Derived Algebraic Geometry IX: Closed Immersions

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Contents

Introduction

Let $\mathbf{C}$ denote the field of complex numbers, and let $\mathcal{V}$ denote the category of schemes of finite type over $\mathbf{C}$. The category $\mathcal{V}$ admits finite limits but does not admit finite colimits. Given a diagram of finite type $\mathbf{C}$-schemes

$$X_0 \stackrel{i}{\leftarrow} X_{01} \stackrel{j}{\rightarrow} X_1,$$

we do not generally expect a pushout $X_0 \coprod_{X_{01}} X_1$ to exist in the category $\mathcal{V}$. However, there is one circumstance in which we can prove the existence of the pushout $X = X_0 \coprod_{X_{01}} X_1$: namely, the case where the maps i and j are both closed immersions. In this case, we can describe the pushout X explicitly: the diagram

$$\begin{array}{ccc} X_{01} & \xrightarrow{i} & X_0 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ X_1 & \xrightarrow{i'} & X \end{array}$$

is a pushout square in the category of topological spaces (that is, the underlying topological space of X is obtained from the underlying topological spaces of X_0 and X_1 by gluing along X_{01}), and the structure sheaf $\mathcal{O}_X$ is given by

$$j'_* \mathcal{O}_{X_0} \times_{(j' \circ i)_* \mathcal{O}_{X_{01}}} i'_* \mathcal{O}_{X_1}.$$

Our main goal in this paper is to develop an analogous picture in the setting of spectral algebraic geometry. We begin in §1 by introducing a very general notion of closed immersion, which makes sense in the context of $\mathcal{T}$-structured ∞-topoi for an arbitrary pregeometry $\mathcal{T}$ (Definition 1.1). To obtain a good theory, we need to assume that the pregeometry $\mathcal{T}$ satisfies the additional condition of being *unramified* in the sense of Definition 1.3. Under this assumption, we will show that the collection of closed immersions behaves well with respect to fiber products (Theorem 1.6). The main step of the proof will be given in §3, using some general approximation results for $\mathcal{T}$-structures which we develop in §2.

In §4 we will focus our attention to closed immersions in the setting of spectral algebraic geometry. We will show that the pregeometry $\mathcal{T}^{\mathrm{Sp}}$ which controls the theory of spectral Deligne-Mumford stacks is unramified (Proposition 4.1). Moreover, we establish a local structure theorem for closed immersions between spectral Deligne-Mumford stacks: if $\mathfrak{X} = \operatorname{Spec} A$ is affine, then a map $f : \mathfrak{Y} \to \mathfrak{X}$ is a closed immersion if and only if $\mathfrak{Y} \simeq \operatorname{Spec} B$ for some A-algebra B for which the underlying map of commutative rings $\pi_0 A \to \pi_0 B$ is surjective (Theorem 4.4).

We next study the operation of gluing geometric objects along closed immersions. In §5 we will show that if $\mathcal{T}$ is an arbitrary pregeometry and we are given a diagram

$$\mathfrak{X}_0 \stackrel{i}{\leftarrow} \mathfrak{X}_{01} \stackrel{j}{\rightarrow} \mathfrak{X}_1$$

of $\mathcal{T}$-structured ∞-topoi where i and j are closed immersions, then there exists a pushout diagram of $\mathcal{T}$-structured ∞-topoi

$$\begin{array}{ccc} \mathfrak{X}_{01} & \longrightarrow & \mathfrak{X}_0 \\ \downarrow & & \downarrow \\ \mathfrak{X}_1 & \longrightarrow & \mathfrak{X} \end{array}$$

(Theorem 5.1). In §6, we specialize to the case where $\mathcal{T} = \mathcal{T}^{\mathrm{Sp}}$. In this case, we show that if $\mathfrak{X}_0$, $\mathfrak{X}_1$, and $\mathfrak{X}_{01}$ are spectral Deligne-Mumford stacks, then $\mathfrak{X}$ is also a spectral Deligne-Mumford stack (Theorem 6.1).

Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ \mathfrak{X}_1 & \xrightarrow{i'} & \mathfrak{X} \end{array}$$

where i and j are closed immersions. In many respects, the geometry of $\mathfrak{X}$ is determined by the geometry of the constituents $\mathfrak{X}_0$ and $\mathfrak{X}_1$. For example, in §7, we will show that a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is canonically determined by its restrictions $\mathcal{F}_0 = j'^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_0)$ and $\mathcal{F}_1 = i'^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_1)$, together with a "clutching datum" $i^* \mathcal{F}_0 \simeq j^* \mathcal{F}_1$ (Theorem 7.1). In §9 we consider a nonlinear analogue of this result: a map of spectral Deligne-Mumford stacks $\mathfrak{Y} \to \mathfrak{X}$ is determined by the fiber products $\mathfrak{Y}_0 = \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}_0$ and $\mathfrak{Y}_1 = \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}_1$, together with a "clutching datum" $\mathfrak{X}_{01} \times_{\mathfrak{X}_0} \mathfrak{Y}_0 \simeq \mathfrak{X}_{01} \times_{\mathfrak{X}_1} \mathfrak{Y}_1$ (Theorem 9.1). Moreover, we will show that many properties of the morphism $\mathfrak{Y} \to \mathfrak{X}$ can be established by verifying the same property for the induced maps $\mathfrak{Y}_0 \to \mathfrak{X}_0$ and $\mathfrak{Y}_1 \to \mathfrak{X}_1$ (Proposition 9.3). In order to prove this, we will need some general remarks about finite presentation conditions on morphisms between spectral Deligne-Mumford stacks, which we collect in §8.

In the last few sections of this paper, we will apply our general theory of closed immersions to develop a "derived" version of complex analytic geometry. In §11, we will introduce a pregeometry $\mathcal{T}_{\mathrm{an}}$. Roughly speaking, a $\mathcal{T}_{\mathrm{an}}$-structure on an ∞-topos $\mathcal{X}$ is a sheaf $\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$ of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ equipped with some additional structure: namely, the ability to obtain a new section of $\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$ by "composing" a finite collection $\{s_i\}_{1 \leq i \leq n}$ of sections of $\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$ with a (possibly only locally defined) holomorphic function $\mathbf{C}^n \to \mathbf{C}$. Our main technical result (Proposition 11.12) asserts that for some purposes (such as forming pullbacks along closed immersions) this additional structure can be ignored. We will deduce this from a more general result about "unramified" transformations between pregeometries (Proposition 10.3), which we prove in §10. In §12 we will introduce a full subcategory $\mathcal{A}n_{\mathbf{C}}^{\mathrm{der}}$ of the ∞-category of $\mathcal{T}_{\mathrm{an}}$-structured ∞-topoi, which we will refer to as the *∞-category of derived complex analytic spaces*. Using Proposition 11.12, we will show verify that our definition is reasonable: for example, the ∞-category $\mathcal{A}n_{\mathbf{C}}^{\mathrm{der}}$ admits fiber products (Proposition 12.12), and contains the usual category of complex analytic spaces as a full subcategory (Theorem 12.8).

Notation and Terminology

Throughout this paper we will make extensive use of the language of ∞-categories, as developed in [40] and [41] and the earlier papers in this series. For convenience, we will adopt the following reference conventions:

- (T) We will indicate references to [40] using the letter T.
- (A) We will indicate references to [41] using the letter A.
- (V) We will indicate references to [42] using the Roman numeral V.
- (VII) We will indicate references to [43] using the Roman numeral VII.
- $(VIII)$ We will indicate references to [44] using the Roman numeral VIII.

