])$ vanishes, it will suffice to show that that the groups $\mathrm{Ext}^i_{R[x]}(R[x]/(y), K[x])$ and $\mathrm{Ext}^{i-1}_{R[x]}(N, K[x])$ are trivial. Since the support of N has Krull dimension strictly less than j, the vanishing of $\mathrm{Ext}^{i-1}_{R[x]}(N, K[x])$ follows from the inductive hypothesis (provided that $i > n + j + 1$). Using the exact sequence $0 \to R[x] \to R[x] \to R[x]/(y) \to 0$, we obtain an exact sequence

$$\mathrm{Ext}^{i-1}_{R[x]}(R[x], K[x]) \to \mathrm{Ext}^i_{R[x]}(R[x]/(y), K[x]) \to \mathrm{Ext}^i_{R[x]}(R[x], K[x]).$$

Note that we have $\mathrm{Ext}^j_{R[x]}(R[x], K[x]) \simeq \mathrm{Ext}^j_R(R, K[x]) \simeq \mathrm{Ext}^j_R(R, K)[x]$. Since $i - 1 > n$, the desired vanishing follows from our assumption that K has injective dimension $\leq n$.

To complete the proof of $(*)$, it will suffice to show that the canonical map $R[x] \to \underline{\mathrm{Map}}_{R[x]}(K[x], K[x])$ is an equivalence. Since R is discrete, the assumption that K has injective dimension $\leq n$ implies that K is $(-n)$-connective. It follows that K is almost perfect, so that the construction $M \mapsto \underline{\mathrm{Map}}_R(K, M)$ commutes with filtered colimits when restricted to m-truncated R-modules for every m. It follows that the canonical map

$$R[x] \simeq \oplus_{n \geq 0} R \simeq \oplus_{n \geq 0} \underline{\mathrm{Map}}_R(K, K) \to \underline{\mathrm{Map}}_R(K, \oplus_{n \geq 0} K) \simeq \underline{\mathrm{Map}}_R(K, K[x]) \simeq \underline{\mathrm{Map}}_{R[x]}(K[x], K[x])$$

is an equivalence, as desired. $\square$

4.4 A Linear Representability Criterion

Let R be a Noetherian $\mathbb{E}_\infty$-ring, let $f : X \to \mathrm{Spec}^{\mathrm{f}} R$ be a natural transformation of functors, and let $X_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ be the functor given by $X_0(A) = \mathrm{fib}(X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A))$. Reasoning as in Example 1.3.15, we see that f admits a relative cotangent complex if and only if it satisfies the following pair of conditions:

- (a) For every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, define $F_\eta : \mathrm{Mod}^{\mathrm{cn}}_A \to \mathcal{S}$ by the formula $F_\eta(M) = X_0(A \oplus M) \times_{X_0(A)} \{\eta\}$. Then F_η is almost corepresentable.
- (b) For every map of connective $\mathbb{E}_\infty$-rings $A \to B$ and every connective B-module M, the diagram of spaces

$$\begin{array}{ccc} X_0(A \oplus M) & \longrightarrow & X_0(B \oplus M) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

  is a pullback square.

Under some mild hypotheses on X, Proposition 4.1.6 asserts that condition (b) is equivalent to the requirement that the tangent complexes of f are preserved by flat base change. Our goal in this section is to prove a result which is useful for verifying condition (a). First, we need a definition.

Definition 4.4.1. Let R be a Noetherian $\mathbb{E}_\infty$-ring. We will say that an R-module M is *coherent* if it is truncated and almost perfect (that is, if and only if the homotopy groups $\pi_i M$ are finitely generated modules over $\pi_0 R$, which vanish for almost every integer i). We let $\mathrm{Mod}_R^{\mathrm{coh}}$ denote the full subcategory of Mod_R spanned by the coherent R-modules.

We can now state our main result.

Theorem 4.4.2. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module K, and let $F : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then F is corepresentable by an almost perfect (not necessarily connective) R-module if and only if the following conditions are satisfied:*

(1) *The functor F is reduced and excisive (and therefore admits an essentially unique extension to a left exact functor $F^+ : \mathrm{Mod}_R^{\mathrm{acn}} \to \mathcal{S}$, by Lemma 1.3.2).*

(2) *For every connective R-module M, the canonical map $F(M) \to \varprojlim F(\tau_{\leq n} M)$ is an equivalence.*

(3) *For every integer n, the restriction $F|(\mathrm{Mod}_R^{\mathrm{cn}})_{\leq n}$ commutes with filtered colimits.*

(4) *There exists an integer $n \geq 0$ such that $F(M)$ is n-truncated for every discrete R-module M.*

(5) *For every coherent R-module M, the set $\pi_0 F^+(M)$ is finitely generated as a module over $\pi_0 R$.*

Remark 4.4.3. Let R be a Noetherian $\mathbb{E}_\infty$-ring. Then $\mathrm{Mod}_R^{\mathrm{coh}}$ is a stable subcategory of Mod_R, and the t-structure on Mod_R restricts to a t-structure $((\mathrm{Mod}_R^{\mathrm{coh}})_{\geq 0}, (\mathrm{Mod}_R^{\mathrm{coh}})_{\leq 0})$ on $\mathrm{Mod}_R^{\mathrm{coh}}$. It follows that the ∞-category $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})$ inherits a t-structure $(\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})_{\geq 0}, \mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})_{\geq 0})$. Since Mod_R admits filtered colimits (and the t-structure on Mod_R is stable under filtered colimits), the inclusion $\mathrm{Mod}_R^{\mathrm{coh}} \to \mathrm{Mod}_R$ induces a t-exact functor $F : \mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}}) \to \mathrm{Mod}_R$. We claim that F induces an equivalence $F_{\leq 0} : \mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})_{\leq 0} \to (\mathrm{Mod}_R)_{\leq 0}$. Note that any 0-truncated coherent R-module is almost perfect, and therefore a compact object of $(\mathrm{Mod}_R)_{\leq 0}$. It follows from Proposition T.5.3.5.11 that the functor $F_{\leq 0}$ is fully faithful. To complete the proof, it will suffice to show that $F_{\leq 0}$ is essentially surjective. Since the image of $F_{\leq 0}$ is closed under filtered colimits. Every object $M \in (\mathrm{Mod}_R)_{\leq 0}$ can be written as a filtered colimit

$$\tau_{\geq 0} M \to \tau_{\geq -1} M \to \tau_{\geq -2} M \to \cdots ;$$

it will therefore suffice to show that every object $M \in (\mathrm{Mod}_R)_{\leq 0} \cap (\mathrm{Mod}_R)_{\geq -n}$ belongs to the essential image of $F_{\leq 0}$. We proceed by induction on n. When $n = 0$, it suffices to observe that every discrete R-module can be written as a filtered colimit of its finitely generated submodules, which we can identify with objects of $(\mathrm{Mod}_R^{\mathrm{coh}})_{\leq 0}$. If $n > 0$, then we have a fiber sequence

$$\Sigma^{-1}(\tau_{\leq -1} M) \xrightarrow{\alpha} \pi_0 M \to M.$$

The inductive hypothesis guarantees that α is the image of a morphism $\overline{\alpha}$ in $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})_{\leq 0}$. Note that the domain of $\overline{\alpha}$ belongs to $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})_{\leq -1}$ (in fact, it belongs to $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})_{\leq -2}$), so that $\mathrm{cofib}(\overline{\alpha}) \in \mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})_{\leq 0}$ Then $M \simeq \mathrm{cofib}(\alpha) = F(\mathrm{cofib}(\overline{\alpha}))$ belongs to the essential image of $F_{\leq 0}$, as desired.

Proof of Theorem 4.4.2. Let K denote a dualizing module for R. Suppose that the functor $F : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathcal{S}$ is given by the formula $F(M) = \mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ for some R-module N. Then conditions (1) and (2) are vacuous. Condition (4) follows from assumption that N is almost connective, and condition (3) from the condition that N is almost perfect. Moreover, if N is almost perfect, then condition (5) follows from Lemma 4.3.6.

Now suppose that conditions (1) through (5) are satisfied. We wish to prove that F^+ is corepresentable by an almost perfect R-module. Fix a dualizing module K for R. For every R-module M, we let $\mathbb{D}(M)$ denote the R-module $\underline{\mathrm{Map}}_R(M, K)$. It follows from Theorem 4.2.7 that the construction $M \mapsto \mathbb{D}(M)$ induces a contravariant equivalence from the ∞-category $\mathrm{Mod}_R^{\mathrm{coh}}$ to itself. We define a functor $G : (\mathrm{Mod}_R^{\mathrm{coh}})^{op} \to \mathcal{S}$ by the formula $G(M) = F^+(\mathbb{D}(M))$. Assumption (1) implies that the functor G is left exact, and can therefore be identified with an object of $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})$.

We first claim that F satisfies the following stronger version of (4):

$(4')$ There exists an integer n such that $F^+(M)$ is $(n+m)$-truncated whenever $M \in \mathrm{Mod}_R^{\mathrm{acn}}$ is m-truncated.

To prove $(4')$, we first apply Proposition A.1.4.2.22 to factor F^+ as a composition

$$\mathrm{Mod}_R^{\mathrm{acn}} \xrightarrow{f} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S},$$

where f is exact. Let $M \in \mathrm{Mod}_R$ be m-truncated and k-connective; we will prove that the spectrum $f(M)$ is $(n+m)$-truncated. The proof proceeds by induction on m, the case $m < k$ being trivial. We have a fiber sequence

$$(\pi_m M)[m] \to M \to \tau_{\leq m-1} M.$$

Since f is exact, to prove that $f(M)$ is $(n+m)$-truncated it will suffice to show that $f(\tau_{\leq m-1} M)$ and $f((\pi_m M)[m])$ are $(n+m)$-truncated. In the first case, this follows from the inductive hypothesis. In the second, we must show that the spectrum $f(\pi_m M)$ is n-truncated. Since $n \geq 0$, this is equivalent to the assertion that the space $\Omega^\infty f(\pi_m M) \simeq F(\pi_m M)$ is n-truncated, which follows from (4).

Let n be an integer satisfying $(4')$. Choose an integer n' such that K is n'-truncated. If $M \in \mathrm{Mod}_R^{\mathrm{coh}}$ is $(n+n'+1)$-connective, then $\mathbb{D}(M)$ is $(-n-1)$-truncated so that condition (5) guarantees that $G(M) = F^+\,\mathbb{D}(M)$ is contractible (note that $F^+\,\mathbb{D}(M)$ is automatically nonempty). It follows that, as an object of $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})$, G belongs to $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}})_{\leq n+n'}$. Applying Remark 4.4.3, we conclude that G is the image of an object $N \in (\mathrm{Mod}_R)_{\leq n+n'}$ under the right adjoint to the functor $\mathrm{Ind}(\mathrm{Mod}_R^{\mathrm{coh}}) \to \mathrm{Mod}_R$ appearing in Remark 4.4.3. Unwinding the definitions, we deduce that N represents the functor G: that is, we have homotopy equivalences

$$G(M) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$$

for $M \in \mathrm{Mod}_R^{\mathrm{coh}}$ which depend functorially on M. In particular, we obtain bijections

$$\pi_i N \simeq \pi_0 \mathrm{Map}_{\mathrm{Mod}_R}(\tau_{\leq n+n'}(R[i]), N) \simeq \pi_0 G(\tau_{\leq n+n'}(R[i])) \simeq \pi_0 F^+\,\mathbb{D}(\tau_{\leq n+n'} R[i]).$$

It follows from (5) that each homotopy group of N is finitely generated as a module over $\pi_0 R$. Using Theorem 4.2.7, we deduce that $\mathbb{D}(N)$ is an almost perfect R-module, and that we have functorial homotopy equivalences

$$F^+(M) = G(\mathbb{D}(M)) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{D}(M), N) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{D}(N), M).$$

Let $F' : \mathrm{Mod}_R^{\mathrm{acn}} \to \mathcal{S}$ be the functor corepresented by $\mathbb{D}(N)$. For every pair of integers a and b, let $\mathcal{C}(a,b)$ denote the full subcategory $(\mathrm{Mod}_R)_{\leq a} \cap (\mathrm{Mod}_R)_{\geq b}$, and let $\mathcal{C}_0(a,b)$ denote the full subcategory spanned by those R-modules $M \in \mathcal{C}(a,b)$ which are coherent. Let $\mathcal{C} = \bigcup_{a,b} \mathcal{C}(a,b)$ denote the full subcategory of Mod_R spanned by those R-modules which are truncated and almost connective. Arguing as in Remark 4.4.3, we deduce that the inclusion $\mathcal{C}_0(a,b) \to \mathcal{C}(a,b)$ extends to an equivalence $\mathrm{Ind}(\mathcal{C}_0(a,b)) \to \mathcal{C}(a,b)$. Since $\mathbb{D}(N)$ is almost perfect, $F' |\, \mathcal{C}(a,b)$ commutes with filtered colimits. Condition (3) implies that $F^+ |\, \mathcal{C}(a,b)$ commutes with filtered colimits. Using Proposition T.5.3.5.10, we deduce that the restriction map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}(a,b),\mathcal{S})}(F' |\, \mathcal{C}(a,b), F^+ |\, \mathcal{C}(a,b)) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_0(a,b),\mathcal{S})}(F' |\, \mathcal{C}_0(a,b), F^+ |\, \mathcal{C}(a,b))$$

is a homotopy equivalence. Passing to the homotopy inverse limit over pairs (a,b), we obtain a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(F' |\, \mathcal{C}, F^+ |\, \mathcal{C}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{Mod}_R^{\mathrm{coh}},\mathcal{S})}(F' |\, \mathrm{Mod}_R^{\mathrm{coh}}, F^+ |\, \mathrm{Mod}_R^{\mathrm{coh}}).$$

In particular, the equivalence $F' |\, \mathrm{Mod}_R^{\mathrm{coh}} \simeq F^+ |\, \mathrm{Mod}_R^{\mathrm{coh}}$ lifts to a natural transformation $\alpha : F' |\, \mathcal{C} \to F^+ |\, \mathcal{C}$. Every object $M \in \mathcal{C}$ belongs to $\mathcal{C}(a,b)$. Since $F' |\, \mathcal{C}(a,b)$ and $F^+ |\, \mathcal{C}(a,b)$ both commute with filtered colimits, we deduce that $\alpha_M : F'(M) \to F^+(M)$ is a filtered colimit of equivalences $F'(M_0) \to F^+(M_0)$, where M_0 is coherent. It follows that α is an equivalence of functors.

To complete the proof that F^+ is corepresentable by an almost perfect R-module, it will suffice to show that α lifts to an equivalence between F^+ and F'. For this, it will suffice to show that F^+ and F' are both right Kan extensions of their restrictions to $\mathcal{C}$. We will need the following criterion:

($*$) Let $H : \mathrm{Mod}_R^{\mathrm{acn}} \to \mathcal{S}$ be a functor. Then H is a right Kan extension of $H|\,\mathcal{C}$ if and only if, for every almost connective R-module M, the canonical map $H(M) \to \varprojlim H(\tau_{\leq n} M)$ is an equivalence.