Notation 0.1. Recall that ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}$ denotes the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are ∞-topoi and whose morphisms are geometric morphisms of ∞-topoi $f^* : \mathcal{X} \to \mathcal{Y}$. We let $\mathcal{T}\mathrm{op}$ denote the ∞-category ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}^{op}$. We will refer to $\mathcal{T}\mathrm{op}$ as the *∞-category of ∞-topoi.* If $\mathcal{T}$ is a pregeometry, we let $\mathcal{T}\mathrm{op}(\mathcal{T})$ denote the ∞-category ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T})^{op}$, where ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T})$ is defined as in Definition V.3.1.9. More informally, the objects of $\mathcal{T}\mathrm{op}(\mathcal{T})$ are pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}_{\mathcal{X}} : \mathcal{T} \to \mathcal{X}$ is a $\mathcal{T}$-structure on $\mathcal{X}$. A morphism from $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ to $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in $\mathcal{T}\mathrm{op}(\mathcal{T})$ is given by a geometric morphism of ∞-topoi $f^* : \mathcal{Y} \to \mathcal{X}$ together with a local natural transformation $f^* \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ of $\mathcal{T}$-structures on $\mathcal{X}$.

Notation 0.2. If R is an $\mathbb{E}_\infty$-ring, we let $\mathrm{Spec}\, R$ denote the nonconnective spectral Deligne-Mumford stack $\mathrm{Spec}^{\text{ét}} R = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, where $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}((\mathrm{CAlg}_R^{\text{ét}})^{op})$ is the ∞-topos of étale sheaves over R, and $\mathcal{O}_{\mathcal{X}}$ is the sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ given by the formula $\mathcal{O}_{\mathcal{X}}(R') = R'$.

1 Unramified Pregeometries and Closed Immersions

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of schemes. Recall that f is said to be a *closed immersion* if the following two conditions are satisfied:

- (a) The underlying map of topological spaces $X \to Y$ determines a homeomorphism from X to a closed subset of Y.
- (b) The map $f^* \mathcal{O}_Y \to \mathcal{O}_X$ is an epimorphism of sheaves of commutative rings on X.

Our goal in this paper is to develop an analogous theory of closed immersions in the setting of spectral algebraic geometry. With an eye towards certain applications (such as the theory of derived complex analytic spaces, which we study in §11), we will work in a more general setting. Let $\mathcal{T}$ be a pregeometry (see Definition V.3.1.1) and let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism in the ∞-category ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T})^{op}$ of $\mathcal{T}$-structured ∞-topoi. Motivated by the case of classical scheme theory, we introduce the following definitions:

Definition 1.1. Let $\mathcal{T}$ be a pregeometry, $\mathcal{X}$ an ∞-topos, and let $\alpha : \mathcal{O} \to \mathcal{O}'$ be a morphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. We will say that α is an *effective epimorphism* if, for every object $X \in \mathcal{T}$, the induced map $\mathcal{O}(X) \to \mathcal{O}'(X)$ is an effective epimorphism in the ∞-topos $\mathcal{X}$.

We will say that a morphism $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in $\mathcal{T}\mathrm{op}(\mathcal{T})$ is a *closed immersion* if the following conditions are satisfied:

- (A) The underlying geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$ is a closed immersion (see Definition T.7.3.2.7).
- (B) The map of structure sheaves $f^* \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is an effective epimorphism.

Remark 1.2. Let $\mathcal{T}$ be a pregeometry, and suppose we are given a commutative diagram

$$\begin{array}{ccc} & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) & \\ {}^{f}\swarrow & & \searrow^{h} \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) & \xrightarrow{\quad g \quad} & (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \end{array}$$

in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T})^{op}$. If g is a closed immersion, then f is a closed immersion if and only if h is a closed immersion.

In order for Definition 1.1 to be of any use, we need an additional assumption on $\mathcal{T}$. To motivate this assumption, we begin with the following observation:

$(*)$ Suppose we are given a pullback diagram of schemes

$$\begin{array}{ccc} (X', \mathcal{O}_{X'}) & \longrightarrow & (Y', \mathcal{O}_{Y'}) \\ \downarrow & & \downarrow \\ (X, \mathcal{O}_X) & \xrightarrow{f} & (Y, \mathcal{O}_Y). \end{array}$$

If f is a closed immersion, then the underlying diagram of topological spaces

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is also a pullback square.

We will say that $\mathcal{T}$ is *unramified* if it the analogue of condition $(*)$ is satisfied for some very simple diagrams of affine $\mathcal{T}$-schemes. The main result of this section (Theorem 1.6) asserts that in this case, one can deduce a version of $(*)$ for arbitrary $\mathcal{T}$-structured ∞-topoi.

Definition 1.3. Let $\mathcal{T}$ be a pregeometry. We will say that $\mathcal{T}$ is *unramified* if the following condition is satisfied: for every morphism $f : X \to Y$ in $\mathcal{T}$ and every object $Z \in \mathcal{T}$, the diagram

$$\begin{CD} X \times Z @>>> X \times Y \times Z \\ @VVV @VVV \\ X @>>> X \times Y \end{CD}$$

induces a pullback square

$$\begin{CD} \mathcal{X}_{X \times Z} @>>> \mathcal{X}_{X \times Y \times Z} \\ @VVV @VVV \\ \mathcal{X}_X @>>> \mathcal{X}_{X \times Y} \end{CD}$$

in $\mathcal{T}\mathrm{op}$.

Example 1.4. Let $\mathcal{T}$ be a discrete pregeometry. For every object $U \in \mathcal{T}$, the ∞-topos $\mathcal{X}_U$ is equivalent to $\mathcal{S}$. It follows that $\mathcal{T}$ is unramified.

Example 1.5. Let $\mathcal{T}$ be a pregeometry. Suppose that the composite functor

$$\mathcal{T} \stackrel{\mathrm{Spec}^{\mathcal{T}}}{\longrightarrow} \mathcal{T}\mathrm{op}(\mathcal{T}) \to \mathcal{T}\mathrm{op}$$

preserves finite products. Then $\mathcal{T}$ is unramified.

The significance of Definition 1.3 is that it allows us to prove the following result:

Theorem 1.6. *Let $\mathcal{T}$ be an unramified pregeometry, and suppose we are given morphisms $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $g : (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ in $\mathcal{T}\mathrm{op}(\mathcal{T})$. Assume that f induces an effective epimorphism $f^* \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}}$. Then:*

(1) *There exists a pullback square*

$$\begin{CD} (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) @>{f'}>> (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \\ @V{g'}VV @VV{g}V \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) @>{f}>> (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{CD}$$

in $\mathcal{T}\mathrm{op}(\mathcal{T})$.

(2) *The map f' induces an effective epimorphism $f'^* \mathcal{O}_{\mathcal{X}'} \to \mathcal{O}_{\mathcal{Y}'}$ in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y}')$.*

(3) *The underlying diagram of ∞-topoi*

$$\begin{CD} \mathcal{Y}' @>>> \mathcal{X}' \\ @VVV @VVV \\ \mathcal{Y} @>>> \mathcal{X} \end{CD}$$

is a pullback square in $\mathcal{T}\mathrm{op}$.

(4) *For every geometric morphism of ∞-topoi $h^* : \mathcal{Y}' \to \mathcal{Z}$, the diagram*

$$\begin{array}{ccc} h^* f'^* g^* \mathcal{O}_{\mathcal{X}} & \longrightarrow & h^* g'^* \mathcal{O}_{\mathcal{Y}} \\ \downarrow & & \downarrow \\ h^* f'^* \mathcal{O}_{\mathcal{X}'} & \longrightarrow & h^* \mathcal{O}_{\mathcal{Y}'} \end{array}$$

is a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$.