It is obvious that the functor F' satisfies the criterion of ($*$), and assumption (2) guarantees that F^+ also satisfies the criterion of ($*$). To prove ($*$), it will suffice to show that for every object $M \in \mathrm{Mod}_R^{\mathrm{acn}}$, the Postnikov tower

$$\cdots \to \tau_{\leq 2} M \to \tau_{\leq 1} M \to \tau_{\leq 0} M$$

determines a right cofinal map

$$\mathrm{N}(\mathbf{Z}_{\geq 0})^{op} \to \mathcal{C} \times_{\mathrm{Mod}_R^{\mathrm{acn}}} (\mathrm{Mod}_R^{\mathrm{acn}})_{M/}.$$

This is equivalent to the assertion that for every object $N \in \mathcal{C}$, the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{Mod}_R}(\tau_{\leq n} M, N) \to \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$$

is a homotopy equivalence. This is clear, since the assumption that N is truncated implies that the map $\mathrm{Map}_{\mathrm{Mod}_R}(\tau_{\leq n} M, N) \to \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$ is a homotopy equivalence for $n \gg 0$. □

4.5 Existence of the Cotangent Complexes

Let R be an $\mathbb{E}_\infty$-ring and suppose we are given a map $f : X \to \mathrm{Spec}^{\mathrm{f}} R$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$. In this section, we will prove that under some reasonable finiteness hypotheses, the existence of a cotangent complex for f can be reformulated in terms of good behavior of the tangent complex of f. Our main result is the following:

Theorem 4.5.1. *Let R be a Noetherian $\mathbb{E}_\infty$-ring which admits a dualizing module (Definition 4.2.5), let $Y = \mathrm{Spec}^{\mathrm{f}} R$, and suppose we are given a morphism $f : X \to Y$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ which is cohesive, nilcomplete, and locally almost of finite presentation. Assume further that f satisfies the hypotheses of Proposition 4.1.6, and that there exists an integer q such that $X(A)$ is q-truncated for every discrete commutative ring A. Then the following conditions are equivalent:*

- (A) *The functor X admits a cotangent complex.*
- (B) *For every Noetherian $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$. Then each homotopy group $\pi_n T_{X/Y}(\eta)$ is a finitely generated module over $\pi_0 A$.*
- (C) *Let A be an integral domain, let $\eta \in X(A)$ exhibit A as a finitely generated module over $\pi_0 R$. Then the homotopy groups $\pi_n T_{X/Y}(\eta)$ are finitely generated as modules over A.*

If these conditions are satisfied, then the relative cotangent complex $L_{X/Y}$ is almost perfect. Moreover, if X is integrable, then (A), (B), and (C) are equivalent to either of the following conditions:

- (D) *Let A be an integral domain and let $\eta \in X(A)$ exhibit A as a finitely generated module over $\pi_0 R$. For every integer n, there exists a finite collection of elements $x_1, x_2, \ldots, x_p \in \pi_n T_{X/Y}(\eta)$ and an element $a \in A$ such that, for every field K and every ring homomorphism $A[a^{-1}] \to K$ carrying η in $\eta_K \in X(K)$ the images of the elements $x_1, \ldots, x_p$ form a basis for the vector space $\pi_n T_{X/Y}(\eta_K)$.*
- (E) *Let A be an integral domain and let $\eta \in X(A)$ exhibit A as a finitely generated module over $\pi_0 R$. For every integer n, there exists a nonzero element $a \in A$ such that $(\pi_n T_{X/Y}(\eta))[a^{-1}]$ is a free $A[a^{-1}]$-module of finite rank.*

The proof of Theorem 4.5.1 will require some preliminaries.

Lemma 4.5.2. *Let A be a Noetherian ring containing an element a and let $\widehat{A}$ denote the completion of A with respect to the ideal (a). Suppose we are given a discrete $A[a^{-1}]$-module N and a discrete $\widehat{A}$-module $\widehat{M}$, together with a map $\alpha : N \simeq \widehat{M}[a^{-1}]$ which induces an isomorphism $\overline{\alpha} : \widehat{A}[a^{-1}] \otimes_{A[a^{-1}]} N \to \widehat{M}[a^{-1}]$. Then the canonical map $\mu : N \oplus \widehat{M} \to \widehat{M}[a^{-1}]$ is surjective. Moreover, if N and $\widehat{M}$ are finitely generated over $A[a^{-1}]$ and $\widehat{A}$, respectively, then $\ker(\mu)$ is a finitely generated A-module.*

Proof. Let x be an arbitrary element of $\widehat{M}[a^{-1}]$; we will show that x lies in the image of μ. Since $\overline{\alpha}$ is an isomorphism, we can write $x = \sum c_i \alpha(y_i)$ for some $y_i \in N$, $c_i \in \widehat{A}$. Choose an integer n such each product $a^n \alpha(y_i)$ is the image of some element $\overline{y}_i \in \widehat{M}$. Choose elements $c'_i \in A$ which represent the images of c_i in $A/(a^n)$. Identifying the elements c'_i with their images in $\widehat{A}$, we have $c_i = c'_i + a^n d_i$, so that

$$x = \sum c_i \alpha(y_i) = \sum c'_i \alpha(y_i) + a^n \sum d_i \alpha(y_i) = \alpha(\sum c'_i y_i) + \sum d_i \overline{y}_i$$

belongs to the image of μ, as desired.

Let $M = \ker(\mu)$, so that we have a pullback diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \alpha} \\ \widehat{M} & \longrightarrow & \widehat{M}[a^{-1}] \end{array}$$

in Mod_A. Suppose that N and $\widehat{M}$ are finitely generated; we wish to prove that M is finitely generated. Since the lower horizontal map is an equivalence after tensoring with $A[a^{-1}]$, the upper horizontal map has the same property: that is, we can identify N with $M[a^{-1}]$. We may therefore choose a finitely generated submodule $M' \subseteq M$ such that ϕ induces an isomorphism $M'[a^{-1}] \to N$. It follows that the map $M' \otimes_A \widehat{A}[a^{-1}] \to \widehat{M}[a^{-1}]$ is also an isomorphism: that is, ψ induces a map $\overline{\psi} : M' \otimes_A \widehat{A} \to \widehat{M}$ which is an isomorphism after inverting a.

Note that $\mathrm{fib}(\psi) \simeq \mathrm{fib}(\alpha)$ has the structure of an $A[a^{-1}]$-module, so that ψ induces an equivalence $A/(a) \otimes_A M \to A/(a) \otimes_A \widetilde{M}$, and in particular an isomorphism $M/aM \to \widehat{M}/a\widehat{M}$. Using Nakayama's Lemma, we deduce that the image of ψ generates $\widehat{M}$ as a module over $\widehat{A}$. Enlarging M' if necessary, we may suppose that $\psi(M')$ generates $\widehat{M}$ as a module over $\widehat{A}$: that is, that the map $\overline{\psi}$ is surjective.

Since $\widehat{A}$ is Noetherian, $\ker(\overline{\psi})$ is a finitely generated module over $\widehat{A}$. It is therefore annihilated by a^n for $n \gg 0$. Let K be the kernel of the map $a^n : M' \to M'$, so that $K \otimes_A \widehat{A}$ is the kernel of the map $a^n : M' \otimes_A \widehat{A} \to M' \otimes_A \widehat{A}$. Since $A/(a^n) \simeq \widehat{A}/(a^n)$, we deduce that the canonical map $K \to K \otimes_A \widehat{A}$ is an isomorphism. In particular, $\ker(\overline{\psi})$ is contained in the image of the composite map $K \to M' \to M' \otimes_A \widehat{A}$. However, since $K \subseteq \ker(\phi)$, the injectivity of the map $M \to N \oplus \widehat{M}$ guarantees that $K \cap \ker(\psi) = 0$. It follows that $\ker(\overline{\psi}) \simeq 0$: that is, the map $\overline{\psi}$ is an isomorphism.