Corollary 1.7. *Let $\mathcal{T}$ be an pregeometry and suppose we are given maps*

$$(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \xrightarrow{f} (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \leftarrow (\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$$

in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T})^{op}$. Assume that $\mathcal{T}$ is unramified and that f is a closed immersion. Then there exists a pullback square

$$\begin{array}{ccc} (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) & \xrightarrow{f'} & (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \\ \downarrow & & \downarrow \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) & \xrightarrow{f} & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T})^{op}$. Moreover, f' is a closed immersion and the diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

is a pullback square in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}^{op}$.

Proof. Combine Theorem 1.6 with Corollary T.7.3.2.14. ☐

It is not difficult to reduce the general case of Theorem 1.6 to the special case in which the ∞-topoi $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{X}'$ are all the same. In §3 we will prove the following result:

Proposition 1.8. *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{T}$ a pregeometry. Suppose we are given morphisms*

$$\overline{\mathcal{O}}_0 \overset{\beta}{\leftarrow} \overline{\mathcal{O}} \overset{\alpha}{\rightarrow} \mathcal{O}$$

in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Assume that α is an effective epimorphism and that $\mathcal{T}$ is unramified. Then:

(1) *There exists a pushout square σ :*

$$\begin{array}{ccc} \overline{\mathcal{O}} & \xrightarrow{\alpha} & \mathcal{O} \\ \downarrow{\scriptstyle \beta} & & \downarrow \\ \overline{\mathcal{O}}_0 & \xrightarrow{\alpha_0} & \mathcal{O}_0 \end{array}$$

in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$.

(2) *The map α_0 is an effective epimorphism.*

(3) *For any geometric morphism of ∞-topoi $f^* : \mathcal{X} \to \mathcal{Y}$, the image $f^*(\sigma)$ is a pullback square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$.*

Let us assume Proposition 1.8 for the moment and explain how it leads to a proof of Theorem 1.6.

Proof of Theorem 1.6. Choose a pullback diagram

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}^{op}$. Let $p : \mathcal{T}\mathrm{op}(\mathcal{T}) \to \mathcal{T}\mathrm{op}$ be the forgetful functor. We claim that σ can be lifted to a p-limit diagram $\overline{\sigma}$:

$$\begin{array}{ccc} (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) & \xrightarrow{f'} & (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) & \xrightarrow{f} & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

in $\mathcal{T}\mathrm{op}(\mathcal{T}_0)$. Using Propositions T.4.3.1.9 and T.4.3.1.10, it suffices to prove the existence of a pushout square τ :

$$\begin{array}{ccc} f'^{*} g^{*} \mathcal{O}_{\mathcal{X}} & \longrightarrow & g'^{*} \mathcal{O}_{\mathcal{Y}} \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ f'^{*} \mathcal{O}_{\mathcal{X}'} & \xrightarrow{\beta} & \mathcal{O}_{\mathcal{Y}'} \end{array}$$

in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y}')$, such that $h^*(\tau)$ remains a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$ for every geometric morphism $h^* : \mathcal{Y}' \to \mathcal{Z}$. Proposition 1.8 guarantees the existence of τ. Using Proposition T.4.3.1.5, we conclude that $\overline{\sigma}$ is a pullback square in $\mathcal{T}\mathrm{op}(\mathcal{T}_0)$. This proves assertion (1); assertions (3) and (4) follow from the construction. Finally, (2) follows from the corresponding part of Proposition 1.8. □

2 Resolutions of $\mathcal{T}$-Structures

Let $\mathcal{T}$ be a discrete pregeometry (that is, a small ∞-category which admits finite products). By definition, the ∞-category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{S})$ is the full subcategory of $\mathrm{Fun}(\mathcal{T}, \mathcal{S})$ spanned by those functors which preserve finite products. We can therefore identify $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{S})$ with the ∞-category $\mathcal{P}_{\Sigma}(\mathcal{T}^{op})$ studied in §T.5.5.8. According to Proposition T.5.5.8.15, the Yoneda embedding $\mathcal{T}^{op} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{S})$ exhibits $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{S})$ as the ∞-category freely generated from $\mathcal{T}^{op}$ by adjoining sifted colimits. In particular, every object of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{S})$ can be constructed from objects belonging to the essential image of j by means of sifted colimits (Lemma T.5.5.8.14).

Our goal in this section is to obtain an analogue of the results cited above, where we replace the ∞-category $\mathcal{S}$ of spaces by an arbitrary ∞-topos $\mathcal{X}$ (and where we do not assume the pregeometry $\mathcal{T}$ to be discrete). Roughly speaking, we would like to say that every $\mathcal{T}$-structure on $\mathcal{X}$ can be "resolved" by $\mathcal{T}$-structures which are of a particularly simple type (namely, which are *elementary* in the sense of Definition 2.6). The basic idea is simple: if $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, then every global section of $\mathcal{O}(X)$ (where X is an object of $\mathcal{T}$) determines a "free" $\mathcal{T}$-structure on $\mathcal{X}$ equipped with a map to $\mathcal{O}$. Unfortunately, these maps do not generally give us enough information to recover $\mathcal{O}$, because the objects $\mathcal{O}(U)$ generally need not admit many global sections. Consequently, we will not be able to describe $\mathcal{O}$ as a colimit of objects of $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$. We can remedy the situation by considering not only global sections of $\mathcal{O}$, but also local sections over each object $X \in \mathcal{X}$. Every map $U \to \mathcal{O}(X)$ determines a "free" $\mathcal{T}$-structure on the ∞-topos $\mathcal{X}^{/U}$, and a map from this "free" $\mathcal{T}$-structure to the restriction $\mathcal{O}\,|U$. We will show that $\mathcal{O}$ can always be recovered as a *relative colimit* of these types of maps. To make this idea precise, we need to introduce a bit of terminology.

Notation 2.1. Let $\mathcal{T}$ be a pregeometry and $\mathcal{X}$ an ∞-topos. We let $\underline{\mathit{Str}}_{\mathcal{T}}(\mathcal{X})$ denote the full subcategory of

$$\mathrm{Fun}(\mathcal{T}, \mathrm{Fun}(\Delta^1, \mathcal{X})) \times_{\mathrm{Fun}(\mathcal{T}, \mathrm{Fun}(\{1\}, \mathcal{X}))} \mathcal{X}$$

spanned by those vertices which correspond to an object $U \in \mathcal{X}$ and a $\mathcal{T}$-structure on the ∞-topos $\mathcal{X}^{/U}$.

The ∞-category $\underline{Str}_{\mathcal{T}}(\mathcal{X})$ comes equipped with a Cartesian fibration $p' : \underline{Str}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$. For each object $U \in \mathcal{X}$, the fiber $p^{-1}\{U\}$ can be identified with $\mathrm{Str}_{\mathcal{T}}(\mathcal{X}^{/U})$. If $f : U \to V$ is a morphism in $\mathcal{X}$, then the associated functor $\mathrm{Str}_{\mathcal{T}}(\mathcal{X}^{/V}) \to \mathrm{Str}_{\mathcal{T}}(\mathcal{X}^{/U})$ is given by composition with the geometric morphism $f^* : \mathcal{X}^{/V} \to \mathcal{X}^{/U}$ given by $V' \mapsto V' \times_V U$.