We now complete the proof that M is finitely generated by showing that $M' = M$. For this, it suffices to show that we have a pullback square σ :

$$\begin{array}{ccc} M' & \longrightarrow & N \\ \downarrow & & \downarrow \\ \widehat{M} & \longrightarrow & \widehat{M}[a^{-1}] \end{array}$$

in the ∞-category Mod_A. The above argument shows that ϕ and ψ induce isomorphisms

$$M'[a^{-1}] \simeq N \qquad M' \otimes_A \widehat{A} \simeq \widehat{M},$$

so that σ is obtained by tensoring M' with the diagram σ_0 :

$$\begin{array}{ccc} A & \longrightarrow & A[a^{-1}] \\ \downarrow & & \downarrow \\ \widehat{A} & \longrightarrow & \widehat{A}[a^{-1}], \end{array}$$

which is evidently a pullback square. □

Proposition 4.5.3. *Let R be a connective $\mathbb{E}_\infty$-ring, let $Y = \mathrm{Spec}^{\mathrm{f}} R$, let $f : X \to Y$ be a morphism in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$*, and let $X_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ denote the functor given by $X_0(A) = \mathrm{fib}(X(A) \to Y(A))$. Suppose that the following conditions are satisfied:*

(a) *For every morphism $A \to B$ in $\mathrm{CAlg}^{\mathrm{cn}}_R$ every connective B-module M, the diagram*

$$\begin{array}{ccc} X_0(A \oplus M) & \longrightarrow & X_0(B \oplus M) \\ \downarrow & & \downarrow \\ X_0(A) & \longrightarrow & X_0(B) \end{array}$$

is a pullback square.

(b) *For every truncated object $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$ and every point $\eta \in X_0(A)$, the functor F_η given by $F_\eta(M) = X_0(A \oplus M) \times_{X_0(A)} \{\eta\}$ is corepresentable by an almost connective A-module $L_{X/Y}(\eta)$.*

(c) *The functor X is nilcomplete.*

Then f admits a cotangent complex.

Proof. Using assumption (a) and the relative analogue of Example 1.3.15, we are reduced to proving that for every $A \in \mathrm{CAlg}^{\mathrm{cn}}_R$ and every point $\eta \in X_0(A)$, the functor F_η described in (b) is almost corepresentable. For every integer $n \geq 0$, let η_n denote the image of η in $X_0(\tau_{\leq n} A)$. Then assumption (b) guarantees the existence of almost connective objects $L_{X/Y}(\eta_n)$ corepresenting the functors F_{η_n}, and (a) gives equivalences

$$\tau_{\leq n-1} A \otimes_{\tau_{\leq n} A} L_{X/Y}(\eta_n) \simeq L_{X/Y}(\eta_{n-1})$$

for $n > 0$. Choose an integer m such that $L_X(\eta_0)$ is m-connective. It follows that each $L_{X/Y}(\eta_n)$ is m-connective, and that the maps $L_{X/Y}(\eta_n) \to L_{X/Y}(\eta_{n-1})$ are $(m+n)$-connective for $n > 0$. Let N denote the limit of the tower

$$\cdots \to L_{X/Y}(\eta_2) \to L_{X/Y}(\eta_1) \to L_{X/Y}(\eta_0)$$

in the ∞-category Mod_A. Then N is m-connective, and the canonical map $N \to L_{X/Y}(\eta_n)$ is $(m+n+1)$-connective for every integer n. Let M be a connective A-module. We may assume without loss of generality that $m \leq 0$. Using assumptions (a) and (c), we obtain homotopy equivalences

$$\begin{aligned} F_\eta(M) &\simeq \varprojlim_k F_\eta(\tau_{\leq k+m} M) \\ &\simeq \varprojlim_k F_{\eta_k}(\tau_{\leq k+m} M) \\ &\simeq \varprojlim_k \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leq k} A}}(L_{X/Y}(\eta_k), \tau_{\leq k+m} M) \\ &\simeq \varprojlim_k \mathrm{Map}_{\mathrm{Mod}_{\tau_{\leq k} A}}(\tau_{\leq k+m} L_{X/Y}(\eta_k), \tau_{\leq k+m} M) \\ &\simeq \varprojlim_k \mathrm{Map}_{\mathrm{Mod}_A}(\tau_{\leq k+m} L_{X/Y}(\eta_k), \tau_{\leq k+m} M) \\ &\simeq \varprojlim_k \mathrm{Map}_{\mathrm{Mod}_A}(N, \tau_{\leq k+m} M) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_A}(N, M) \end{aligned}$$

depending functorially on M. It follows that F_η is corepresented by N. □

Proof of Theorem 4.5.1. Note that if f admits a relative cotangent complex $L_{X/Y}$, then $L_{X/Y}$ is almost perfect (since f is locally almost of finite presentation; see Corollary 2.3.7).

Suppose first that (A) is satisfied; we will prove (B). Assumption (A) implies that the cotangent complex $L_{X/Y}$ exists and is almost perfect. For each point $\eta \in X(A)$, the tangent complex $T_{X/Y}(\eta)$ is given by the A-linear dual of $\eta^* L_{X/Y}$ (Example 4.1.2). In particular, we have isomorphisms $\pi_n T_{X/Y}(\eta) \simeq \mathrm{Ext}_A^{-n}(\eta^* L_{X/Y}, A)$, so that $\pi_n T_{X/Y}(\eta)$ is a finitely generated module over $\pi_0 A$ whenever A is Noetherian (Lemma 4.3.6).

The implication $(B) \Rightarrow (C)$ is obvious. We next prove that (C) implies (A). Suppose that condition (C) is satisfied. To prove that X admits a cotangent complex, it will suffice to show that the morphism f admits a cotangent complex (Proposition 2.2.9). Using Proposition 4.1.6, see that f satisfies conditions (a) and (c) of Proposition 4.5.3. It will therefore suffice to show that for every truncated object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, the functor $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ given by $F_\eta(M) = X_0(A \oplus M) \times_{X_0(A)} \{\eta\}$ is almost corepresentable. Write A as the colimit of a filtered diagram $\{A_\alpha\}$ of connective $\mathbb{E}_\infty$-algebras which are of finite presentation over R. Choose $m \geq 0$ such that A is m-truncated, so that $A \simeq \varinjlim \tau_{\leq m} A_\alpha$. Since f is locally almost of finite presentation, we can assume that η is the image of a point $\eta_\alpha \in X_0(A_\alpha)$ for some index α. Using condition (a) of Proposition 4.5.3, we see that F_η factors as a composition

$$\mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{Mod}_{\tau_{\leq n} A_\alpha}^{\mathrm{cn}} \overset{F_{\eta_\alpha}}{\to} \mathcal{S}\,.$$

We may therefore replace A by A_α, and thereby reduce to the case where A is almost of finite presentation over R. Then A admits a dualizing module (Theorem 4.3.14). We will show that F_η is almost corepresentable by verifying conditions (1) through (5) of Theorem 4.4.2:

(1) The functor $F_\eta : \mathrm{Mod}_A^{\mathrm{cn}} \to \mathcal{S}$ is obviously reduced. Since X is infinitesimally cohesive, the canonical map $F_\eta(M) \to \Omega F_\eta(\Sigma M)$ is an equivalence for every connective A-module M, so that F_η is excisive by Proposition A.1.4.2.13.

(2) For every connective A-module M, we claim that the canonical map $F_\eta(M) \to \varprojlim F_\eta(\tau_{\leq n} M)$ is a homotopy equivalence. This follows immediately from the nilcompleteness of the functor X.

(3) We claim that F_η commutes with filtered colimits when restricted to $(\mathrm{Mod}_A)_{\leq n}$. This is an immediate consequence of our assumption that the map f is locally almost of finite presentation.

(4) Choose an integer n such that $X(B)$ is n-truncated for every commutative ring B. We claim that $F_\eta(M)$ is n-truncated for every discrete A-module M. Since X is cohesive, we can replace A by $\pi_0 A$ and thereby reduce to the case where A is discrete. Then $F_\eta(M)$ is the fiber of a map

$$X(A \oplus M) \to X(A) \times_{Y(A)} Y(A \oplus M)$$

between n-truncated spaces, and therefore n-truncated.

(5) Using Lemma 1.3.2, we can extend F_η to an excisive $F_\eta^+ : \mathrm{Mod}_A^{\mathrm{acn}} \to \mathcal{S}$. We wish to prove that for every coherent A-module M, $\pi_0 F_\eta^+(M)$ is finitely generated as a discrete module over $\pi_0 A$. Given a fiber sequence of coherent A-modules

$$M' \to M \to M'',$$

we obtain an exact sequence of $\pi_0 A$-modules

$$\pi_0 F_\eta^+(M') \to \pi_0 F_\eta^+(M) \to \pi_0 F_\eta^+(M'').$$

Consequently, to prove that $\pi_0 F_\eta^+(M)$ is finitely generated, it suffices to prove the corresponding assertions for M' and M''. We may therefore reduce to the case where the module M is concentrated in a single degree k. Then $\pi_k M$ is a finitely generated module over the Noetherian ring $\pi_0 A$, and therefore admits a finite composition series with successive quotients of the form $(\pi_0 A)/\mathfrak{p}$, where $\mathfrak{p} \subseteq \pi_0 A$ is a prime ideal. We may therefore assume that $\pi_k M$ has the form $(\pi_0 A)/\mathfrak{p}$. Since X is cohesive, we can replace A by the integral domain $(\pi_0 A)/\mathfrak{p}$, so that $M \simeq \Sigma^k A]$. In this case, $\pi_0 F_\eta^+(M) \simeq \pi_k T_{X/Y}(\eta)$ is finitely generated by virtue of assumption (C).

We next show that $(A) \Rightarrow (D)$. Let A be a Noetherian integral domain equipped with a point $\eta \in X(A)$, and let n be an integer. Corollary 2.3.7 implies that $\eta^* L_{X/Y}$ is an almost perfect A-module. In particular, the homotopy groups $\pi_m \eta^* L_{X/Y}$ are finitely generated A-modules, which vanish for $m \ll 0$. We may therefore choose a nonzero element $a \in A$ such that $(\pi_m \eta^* L_{X/Y})[a^{-1}]$ is a finitely generated free module over $A[a^{-1}]$ of rank r_m for $m \leq -n$. For each $m \leq -n$, choose a collection of elements $\{x_{i,m} \in \pi_m \eta^* L_{X/Y}\}_{1 \leq i \leq r_m}$ whose images form a basis for $(\pi_m \eta^* L_{X/Y})[a^{-1}]$ as a module over $A[a^{-1}]$. These choices determine a map

$$\tau_{\geq 1-n} \eta^* L_{X/Y} \oplus \bigoplus_{m \leq -n} (\Sigma^m A)^{r_m} \to \eta^* L_{X/Y}$$

which is an equivalence after inverting the element a. It follows that for every A-module M in which a is invertible, the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_A}(\eta^* L_{X/Y}, M) \to \mathrm{Map}_{\mathrm{Mod}_A}(\tau_{\geq 1-n} \eta^* L_{X/Y}, M) \times \prod_{m \leq -n} (\Omega^{\infty - m} M)^{r_m}.$$

In particular, given a ring homomorphism $A[a^{-1}] \to K$ carrying η to $\eta_K \in X(K)$, taking $M = \Sigma^{-n} K$ gives a vector space isomorphism

$$\pi_n T_{X/Y}(\eta_K) \simeq \mathrm{Ext}^{-n}_A(\eta^* L_{X/Y}, K) \simeq K^{r_n},$$

given by evaluation on the elements $\{x_{i,n}\}_{1 \leq i \leq r_n}$.

We now show that $(D) \Rightarrow (E)$. Assume that X satisfies (D), let A be an integral domain, and let $\eta \in X(A)$ exhibit A as a finitely generated algebra over $\pi_0 R$. Since R admits a dualizing module, so does A (Theorem 4.3.14), so that A has finite Krull dimension d (Remark 4.3.13). Using (D), we can choose $a \in A$ and, for $n - 1 \leq m \leq n + d + 1$, a finite collection of elements $\{y_{i,m} \in \pi_m T_{X/Y}(\eta)\}_{1 \leq i \leq r_m}$ with the following property: for every field K equipped with a map $A[a^{-1}] \to K$ carrying η to $\eta_K \in X(K)$, the images of the elements $\{y_{i,m}\}_{1 \leq i \leq r_m}$ form a basis for the K-vector space $\pi_m(T_{X/Y}(\eta_K))$. For every commutative $A[a^{-1}]$-algebra B, let η_B denote the image of η in $X(B)$, so that the elements $\{y_{i,m}\}$ determines a map of B-modules $\bigoplus_{n-1 \leq m \leq n+d+1} (\Sigma^m B)^{r_m} \to T_{X/Y}(\eta_B)$. Let us denote the fiber of this map by F_B. Note that if B is a field, then the homotopy groups $\pi_i F_B$ vanish for $n - 1 \leq i \leq n + d$. We will prove the following assertion:

- $(*)$ Let B be a quotient ring of $A[a^{-1}]$ having Krull dimension $\leq d'$. Then the homotopy groups $\pi_i F_B$ vanish for $n - 1 \leq i \leq n + d - d'$.

Taking $B = A[a^{-1}]$ and $d = d'$, we deduce that $\pi_{n-1} F_{A[a^{-1}]} \simeq \pi_n F_{A[a^{-1}]} \simeq 0$, so that the map

$$\pi_n(\bigoplus_{n-1 \leq m \leq n+d+1} (\Sigma^m A[a^{-1}])^{r_m} \to \pi_n T_{X/Y}(\eta_{A[a^{-1}]})$$

is an isomorphism: that is, the images of the elements $\{y_{i,n}\}_{1 \leq i \leq r_n}$ freely generate $(\pi_n T_{X/Y}(\eta))[a^{-1}] \simeq \pi_n T_{X/Y}(\eta_{A[a^{-1}]})$ as a module over $A[a^{-1}]$.

It remains to prove $(*)$. We proceed by Noetherian induction on B. If $B = 0$, there is nothing to prove. Otherwise, let $\mathfrak{p}$ be an associated prime of B, so that there exists a nonzero ideal $I \subseteq B$ which is isomorphic, as a B-module, to $B/\mathfrak{p}$. We then have an exact sequence of B-modules

$$B/\mathfrak{p} \to B \to B/I$$

which determines a fiber sequence

$$F_{B/\mathfrak{p}} \to F_B \to F_{B/I}.$$

It follows from the inductive hypothesis that the homotopy groups $\pi_i F_{B/I}$ vanish for $n - 1 \leq i \leq n + d - d'$. It will therefore suffice to show that the homotopy groups $\pi_i F_{B/\mathfrak{p}}$ vanish for $n - 1 \leq i \leq n + d - d'$. Replacing B by $B/\mathfrak{p}$, we can reduce to the case where B is an integral domain. For every nonzero element $b \in B$, the quotient ring $B/(b)$ has Krull dimension $\leq d' - 1$. Applying the inductive hypothesis, we deduce that the homotopy groups $\pi_i F_{B/(b)}$ vanish for $n - 1 \leq i \leq n + 1 + d - d'$. Using the fiber sequence

$$F_B \xrightarrow{b} F_B \to F_{B/(b)},$$

we deduce that multiplication by b induces an isomorphism from $\pi_i F_B$ to itself for $n - 1 \leq i \leq n + d - d'$. It will therefore suffice to show that $K \otimes_B \pi_i F_B$ vanishes for $n - 1 \leq i \leq n + d - d'$, where K denotes the fraction field of B. This follows from our construction, since Proposition 4.1.6 supplies an equivalence $K \otimes_B \pi_i F_B \simeq \pi_i(K \otimes_B F_B) \simeq \pi_i F_K$.