The objects of $\underline{Str}_{\mathcal{T}}(\mathcal{X})$ can be thought of as pairs $(U, \mathcal{O})$, where $U \in \mathcal{X}$ and $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X}^{/U})$. A morphism from $(U, \mathcal{O})$ to $(V, \mathcal{O}')$ in $\underline{Str}_{\mathcal{T}}(\mathcal{X})$ can be identified with a pair consisting of a map $f : U \to V$ in $\mathcal{X}$ and a map $\alpha : \mathcal{O} \to f^* \mathcal{O}'$ in $\mathrm{Str}_{\mathcal{T}}(\mathcal{X}^{/U})$. We let $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ be the subcategory of $\underline{Str}_{\mathcal{T}}(\mathcal{X})$ spanned by all the objects of $\underline{Str}_{\mathcal{T}}(\mathcal{X})$ together with those morphisms (f, α) in $\underline{Str}_{\mathcal{T}}(\mathcal{X})$ for which α is local. The functor p' restricts to a Cartesian fibration $p : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$, whose fiber over an object $U \in \mathcal{X}$ can be identified with $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/U})$.

Let $\mathcal{T}$ be a pregeometry and $\mathcal{X}$ an ∞-topos. According to Proposition V.3.3.1, the ∞-category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ admits sifted colimits, which are computed pointwise: that is, for each $U \in \mathcal{T}$, evaluation at U induces a functor $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$ which preserves sifted colimits. Our first goal is to establish an analogous statement for colimits relative to the projection $p : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$.

Proposition 2.2. *Let $\mathcal{X}$ be an ∞-topos, $\mathcal{T}$ a pregeometry, and $q : \mathcal{C} \to \mathcal{X}$ a Cartesian fibration, where the fibers of q are essentially small sifted ∞-categories. Let $\overline{q} : \mathcal{C}^{\triangleright} \to \mathcal{X}$ be an extension of q which carries the cone point of $\mathcal{C}^{\triangleright}$ to a final object of $\mathcal{X}$, and let $p : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$ be the forgetful functor of Notation 2.1. Suppose that we are given a map $Q : \mathcal{C} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ such that $q = p \circ Q$ and Q carries each q-Cartesian morphism in $\mathcal{C}$ to a p-Cartesian morphism in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Then:*

(1) *There exists a p-colimit diagram $\overline{Q} : \mathcal{C}^{\triangleright} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ extending Q with $p \circ \overline{Q} = \overline{q}$.*

(2) *For every geometric morphism of ∞-topoi $f^* : \mathcal{X} \to \mathcal{Y}$, the induced map $f^*\overline{Q} : \mathcal{C}^{\triangleright} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$ is a p'-colimit diagram, where $p' : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y}) \to \mathcal{Y}$ is the projection map.*

(3) *Let $\overline{Q} : \mathcal{C}^{\triangleright} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ be an arbitrary map extending Q with $p \circ \overline{Q} = \overline{q}$. Then $\overline{Q}$ is a p-colimit diagram if and only if, for each object $U \in \mathcal{T}$, the composite map $\mathcal{C}^{\triangleright} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$ is a colimit diagram in $\mathcal{X}$.*

The proof of Proposition 2.2 will require the following technical lemma:

Lemma 2.3. *Let $\mathcal{X}$ be an ∞-topos and $q : \mathcal{C} \to \mathcal{X}$ a Cartesian fibration. Assume that each fiber of q is an essentially small sifted ∞-category. Then:*

(1) *Let $q' : \mathcal{C} \to \mathcal{X}$ be another functor equipped with a natural transformation $\alpha : q' \to q$ with the following property:*

   $(*)$ *For every q-Cartesian morphism $C \to D$ in $\mathcal{C}$, the diagram*

$$\begin{array}{ccc} q'(C) & \longrightarrow & q'(D) \\ \downarrow & & \downarrow \\ q(C) & \longrightarrow & q(D) \end{array}$$

   is a pullback square in $\mathcal{X}$. Then q' has a colimit in $\mathcal{X}$.

(2) *Let $\{\alpha_i : q_i \to q\}_{i \in I}$ be a finite collection of natural transformations of functors $\mathcal{C} \to \mathcal{X}$ which satisfy the hypotheses of (1), and let $\alpha : q' \to q$ be their product (in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{X})_{/q}$). Then the natural map $\varinjlim(q') \to \prod_{i \in I} \varinjlim(q_i)$ is an equivalence in $\mathcal{X}$.*

Proof. We first prove (1). Since $\mathcal{X}$ is presentable, we can choose an essentially small subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ such that the identity functor $\mathrm{id}_{\mathcal{X}}$ is a left Kan extension of $\mathrm{id} \,|\, \mathcal{X}_0$. Let $\mathcal{C}_0 = \mathcal{C} \times_{\mathcal{X}} \mathcal{X}_0$. Since the fibers of q are essentially small, the ∞-category $\mathcal{C}_0$ is essentially small. We will prove that q' is a left Kan extension of $q' | \, \mathcal{C}_0$. The existence of a colimit for q' will then follow from Lemma T.4.3.2.7, and the equivalence $q'(C) \rightarrow \varinjlim(q') \times_{\varinjlim(q)} q(C)$ will follow from Theorem T.6.1.3.9 (since we may assume without loss of generality that $C \in \mathcal{C}_0$).

Fix an object $C \in \mathcal{C}$; we wish to prove that q' exhibits $q'(C)$ as a colimit of the diagram

$$\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/C} \rightarrow \mathcal{C} \stackrel{q'}{\rightarrow} \mathcal{X}.$$

Let $\mathcal{C}'$ denote the full subcategory of $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/C}$ spanned by those objects which correspond to q-Cartesian morphisms $D \rightarrow C$. The inclusion $\mathcal{C}' \subseteq \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/C}$ admits a left adjoint and is therefore left cofinal. It will therefore suffice to show that q' exhibits $q'(C)$ as a colimit of $q' | \, \mathcal{C}'$. Note that q induces an equivalence $\mathcal{C}' \rightarrow \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_{/q(C)}$. The assumption on α guarantees that $q' | \, \mathcal{C}'$ can be identified with the functor $D \mapsto q(D) \times_{q(C)} q'(C)$. Since colimits in $\mathcal{X}$ are universal, it suffices to show that $q(C)$ is the colimit of the forgetful functor $\mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_{/q(C)} \rightarrow \mathcal{X}$; this follows from our assumption that $\mathrm{id}_{\mathcal{X}}$ is a left Kan extension of $\mathrm{id}_{\mathcal{X}} \,|\, \mathcal{X}_0$.

We now prove (2). We work by induction on the cardinality of I. Suppose first that I is empty; in this case we must prove that $\varinjlim(q)$ is a final object of $\mathcal{X}$. We first show that q is left cofinal. Lemma T.4.1.3.2, we are reduced to proving that each fiber of q is weakly contractible. This follows from our assumption that the fibers of q are sifted (Proposition T.5.5.8.7). Since q is left cofinal, we have an equivalence $\varinjlim(q) \simeq \varinjlim(\mathrm{id}_{\mathcal{X}})$. Since $\mathcal{X}$ has a final object $\mathbf{1}$, we conclude that $\varinjlim(\mathrm{id}_{\mathcal{X}}) \simeq \mathbf{1}$ as desired.

In the case where I has a single element, there is nothing to prove. If I has more than one element, we can write I as a disjoint union of proper subsets I_- and I_+. Let $q_- = \prod_{i \in I_-} q_i$ and $q_+ = \prod_{i \in I_+} q_i$ (the product taken in $\mathrm{Fun}(\mathcal{C}, \mathcal{X})_{/q}$), so that $q' \simeq q_- \times_q q_+$. Using the inductive hypothesis, we are reduced to proving that $\varinjlim(q') \simeq \varinjlim(q_-) \times \varinjlim(q_+)$.