We now complete the proof by showing that if X is integrable, then condition (E) implies condition (C). Assume that condition (E) is satisfied. We will show that for every Noetherian commutative ring A, every point $\eta \in X(A)$ which exhibits A as a finitely generated algebra over $\pi_0 R$, and every finitely generated A-module M, the homotopy groups $\pi_n T_{X/Y}(\eta; M)$ are finitely generated A-modules. Proceeding by Noetherian induction, we may suppose that this condition is satisfied for every quotient M/M' of M by a nonzero submodule M'.

If $M \simeq 0$ there is nothing to prove. Otherwise, M has an associated prime ideal: that is, there exists a nonzero element $x \in M$ whose annihilator is a prime ideal $\mathfrak{p} \subseteq A$.

$$0 \to Ax \to M \to M/Ax \to 0.$$

Using the inductive hypothesis, we can replace M by Ax and thereby reduce to the case where M has the form $A/\mathfrak{p}$. Using our assumption that f is cohesive, we can replace A by $A/\mathfrak{p}$ and thereby reduce to the case where A is an integral domain and $M = A$. For every ideal $I \subseteq A$, let $F_I : \mathrm{Mod}^{\mathrm{cn}}_{A/I} \to \mathcal{S}$ denote the functor given by $F_I(M) = X_0(A/I \oplus M) \times_{X_0(A/I)} \{\eta_I\}$. Using the inductive hypothesis and the proof of the implication $(C) \Rightarrow (A)$, we see that F_I is corepresented by an almost perfect module over A/I for every nonzero ideal $I \subseteq A$.

Fix an integer n; we wish to show that $\pi_n T_{X/Y}(\eta)$ is finitely generated. Using condition (E), we can choose a nonzero element $a \in A$ such that the modules $\pi_{n+1} T_{X/Y}(\eta')$ and $\pi_n T_{X/Y}(\eta')$ are finitely generated free modules over $A[a^{-1}]$, where η' denotes the image of η in $X(A[a^{-1}])$. Let $\widehat{A} = \varinjlim A/(a^n)$ denote the completion of A with respect to the principal ideal (a). We have a pullback diagram of A-modules

$$\begin{array}{ccc} A & \longrightarrow & A[a^{-1}] \\ \downarrow & & \downarrow \\ \widehat{A} & \longrightarrow & \widehat{A}[a^{-1}]. \end{array}$$

Let $\widehat{\eta}$ denote the image of η in $X(\widehat{A})$, and define $\widehat{\eta}' \in X(\widehat{A}[a^{-1}])$ similarly. Since X is cohesive, we have a pullback square of tangent complexes

$$\begin{array}{ccc} T_{X/Y}(\eta) & \longrightarrow & T_{X/Y}(\eta') \\ \downarrow & & \downarrow \\ T_{X/Y}(\widehat{\eta}) & \longrightarrow & T_{X/Y}(\widehat{\eta}') \end{array}$$

and therefore a long exact sequence of A-modules

$$\pi_{n+1} T_{X/Y}(\eta') \oplus \pi_{n+1} T_{X/Y}(\widehat{\eta}) \xrightarrow{\mu} \pi_{n+1} T_{X/Y}(\widehat{\eta}') \to \pi_n T_{X/Y}(\eta) \to \pi_n T_{X/Y}(\eta') \oplus \pi_n T_{X/Y}(\widehat{\eta}) \xrightarrow{\nu} \pi_n T_{X/Y}(\widehat{\eta}').$$

We will prove that $\pi_n T_{X/Y}(\widehat{\eta})$ and $\pi_{n+1} T_{X/Y}(\widehat{\eta})$ are finitely generated modules over $\widehat{A}$. Since $\widehat{A}[a^{-1}]$ is flat over $\widehat{A}$ and $A[a^{-1}]$, it will then follow from (B_1) (which is satisfied by Proposition 4.1.6) and Lemma 4.5.2

that μ is surjective and $\ker(\nu)$ is a finitely generated A-module, thereby showing that $\pi_n T_{X/Y}(\eta)$ is finitely generated as an A-module.

For every integer $k \geq 0$, let $L_k \in \mathrm{Mod}_{A/(a^k)}$ corepresent the functor $F_{(a^k)}$. Since X is locally almost of finite presentation, each L_k is almost perfect. Corollary XII.5.1.14 supplies an equivalence of ∞-categories $\mathrm{Mod}_{\widehat{A}}^{\mathrm{aperf}} \simeq \varprojlim_k \mathrm{Mod}_{A/(a^k)}^{\mathrm{aperf}}$. Under this equivalence, we can identify the inverse system $\{L_k\}_{k \geq 0}$ with an almost perfect $\widehat{A}$-module $\widehat{L}$. For each $m \geq 0$, let η_k denote the image of η in $X(A/(a^k))$. Set $T = \underline{\mathrm{Map}}_{\widehat{A}}(\widehat{L}, \widehat{A})$, so that we have a canonical identification

$$\begin{aligned}
\varprojlim_k T_{X/Y}(\eta_k) &\simeq \varprojlim_k \underline{\mathrm{Map}}_{A/(a^k)}(L_k, A/(a^k)) \\
&\simeq \varprojlim_k \underline{\mathrm{Map}}_{\widehat{A}}(\widehat{L}, A/(a^k)) \\
&\simeq \underline{\mathrm{Map}}_{\widehat{A}}(\widehat{L}, \widehat{A}) \\
&\simeq T.
\end{aligned}$$

It follows from Lemma 4.3.6 that the homotopy groups of T are finitely generated modules over $\widehat{A}$. We will complete the proof by showing that the map $\rho : T_{X/Y}(\widehat{\eta}) \to T$ is an equivalence.

Since f is cohesive, the canonical map $\widehat{A}/(a) \otimes_A T_{X/Y}(\widehat{\eta}) \to T_{X/Y}(\eta_1)$ is an equivalence. It follows that ρ induces an equivalence after tensoring with $\widehat{A}/(a)$: that is, the homotopy groups of $\mathrm{fib}(\rho)$ are modules over $\widehat{A}[a^{-1}]$. Fix an integer m; we wish to show that $\pi_m \mathrm{fib}(\rho) \simeq 0$. For this, we study the exact sequence

$$\pi_{m+1} T \xrightarrow{\mu} \pi_m \mathrm{fib}(\rho) \xrightarrow{\nu} \pi_m T_{X/Y}(\widehat{\eta}).$$

We will show that ν is injective, so that μ is surjective. It then follows that $\pi_m \mathrm{fib}(\rho)$ is a finitely generated module over $\widehat{A}$. Since a acts invertibly on $\pi_m \mathrm{fib}(\rho)$, it then follows from Nakayama's lemma that $\pi_m \mathrm{fib}(\rho) = 0$, as desired.

Choose an element $y_0 \in \pi_m T_{X/Y}(\widehat{\eta})$ belonging to the image of ν; we wish to show that $y_0 = 0$. Note that y_0 is a-divisible: that is, we can find elements $y_1, y_2, \ldots \in \pi_m T_{X/Y}(\widehat{\eta})$ such that $a y_{i+1} = y_i$. If $y_0 \neq 0$, then we can choose a maximal ideal $\mathfrak{m} \subseteq \widehat{A}$ such that the image of y_0 is nonzero in the localization $(\pi_m T_{X/Y}(\widehat{\eta}))_{\mathfrak{m}}$. Let B denote the completion of $\widehat{A}$ at the maximal ideal $\mathfrak{m}$, and let η_B denote the image of η in $X(B)$. Then B is faithfully flat over $\widehat{A}_{\mathfrak{m}}$, so that the image of y_0 is nonzero in $B \otimes_{\widehat{A}} \pi_m T_{X/Y}(\widehat{\eta}) \simeq \pi_m T_{X/Y}(\eta_B)$.

Let $\mathfrak{m}_B$ denote the maximal ideal of B, and choose a tower of $\mathbb{E}_\infty$-algebras

$$\cdots \to B_2 \to B_1 \to B_0$$

satisfying the requirements of Lemma XII.5.1.5. Using Lemma 2.1.18, we see that for every pair of integers $p \geq q \geq 0$, we have an equivalence

$$\{\tau_{\leq p}(B_j \oplus \Sigma^q B_j)\}_{j \geq 0} \simeq \{B/\mathfrak{m}_B^j \oplus \Sigma^q B/\mathfrak{m}_B^j\}$$

of pro-objects of CAlg. Since f is nilcomplete and integrable, it follows that the canonical map $T_{X/Y}(\eta_B) \to \varprojlim_j T_{X/Y}(\eta_{B,j})$ is an equivalence, where $\eta_{B,j}$ denotes the image of η in $X(B/\mathfrak{m}_B^j)$. We therefore obtain an equivalence

$$\begin{aligned}
T_{X/Y}(\eta_B) &\simeq \varprojlim_j \underline{\mathrm{Map}}_{B/\mathfrak{m}_B^j}((B/\mathfrak{m}_B^j) \otimes_{\widehat{A}} \widehat{L}, B/\mathfrak{m}_B^j) \\
&\simeq \varprojlim_j \underline{\mathrm{Map}}_B(B \otimes_{\widehat{A}} \widehat{L}, B/\mathfrak{m}_B^j) \\
&\simeq \underline{\mathrm{Map}}_B(B \otimes_{\widehat{A}} \widehat{L}, B).
\end{aligned}$$

Since $\widehat{L}$ is almost perfect over $\widehat{A}$, $B \otimes_{\widehat{A}} \widehat{L}$ is almost perfect over B, so that the homotopy groups of $T_{X/Y}(\eta_B)$ are finitely generated B-modules by Lemma 4.3.6. Since the image of a is contained in the maximal ideal B, it follows from Nakayama's Lemma that $\pi_m T_{X/Y}(\eta_B)$ does not contain any nonzero a-divisible elements. It follows that the image of y_0 in $\pi_m T_{X/Y}(\eta_B) \simeq B \otimes_{\widehat{A}} \pi_m T_{X/Y}(\widehat{A})$ is zero, contrary to our earlier assumption. $\square$

References

[1] Artin, M. *Versal deformations and algebraic stacks.* Inventiones Mathematicae 27 (1974), 165-189.

[2] Artin, M. and B. Mazur. *Étale Homotopy.* Lecture Notes in Mathematics 100, Springer-Verlag, Berlin and New York, 1969.

[3] Atiyah, M. and I. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ontario.

[4] Behrend, K. and B. Fantechi. *The intrinsic normal cone.* Inventiones Mathematicae 128 (1997) no. 1, 45-88.

[5] Ben-Zvi, D., Francis, J., and D. Nadler. *Integral Transforms and and Drinfeld Centers in Derived Algebraic Geometry.*

[6] Bergner, J. *Three models for the homotopy theory of homotopy theories.* Topology 46 (2007), no. 4, 397–436.

[7] Boardman, J. and R. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces.* Lecture Notes in Mathematics, 347, Springer-Verlag (1973).

[8] Conrad, B. and A.J. de Jong. *Approximation of Versal Deformations.* Journal of Algebra 225 (2002) no.2, 489-515.

[9] Crane, L. and D.N. Yetter. *Deformations of (bi)tensor categories.* Cahiers Topologie Geom. Differentielle Categ. 39 (1998), no. 3, 163–180.

[10] Deligne, P. *Catégories tannakiennes.* The Grothendieck Festschrift, Vol. II, 111195, Progr. Math., 87, Birkhuser Boston, Boston, MA, 1990.

[11] Grothendieck, A. *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents, Première partie.* Publications Mathématiques de l'IHÉS, 11 (1961), p. 5-167.

[12] Grothendieck, A. *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): IV. Étude locale des schémas et des morphismes de schémas, Troisième partie.* Publications Mathmatiques de l'IHS 28: 5255.

[13] Demazure, M. and Grothendieck, A., eds. *Schémas en groupes vol. 1-3.* Séminaire de Géométrie Algbrique du Bois Marie 1962/64 (SGA 3). Lecture Notes in Mathematics, Vol. 151-153 Springer-Verlag, Berlin-New York.

[14] Efimov, A., Lunts, V., and D. Orlov. *Deformation theory of objects in homotopy and derived categories. I: General Theory.* Adv. Math. 222 (2009), no. 2, 359–401.

[15] Efimov, A., Lunts, V., and D. Orlov. *Deformation theory of objects in homotopy and derived categories. II: Pro-representability of the deformation functor.* Available at arXiv:math/0702839v3 .

[16] Efimov, A., Lunts, V., and D. Orlov. *Deformation theory of objects in homotopy and derived categories. III: Abelian categories.* Available as arXiv:math/0702840v3 .

[17] Elmendorf, A.D., Kriz, I. , Mandell, M.A., and J.P. May. *Rings, modules and algebras in stable homotopy theory.* Mathematical Surveys and Monographs 47, American Mathematical Society, 1997.

[18] Etingof, P., Nikshych, D., and V. Ostrik. *On fusion categories.* Ann. of Math. (2) 162 (2005), no. 2, 581–642.

[19] Francis, J. *Derived Algebraic Geometry over $\mathcal{E}_n$-Rings.* Unpublished MIT PhD dissertation.

[20] Frenkel, E., Gaitsgory, D., and K. Vilonen. *Whittaker patterns in the geometry of moduli spaces of bundles on curves.* Ann. of Math. (2) 153 (2001), no. 3, 699–748.

[21] Fresse, B. *Koszul duality of E_n-operads.* Available as arXiv:0904.3123v6 .

[22] Fukuyama, H. and I. Iwanari. *Monoidal Infinity Category of Complexes from Tannakian Viewpoint.* Available at http://arxiv.org/abs/1004.3087.

[23] Fulton, W. *Algebraic curves.* W.A. Benjamin, Inc., New York, 1969.

[24] Fulton, W. and R. Pandharipande. *Notes on stable maps and quantum cohomology.* Algebraic geometry—Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[25] Getzler, E. *Lie theory for L_∞-algebras.* Ann. of Math. (2) 170 (2009), no. 1, 271–301.

[26] Goldman, W. and J. Millson. *Deformations of flat bundles over Kähler manifolds.* Geometry and Topology, Manifolds, Varieties and Knots (C. McCrory and T. Shifrin, eds.), Lecture Notes in Pure and Applied Math., vol. 105, Marcel Dekker, New York/Basel, 1987, pp. 129-145.

[27] Goldman, W. and J. Millson. *The deformation theory of representations of fundamental groups of compact Kähler manifolds.* Bull. Amer. Math. Soc. (N.S.), Volume 18, Number 2 (1988), 153-158.

[28] Goodwillie, T. *Calculus III: Taylor Series.* Geometry and Topology 7 (2003), 645-711.

[29] Grauert, H. and R. Remmert. *Coherent Analytic Sheaves.* Springer, 1984.

[30] Grothendieck, A. *Sur quelques points d'algebra homologique.* Tohoku Math. J. 9, 1957, 119–221.

[31] Hartshorne, R. *Residues and Duality.* Lecture Notes in Mathematics 20, 1966.

[32] Hinich, V. *DG coalgebras as formal stacks.* J. Pure Appl. Algebra, 162 (2001), 209-250.

[33] Hinich, V. *Deformations of homotopy algebras.* Communication in Algebra, 32 (2004), 473-494.

[34] Hopkins, M. and J. Lurie. *Vanishing Theorems for Higher Tate Cohomology.* In preparation.

[35] Illusie, L. *Complexe cotangent et déformations I.* Lecture Notes in Mathematics 239, Springer-Verlag, 1971.