We next prove that the diagonal map $\delta : \mathcal{C} \rightarrow \mathcal{C} \times_{\mathcal{X}} \mathcal{C}$ is left cofinal. According to Theorem T.4.1.3.1, it will suffice to show that for every pair of objects $(C, D) \in \mathcal{C}$ having the same image $U \in \mathcal{X}$, the fiber product $\mathcal{C}'' = \mathcal{C} \times_{\mathcal{C} \times_{\mathcal{X}} \mathcal{C}} (\mathcal{C} \times_{\mathcal{X}} \mathcal{C})_{(C,D)/}$ is weakly contractible. Let $\mathcal{C}'''$ denote the fiber product $\mathcal{C}'' \times_{\mathcal{X}} \{U\}$. Since q is a Cartesian fibration, the inclusion $\mathcal{C}''' \subseteq \mathcal{C}''$ admits a right adjoint; it will therefore suffice to show that $\mathcal{C}''$ is contractible. This follows from Theorem T.4.1.3.1, since the diagonal map $\mathcal{C}_U \rightarrow \mathcal{C}_U \times \mathcal{C}_U$ is left cofinal by virtue of our assumption that $\mathcal{C}_U$ is sifted.

We define functors $f, f_-, f_+ : \mathcal{C} \times_{\mathcal{X}} \mathcal{C} \rightarrow \mathcal{X}$ by the formulas

$$f_-(C, D) = q_-(C) \qquad f(C, D) = q_-(C) \times_{q(C)} q_+(D) \qquad f_+(C, D) = q_+(D).$$

There are evident natural transformations $f_- \leftarrow f \rightarrow f_+$. Note that $q' = f \circ \delta$, $q_- = f_- \circ \delta$, and $q_+ = f_+ \circ \delta$. Since δ is left cofinal, we are reduced to proving that the canonical map $\varinjlim(f) \rightarrow \varinjlim(f_-) \times \varinjlim(f_+)$ is an equivalence.

Let $\epsilon : \mathcal{C} \rightarrow \mathcal{C} \rightarrow \mathcal{C} \times_{\mathcal{X}} \mathcal{C}$ denote a right adjoint to the inclusion $\mathcal{C} \times_{\mathcal{X}} \mathcal{C} \hookrightarrow \mathcal{C} \times \mathcal{C}$. Concretely, ϵ is given by the formula $(C, D) \mapsto (\pi_1^* C, \pi_2^* D)$, where π_1 and π_2 are the projection maps $q(C) \times q(D) \rightarrow q(C)$ and $q(C) \times q(D) \rightarrow q(D)$. Since ϵ admits a left adjoint, it is left cofinal. We are therefore reduced to proving that the map

$$\varinjlim(f \circ \epsilon) \rightarrow \varinjlim(f_- \circ \epsilon) \times \varinjlim(f_+ \circ \epsilon)$$

is an equivalence. Since the natural transformations $q_- \rightarrow q \leftarrow q_+$ satisfy $(*)$, we see that these functors are given informally by the formulas

$$(f_- \circ \epsilon)(C, D) = q_-(C) \times q(D) \qquad (f \circ \epsilon)(C, D) = q_-(C) \times q_+(D) \qquad (f_+ \circ \epsilon)(C, D) = q(C) \times q_+(D).$$

Since colimits in $\mathcal{X}$ are universal, we are reduced to proving that

$$\varinjlim(q_-) \times \varinjlim(q) \leftarrow \varinjlim(q_-) \times \varinjlim(q_+) \rightarrow \varinjlim(q) \times \varinjlim(q_+)$$

is a limit diagram in $\mathcal{X}$. This follows immediately from our observation that $\varinjlim(q)$ is a final object of $\mathcal{X}$. $\square$

Remark 2.4. Let $\mathcal{C}$ be a nonempty ∞-category such that every pair of objects of $\mathcal{C}$ admits a coproduct in $\mathcal{C}$. Then $\mathcal{C}$ is sifted: this is an immediately consequence of Theorem T.4.1.3.1.

Proof of Proposition 2.2. Let $\mathcal{E} = \mathcal{C}^{\triangleright} \times \mathcal{T} \times \Delta^1$, and let $\mathcal{E}_0$ denote the full subcategory of $\mathcal{E}$ spanned by $\mathcal{C} \times \mathcal{T} \times \Delta^1$ and $\mathcal{C}^{\triangleright} \times \mathcal{T} \times \{1\}$. Let v denote the cone point of $\mathcal{C}^{\triangleright}$. The functors $\overline{q}$ and Q together determine a map $F : \mathcal{E} \to \mathcal{X}$. We first claim that there exists a functor $\overline{F} : \mathcal{E} \to \mathcal{X}$ which is a left Kan extension of F. In view of Lemma T.4.3.2.13, it will suffice to prove that for every object $U \in \mathcal{T}$, the diagram $F|(\mathcal{E}_0)_{/(v,U,0)}$ admits a colimit in $\mathcal{X}$. Let q' denote the composition of this diagram with the left cofinal inclusion $\mathcal{C} \times \{U\} \times \{0\} \to (\mathcal{E}_0)_{/(v,U,0)}$; we are therefore reduced to proving that q' has a colimit in $\mathcal{X}$. The restriction of F to $\mathcal{C} \times \{U\} \times \Delta^1$ determines a natural transformation $\alpha : q' \to q$. Since Q carries q-Cartesian morphisms in $\mathcal{C}$ to p-Cartesian morphisms in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, the natural transformation α satisfies condition $(*)$ of Lemma 2.3, so that q' admits a colimit in $\mathcal{X}$. This proves the existence of $\overline{F}$, which is in turn equivalent to giving a functor $\overline{Q} : \mathcal{C}^{\triangleright} \to \mathrm{Fun}(\mathcal{T}, \mathrm{Fun}(\Delta^1, \mathcal{X})) \times_{\mathrm{Fun}(\mathcal{T}, \mathrm{Fun}(\{1\}, \mathcal{X}))} \mathcal{X}$ satisfying the hypothesis of (3).

We next claim:

(*a*) For every admissible morphism $U \to X$ in $\mathcal{T}$ and every object $C \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} F(C,U,0) & \longrightarrow & \overline{F}(v,U,0) \\ \downarrow & & \downarrow \\ F(C,X,0) & \longrightarrow & \overline{F}(v,X,0) \end{array}$$

is a pullback square in $\mathcal{X}$.

To prove (*a*), it will suffice (by Theorem T.6.1.3.9) to show that for every morphism $C \to D$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(C,U,0) & \longrightarrow & F(D,U,0) \\ \downarrow & & \downarrow \\ F(C,X,0) & \longrightarrow & F(D,X,0) \end{array}$$

is a pullback square in $\mathcal{X}$. This follows from our assumption that the map $Q(C) \to Q(D)$ is a morphism in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$.

We now claim that $\overline{Q}$ factors through $\underline{Str}_{\mathcal{T}}(\mathcal{X})$: that is, that $\overline{Q}(v)$ is a $\mathcal{T}$-structure on $\mathcal{X}^{/\mathbf{1}}$. We must verify that $\overline{Q}(v) : \mathcal{T} \to \mathcal{X}$ satisfies the conditions listed in Definition V.3.1.4:

(*i*) The functor $\overline{Q}(v)$ preserves finite products. This follows immediately from the second assertion of Lemma 2.3.

(*ii*) Suppose we are given a pullback diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

in $\mathcal{T}$, where the vertical maps are admissible. We wish to prove that the induced diagram

$$\begin{array}{ccc} \overline{F}(v,U',0) & \longrightarrow & \overline{F}(v,U,0) \\ \downarrow & & \downarrow \\ \overline{F}(v,X',0) & \longrightarrow & \overline{F}(v,X,0) \end{array}$$

is a pullback square in $\mathcal{X}$. In view of Theorem T.6.1.3.9, it will suffice to show that for each $C \in \mathcal{C}$, the outer square in the diagram

$$\begin{array}{ccccc} F(C,U',0) & \longrightarrow & F(C,U,0) & \longrightarrow & \overline{F}(v,U,0) \\ \downarrow & & \downarrow & & \downarrow \\ F(C,X',0) & \longrightarrow & F(C,X,0) & \longrightarrow & \overline{F}(v,X,0) \end{array}$$

is a pullback. The square on the left is a pullback since $Q(C) \in \underline{Str}_{\mathcal{T}}(\mathcal{X})$, and the square on the right is a pullback by (a).

(iii) Suppose we are given an admissible covering $\{U_i \to U\}$ in $\mathcal{T}$. We wish to prove that the induced map $\coprod_i \overline{F}(v,U_i,0) \to \overline{F}(v,U,0)$ is an effective epimorphism. For this, it suffices to show that the composite map $\coprod_{i,C} F(C,U_i,0) \to \overline{F}(v,U,0)$ is an effective epimorphism. This map factors as a composition

$$\coprod_{i,C} F(C,U_i,0) \to \coprod_C F(C,U,0) \to \overline{F}(v,U,0)$$

where the first map is an effective epimorphism (since each $Q(C)$ belongs to $\underline{Str}_{\mathcal{T}}(\mathcal{X})$) and the second map is an effective epimorphism because $\overline{F}(v,U,0) \simeq \varinjlim_C F(C,U,0)$.

We next claim that $\overline{Q}$ factors through $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. In other words, we claim that for each $C \in \mathcal{C}$, the map $Q(C) \to \overline{Q}(v)$ belongs to $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. This follows immediately from (a).

To complete the proof of (1) and (2) and the "if" direction of (3), it will suffice to show that for every geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$, the pullback $f^*\overline{Q}$ is a p'-colimit diagram in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$, where $p' : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y}) \to \mathcal{Y}$ denotes the projection map. Note that p' is the restriction of a map $p'' : \underline{Str}_{\mathcal{T}}(\mathcal{Y}) \to \mathcal{Y}$, where $\underline{Str}_{\mathcal{T}}(\mathcal{Y})$ is a full subcategory of $\mathrm{Fun}(\mathcal{T}, \mathrm{Fun}(\Delta^1, \mathcal{Y})) \times_{\mathrm{Fun}(\mathcal{T},\mathrm{Fun}(\{1\},\mathcal{Y}))} \mathcal{Y}$. Since f^* preserves small colimits, the proof of Lemma 2.3 shows that $f^* \circ \overline{F}$ is a left Kan extension of $f^* \circ F$, so that $f^*\overline{Q}$ is a p''-colimit diagram. We wish to show that this p''-colimit diagram is also a p'-colimit diagram: in other words, that if $\mathcal{O} \in \underline{Str}_{\mathcal{T}}(\mathcal{Y})$, then a morphism $f^*\overline{Q}(v) \to \mathcal{O}$ belongs to $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$ if and only if each of the composite maps $f^*Q(C) \to f^*\overline{Q}(v) \to \mathcal{O}$ belongs to $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$. The "only if" direction is clear. Conversely, suppose that each composite map $f^*Q(C) \to \mathcal{O}$ belongs to $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$, and let $U \to X$ be an admissible morphism in $\mathcal{T}$. For each $C \in \mathcal{C}$, the canonical map $f^*F(C,U,0) \to f^*F(C,X,0) \times_{\mathcal{O}(X)} \mathcal{O}(U)$ is a pullback square. Since colimits in $\mathcal{Y}$ are universal, we conclude that the canonical map $f^*\overline{F}(v,U,0) \to f^*\overline{F}(v,X,0) \times_{\mathcal{O}(X)} \mathcal{O}(U)$ is an equivalence, so that $f^*\overline{Q}(v) \to \mathcal{O}$ belongs to $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$ as desired.

The "only if" direction of (3) now follows from the uniqueness of p-limit diagrams. □

Let us now suppose that $\mathcal{T}$ is an arbitrary pregeometry, and that $\mathcal{O}$ is a $\mathcal{T}$-structure on an ∞-topos $\mathcal{X}$. We would like to show that there exists a diagram $Q : \mathcal{C} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ satisfying the hypotheses of Proposition 2.2, where each $Q(C)$ is a reasonably simple object of $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, and $\mathcal{O}$ is a colimit of the diagram Q relative to the projection $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$. Our first goal is to formulate precisely what it means for the objects $Q(C)$ to be "reasonably simple."

Notation 2.5. Let $\mathcal{T}$ be a pregeometry. We let $\mathrm{Spec}^{\mathcal{T}} : \mathcal{T} \to \mathcal{T}\mathrm{op}(\mathcal{T})$ be the absolute spectrum functor of Remark V.3.5.2. For every object $U \in \mathcal{T}$, we will denote the $\mathcal{T}$-structured ∞-topos $\mathrm{Spec}^{\mathcal{T}}(U)$ by $(\mathcal{X}_U, \mathcal{O}_U)$.

Definition 2.6. Let $\mathcal{X}$ be an ∞-topos, $\mathcal{T}$ a pregeometry, and $\mathcal{O}$ an object of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. We will say that $\mathcal{O}$ is *elementary* if there exists an object $X \in \mathcal{T}$ and a morphism $f : (\mathcal{X}, \mathcal{O}) \to \mathrm{Spec}^{\mathcal{T}}(X)$ in $\mathcal{T}\mathrm{op}(\mathcal{T})$ which induces an equivalence $f^* \mathcal{O}_X \simeq \mathcal{O}$. More generally, we say that an object $(\mathcal{O}, U) \in \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ is *elementary* if $\mathcal{O}$ is elementary when viewed as an object of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/U})$.

Remark 2.7. Let $\mathcal{X}$ be an ∞-topos, $\mathcal{T}$ a pregeometry, and fix an object $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/\mathbf{1}})$ (where $\mathbf{1}$ denotes a final object of $\mathcal{X}$). For each object $X \in \mathcal{T}$ and every map $\eta : U \to \mathcal{O}(X)$ in $\mathcal{X}$, we can identify η with a global section of $\mathcal{O}(X)|\,\mathcal{X}^{/U}$, which is classified by a map $f : (U, \mathcal{O}\,|U) \to \mathrm{Spec}^{\mathcal{T}}(X)$ in $\mathcal{T}\mathrm{op}(\mathcal{T})$. We let $\mathcal{O}_\eta$ denote the pullback $f^* \mathcal{O}_X \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/U})$. Note that f induces a map $\alpha : (U, \mathcal{O}_\eta) \to (\mathbf{1}, \mathcal{O})$ in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, and that $(U, \mathcal{O}_\eta)$ is elementary. Unwinding the definitions, we see that a morphism $\alpha : (U, \mathcal{O}') \to (\mathbf{1}, \mathcal{O})$ in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ arises via this construction (for some $X \in \mathcal{X}$ and some map $\eta : U \to \mathcal{O}(X)$) if and only if $(U, \mathcal{O}')$ is elementary. Note however that the object $X \in \mathcal{T}$ and the morphism $\eta : U \to \mathcal{O}(X)$ are generally not determined by α.