[36] Illusie, L. *Complexe cotangent et déformations II.* Lecture Notes in Mathematics 283, Springer-Verlag, 1972.

[37] Johnstone, P. *Stone Spaces.* Cambridge University Press, Cambridge, 1982.

[38] Kapranov, M. and V. Ginzburg. *Koszul duality for Operads.* Duke Math. J., 1994.

[39] Keller, B. and W. Lowen. *On Hochschild cohomology and Morita deformations.* Int. Math. Res. Not. IMRN 2009, no. 17, 3221–3235.

[40] Knutson, D. *Algebraic spaces.* Lecture Notes in Mathematics 203, Springer-Verlag, 1971.

[41] Kontsevich, M. and Y. Soibelman. *Deformations of algebras over operads and the Deligne conjecture.* Conference Moshe Flato 1999, Vol. I (Dijon), 255–307, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000.

[42] Kontsevich, M. and Y. Soibelman. *Deformation Theory.* Unpublished book available at http://www.math.ksu.edu/~soibel/Book-vol1.ps .

[43] Kontsevich, M. and Y. Soibelman. *Notes on A-infinity algebras, A-infinity categories and noncommutative geometry.* Available as math.RA/060624.

[44] Laplaza, M. *Coherence for distributivity.* Coherence in categories, 29-65. Lecture Notes in Mathematics 281, Springer-Verlag, 1972.

[45] Laumon, G. and L. Moret-Bailly. *Champs algebriques.* Springer-Verlag, 2000.

[46] Lowen, W. *Obstruction theory for objects in abelian and derived categories.* Comm. Algebra 33 (2005), no. 9, 3195–3223.

[47] Lowen, W. *Hochschild cohomology, the characteristic morphism, and derived deformations.* Compos. Math. 144 (2008), no. 6, 1557–1580.

[48] Lurie, J. *Tannaka Duality for Geometric Stacks.* Available for download at http://www.math.harvard.edu/ lurie/ .

[49] Lurie, J. *Higher Topos Theory.* Available for download at http://www.math.harvard.edu/ lurie/ .

[50] Lurie, J. *Higher Algebra.* Available for download.

[51] Lurie, J. *Derived Algebraic Geometry V: Structured Spaces.* Available for download.

[52] Lurie, J. *Derived Algebraic Geometry VII: Spectral Schemes.* Available for download.

[53] Lurie, J. *Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems.* Available for download.

[54] Lurie, J. *Derived Algebraic Geometry IX: Closed Immersions.* Available for download.

[55] Lurie, J. *Derived Algebraic Geometry X: Formal Moduli Problems.* Available for download.

[56] Lurie, J. *Derived Algebraic Geometry XI: Descent Theorems*

[57] Lurie, J. *Derived Algebraic Geometry XII: Proper Morphisms, Completions, and the Grothendieck Existence Theorem.*

[58] Lurie, J. *Derived Algebraic Geometry XIV: Representability Theorems.*

[59] MacLane, S., and I. Moerdijk. *Sheaves in Geometry and Logic.* Springer-Verlag, Berlin and New York, 1992.

[60] Mandell, M. *E_∞-Algebras and p-adic Homotopy Theory.* Topology 40 (2001), no. 1, 43-94.

[61] Manetti, M. *Extended deformation functors.* Int. Math. Res. Not. 2002, no. 14, 719–756.

[62] Matsumura, H. *Commutative algebra, 2nd Edition.* Benjamin-Cummings Pub Co, 1980.

[63] Matsumura, H. *Commutative ring theory.* Cambridge University Press, 1986.

[64] May, P. *The Geometry of Iterated Loop Spaces.* Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972. viii+175 pp.

[65] Nisnevich, Y. *The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory.* NATO Advanced Study Institute Series, Ser. C, v. 279, Kluwer, 1989, pp. 241-342.

[66] Pridham, J. *Unifying derived deformation theories.* Adv. Math. 224 (2010), no.3, 772-826.

[67] Quillen, D. *Homotopical Algebra.* Lectures Notes in Mathematics 43, SpringerVerlag, Berlin, 1967.

[68] Raynaud, M. *Anneaux locaux henséliens.* Lecture Notes in Mathematics, 169, Berlin-New York: Springer-Verlag.

[69] Rognes, J. *Galois extensions of structured ring spectra. Stably dualizable groups.* Mem. Amer. Math. Soc. 192 (2008), no. 898, viii+137 pp.

[70] Schlessinger, M. and J. Stasheff. *The Lie algebra structure of tangent cohomology and deformation theory.* Journal of Pure and Applied Algebra 38 (1985), 313-322.

[71] Schlessinger, M. *Functors of Artin Rings.* Trans. Amer. Math. Soc. 130 1968 208–222.

[72] Serre, J-P. *Local algebra.* Springer-Verlag, 2000.

[73] Spivakovsky, M. *A New Proof of D. Popescu's Theorem on Smoothing of Ring Homomorphisms.* Journal of the American Mathematical Society, Volume 12, Number 2, 1999, 381-444.

[74] Thomason, R. W. and T. Trobough. *Higher Algebraic K-theory of schemes and of derived categories.* The Grothendieck Festschrift III, Progress in Math., vol. 88, Birkhuser, 1990, pp. 247435.

[75] Tits, J. *Sur les analogues algbriques des groupes semi-simples complexes.* Colloque dalgebre superieure, tenu a Bruxelles du 19 au 22 decembre 1956, Centre Belge de Recherches Mathematiques Etablissements Ceuterick, Louvain, Paris: Librairie Gauthier-Villars, pp. 261289.

[76] Toën, B. *Champs affines.* Selecta mathematica, new series, 12, 2006.

[77] Toën, B. *Vers une axiomatisation de la théorie des catégories supériures.* K-theory 34 (2005), no. 3, 233-263.

[78] Toën, B. *Finitude homotopique des dg-algèbres propres et lisses.* Available at math.AT/0609762.

[79] Toën, B. *The homotopy theory of dg-categories and derived Morita theory.* Invent. Math. 167 (2007), no. 3, 615–667.

[80] Toën, B. *Derived Azumaya's algebras and generators for twisted derived categories.*

[81] Toën, B., and M. Vaquié. *Moduli of objects in dg-categories.* Ann. Sci. cole Norm. Sup. (4) 40 (2007), no. 3, 387–444.

[82] Toën, B. and G. Vezzosi. *Brave new algebraic geometry and global derived moduli spaces of ring spectra.* Elliptic cohomology, 325359, London Math. Soc. Lecture Note Ser., 342, Cambridge Univ. Press, Cambridge, 2007.

[83] Toën, B. and G. Vezzosi. *Homotopical algebraic geometry. I. Topos theory.* Adv. Math. 193 (2005), no. 2, 257372.

[84] Toën, B. and G. Vezzosi. *Homotopical algebraic geometry. II. Geometric stacks and applications.* Mem. Amer. Math. Soc. 193 (2008), no. 902, x+224 pp.

[85] Toën, B. and G. Vezzosi. *From HAG to DAG: derived moduli stacks.* Axiomatic, enriched and motivic homotopy theory, 173216, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer Acad. Publ., Dordrecht, 2004.

[86] Toën, B. and G. Vezzosi. *Algebraic geometry over model categories.* Available for download: math.AG/0110109.

[87] Quick, G. *Profinite homotopy theory.* Documenta Mathematica, 13, (2008), 585612.

[88] Quillen, D. *Rational homotopy theory.* Ann. of Math. (2) 90 1969 205–295.

[89] Wallbridge, J. *Higher Tannaka duality.* Doctoral dissertation.

[90] Yetter, D. *Braided deformations of monoidal categories and Vassiliev invariants.* Higher category theory (Evanston, IL, 1997), 117–134, Contemp. Math., 230, Amer. Math. Soc., Providence, RI, 1998.