Remark 2.8. Let $\mathcal{X}$, $\mathcal{T}$, and $\mathcal{O}$ be as in Remark 2.7. Choose an object $X \in \mathcal{T}$ and a map $\eta : U \to \mathcal{O}(X)$ in $\mathcal{X}$, so that the object $(U, \mathcal{O}_\eta) \in \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ is defined. By construction, the map η lifts canonically to a map $\overline{\eta} : U \to \mathcal{O}_\eta(X)$ (in $\mathcal{X}^{/U}$). Using the universal property of $\mathrm{Spec}^{\mathcal{T}}(X)$, we obtain the following universal property $\mathcal{O}_\eta$: for every map $\mathcal{O}' \to \mathcal{O}\,|U$ in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/U})$, evaluation on $\overline{\eta}$ determines a homotopy fiber sequence

$$\mathrm{Map}_{\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/U})_{/\,\mathcal{O}\,|U}}(\mathcal{O}_\eta, \mathcal{O}') \to \mathrm{Map}_{\mathcal{X}^{/U}}(U, \mathcal{O}'(X)) \to \mathrm{Map}_{\mathcal{X}}(U, \mathcal{O}(X))$$

(where the fiber is taken over the point $\eta \in \mathrm{Map}_{\mathcal{X}}(U, \mathcal{O}(X))$).

Remark 2.9. Let $\mathcal{X}$, $\mathcal{T}$, and $\mathcal{O}$ be as in Remark 2.7. Let $X \in \mathcal{T}$ be a product of finitely many objects $X_i \in \mathcal{T}$, and let $\eta : U \to \mathcal{O}(X)$ be a morphism in $\mathcal{X}$ which determines maps $\eta_i : U \to \mathcal{O}(X_i)$. The universal property of Remark 2.8 shows that $\mathcal{O}_\eta$ can be identified with the coproduct of the objects $\mathcal{O}_{\eta_i}$ in the ∞-category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/U})_{/\,\mathcal{O}\,|U}$.

Remark 2.10. Let $\mathcal{X}$, $\mathcal{T}$, and $\mathcal{O}$ be as in Remark 2.7. Let $X \in \mathcal{T}$ be an object and let $\eta : U \to \mathcal{O}(X)$ be a morphism in $\mathcal{X}$. Suppose that $f : V \to U$ is another morphism in $\mathcal{X}$, and let $f^* : \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/U}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/V})$ be the induced pullback map. Then there is a canonical equivalence $f^* \mathcal{O}_\eta \simeq \mathcal{O}_{\eta \circ f}$ in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/V})$.

We now formulate a precise sense in which the ∞-category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ is "locally generated" by elementary objects under sifted colimits.

Proposition 2.11. *Let $\mathcal{X}$ be an ∞-topos, $\mathcal{T}$ a pregeometry, and $\mathcal{O}$ an object of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Then there exists a Cartesian fibration $q : \mathcal{C} \to \mathcal{X}$ and a diagram $Q : \mathcal{C}^{\triangleright} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ with the following properties:*

1. *Let $p : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$ be the forgetful functor. The composition $p \circ Q$ is an extension of q which carries the cone point of $\mathcal{C}^{\triangleright}$ to a final object $\mathbf{1} \in \mathcal{X}$.*
2. *For each object $C \in \mathcal{C}$, the object $Q(C) \in \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ is elementary.*
3. *The fibers of q are essentially small sifted ∞-categories.*
4. *The functor Q carries q-Cartesian morphisms in $\mathcal{C}$ to p-Cartesian morphisms in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$.*
5. *The map Q is a p-colimit diagram (see Proposition 2.2).*
6. *The map Q carries the cone point of $\mathcal{C}^{\triangleright}$ to an object of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/\mathbf{1}}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ equivalent to $\mathcal{O}$.*

Proposition 2.11 is an immediate consequence of the following pair of lemmas:

Lemma 2.12. *Let $\mathcal{X}$ be an ∞-topos, $\mathcal{T}$ a pregeometry, and $\mathcal{O} : \mathcal{T} \to \mathcal{X}$ an object of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, which we identify $\mathcal{O}$ with an object of $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ lying over the final object $\mathbf{1}$ of $\mathcal{X}$. Let $\mathcal{E} \subseteq \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1}, \mathcal{O})}$ be the full subcategory spanned by those morphisms $(U, \mathcal{O}') \to (\mathbf{1}, \mathcal{O})$ such that $\mathcal{O}'$ is elementary. Then the forgetful functor $q : \mathcal{E} \to \mathcal{X}$ is a Cartesian fibration, and the fibers of q are essentially small ∞-categories which admit finite coproducts (and are therefore sifted; see Remark 2.4).*

Lemma 2.13. *Let $\mathcal{X}$ be an ∞-topos, $\mathcal{T}$ a pregeometry, and $\mathcal{O} : \mathcal{T} \to \mathcal{X}$ an object of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Let $p : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$ be the canonical map, and let $\mathcal{E}$ be defined as in Lemma 2.12. Then the map*

$$\mathcal{E}^{\triangleright} \subseteq \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})^{\triangleright}_{/(\mathbf{1},\mathcal{O})} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$$

is a p-colimit diagram.

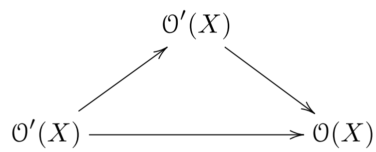
Remark 2.14. More informally, Lemma 2.13 asserts that if $\mathcal{O} : \mathcal{T} \to \mathcal{X}$ is a $\mathcal{T}$-structure on $\mathcal{X}$, then $\mathcal{O}$ can be identified with the p-colimit of $\{(U, \mathcal{O}')\}$, where $(U, \mathcal{O}')$ ranges over all locally elementary objects of $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ equipped with a map $(U, \mathcal{O}') \to (\mathbf{1}, \mathcal{O})$.

Proof of Lemma 2.12. It follows from Remark 2.10 that q is a Cartesian fibration, and Remark 2.9 guarantees that the fibers of q admit finite coproducts. It remains to show that for each object $U \in \mathcal{X}$, the fiber $q^{-1}\{U\}$ essentially small. According to Remark 2.7, every object of $q^{-1}\{U\}$ is given by the morphism $(U, \mathcal{O}_\eta) \to (\mathbf{1}, \mathcal{O})$ determined by an object $X \in \mathcal{T}$ and a morphism $\eta : U \to \mathcal{O}(X)$ in $\mathcal{X}$. Since $\mathcal{T}$ is essentially small and each mapping space $\mathrm{Map}_{\mathcal{X}}(U, \mathcal{O}(X))$ is essentially small, we conclude that $q^{-1}\{U\}$ is essentially small. □

Proof of Lemma 2.13. Using Lemma 2.12 and the description of p-colimit diagrams given by Proposition 2.2, we are reduced to proving the following assertion, for every $X \in \mathcal{T}$:

- $(*)$ Let $e_X : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$ be the map given by evaluation at $X \in \mathcal{T}$ and the vertex $\{1\} \subseteq \Delta^1$. Then the canonical map
$$\mathcal{E}^{\triangleright} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \stackrel{e_X}{\to} \mathcal{X}$$
is a colimit diagram.

To prove $(*)$, consider the canonical functor $\mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{X})$, which carries a morphism $(U, \mathcal{O}') \to (\mathbf{1}, \mathcal{O})$ to the map $\mathcal{O}'(X) \to \mathcal{O}(X)$ in $\mathcal{X}$. Let $\mathcal{E}' = \mathcal{E} \times_{\mathrm{Fun}(\Delta^{\{1,2\}}, \mathcal{X})} \mathrm{Fun}(\Delta^2, \mathcal{X})$. There is an evident projection map $q : \mathcal{E}' \to \mathcal{E}$. This projection map admits a section s, which carries an object $\alpha : (U, \mathcal{O}') \to (\mathbf{1}, \mathcal{O})$ of $\mathcal{E}$ to the pair (α, σ), where σ is the 2-simplex



in $\mathcal{X}$. Let e' denote the composite map

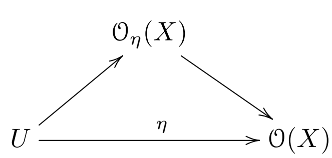
$$\mathcal{E}' \to \mathrm{Fun}(\Delta^2, \mathcal{X}) \to \mathrm{Fun}(\{0\}, \mathcal{X}) \simeq \mathcal{X},$$

so that $e_X | \mathcal{E} \simeq e' \circ s$. Consequently, to prove $(*)$, it will suffice to show that the canonical map $\varinjlim(e' \circ s) \to \mathcal{O}(X)$ is an equivalence in $\mathcal{X}$.

We note that the identity transformation $p \circ s \simeq \mathrm{id}_{\mathcal{E}}$ is the counit of an adjunction between p and s. In particular, s is left cofinal; it will therefore suffice to show that the canonical map $\varinjlim(e') \to \mathcal{O}(X)$ is an equivalence in $\mathcal{X}$. Note that the inclusion $\Delta^{\{0,2\}} \subseteq \Delta^2$ determines a factorization of e' as a composition

$$\mathcal{E}' \stackrel{e'_0}{\to} \mathcal{X}^{/\,\mathcal{O}(X)} \stackrel{e''_0}{\to} \mathcal{X}.$$

The functor e'_0 admits a left adjoint, which carries a morphism $\eta : U \to \mathcal{O}(X)$ to the pair $(\alpha, \sigma) \in \mathcal{E}'$, where $\alpha : (U, \mathcal{O}_\eta) \to (\mathbf{1}, \mathcal{O})$ is as in Remark 2.7 and σ is the diagram



It follows that e'_0 is left cofinal. We are therefore reduced to showing that the canonical map $\varinjlim(e''_0) \to \mathcal{O}(X)$ is an equivalence. This is clear, since $\mathcal{X}^{/\,\mathcal{O}(X)}$ contains $\mathcal{O}(X)$ as a final object. □

3 The Proof of Proposition 1.8

Our goal in this section is to give a proof of Proposition 1.8, which was stated in §1. The basic idea is to use the results of §2 to reduce the proof to an easy special case:

Lemma 3.1. *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{T}$ an unramified pregeometry. Suppose we are given a morphism $f : X \to Y$ in $\mathcal{T}$ and another object $Z \in \mathcal{T}$. Let $g^*_{X,Z} : \mathcal{X}_{X\times Z} \to \mathcal{X}$ be a geometric morphism inducing further geometric morphisms*

$$g^*_{X,Y,Z} : \mathcal{X}_{X\times Y\times Z} \to \mathcal{X} \qquad g^*_X : \mathcal{X}_X \to \mathcal{X} \qquad g^*_{X,Y} : \mathcal{X}_{X\times Y} \to \mathcal{X}.$$

Then:

(1) *The diagram σ :*

$$\begin{array}{ccc} g^*_{X,Y}\,\mathcal{O}_{X\times Y} & \longrightarrow & g^*_X\,\mathcal{O}_X \\ \downarrow & & \downarrow \\ g^*_{X,Y,Z}\,\mathcal{O}_{X\times Y\times Z} & \xrightarrow{\beta} & g^*_{X,Z}\,\mathcal{O}_{X\times Z} \end{array}$$

is a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$*.*

(2) *For every geometric morphism $g^* : \mathcal{X} \to \mathcal{Y}$, the image $g^*(\sigma)$ is a pushout square in* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$*.*

(3) *The map β is an effective epimorphism.*

Proof. Let p denote the forgetful functor $\mathcal{T}\mathrm{op}(\mathcal{T}) \to \mathcal{T}\mathrm{op}(\mathcal{T})$, and let τ denote the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}^{\mathcal{T}}(X\times Z) & \longrightarrow & \mathrm{Spec}^{\mathcal{T}}(X) \\ \downarrow & & \downarrow \\ \mathrm{Spec}^{\mathcal{T}}(X\times Y\times Z) & \longrightarrow & \mathrm{Spec}^{\mathcal{T}}(X\times Y) \end{array}$$

in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T})^{op}$. Then τ is a pullback square. The assumption that $\mathcal{T}$ is unramified guarantees that $p(\tau)$ is also a pullback square. It follows from Proposition T.4.3.1.5 that τ is a p-limit diagram. Assertions (1) and (2) now follow from Propositions T.4.3.1.9 and T.4.3.1.10. Assertion (3) is obvious, since β admits a section (induced by the projection map $X \times Y \times Z \to X \times Z$ in $\mathcal{T}$). □

Our strategy now is to reduce the general case of Proposition 1.8 to the special case treated in Lemma 3.1 by choosing a suitable resolution of the diagram

$$\overline{\mathcal{O}}_0 \xleftarrow{\beta} \overline{\mathcal{O}} \xrightarrow{\alpha} \mathcal{O}.$$

To accomplish this, we will prove a slightly more complicated version of Proposition 2.11. First, we need some terminology.

Definition 3.2. Let $\mathcal{T}$ be a pregeometry and $\mathcal{X}$ an ∞-topos. Suppose we are given a diagram τ :

$$\overline{\mathcal{O}}_0 \xleftarrow{\beta} \overline{\mathcal{O}} \xrightarrow{\alpha} \mathcal{O}$$

in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. We will say that τ is *elementary* if there exists a morphism $f : X \to Y$ in $\mathcal{T}$, an object $Z \in \mathcal{T}$, and a geometric morphism $g^*_{X,Z} : \mathcal{X}_{X\times Z} \to \mathcal{X}$ such that α can be identified with the induced map $g^*_{X,Y}\,\mathcal{O}_{X\times Y} \to g^*_X\,\mathcal{O}_X$ and β with the map $g^*_{X,Y}\,\mathcal{O}_{X\times Y} \to g^*_{X,Y,Z}\,\mathcal{O}_{X\times Y\times Z}$, where g_X and $g_{X,Y,Z}$ are defined as in the statement of Lemma 3.1.

Remark 3.3. If $\overline{\mathcal{O}}_0 \xleftarrow{\beta} \overline{\mathcal{O}} \xrightarrow{\alpha} \mathcal{O}$ is an elementary diagram in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, then $\overline{\mathcal{O}}_0$, $\overline{\mathcal{O}}$, and $\mathcal{O}$ are elementary objects of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Conversely, if $\mathcal{O}$ is an elementary object of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, then the constant diagram $\mathcal{O} \xleftarrow{\mathrm{id}} \mathcal{O} \xrightarrow{\mathrm{id}} \mathcal{O}$, is elementary.

Remark 3.4. Any morphism $\alpha : \mathcal{O} \to \mathcal{O}'$ in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ which admits a section is an effective epimorphism. In particular, if we are given an elementary diagram $\overline{\mathcal{O}}_0 \xleftarrow{\beta} \overline{\mathcal{O}} \xrightarrow{\alpha} \mathcal{O}$ in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, then α is an effective epimorphism.

We will need the following converse of Remark 3.4:

Proposition 3.5. *Let $\mathcal{X}$ be an ∞-topos, $\mathcal{T}$ a pregeometry, and $\tau : \Lambda^2_0 \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/\mathbf{1}}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ a diagram (here $\mathbf{1}$ denotes the final object of $\mathcal{X}$), which we depict as*

$$\overline{\mathcal{O}}_0 \xleftarrow{\beta} \overline{\mathcal{O}} \xrightarrow{\alpha} \mathcal{O}\,.$$

Assume that α is an effective epimorphism. Then there exists a Cartesian fibration $q : \mathcal{C} \to \mathcal{X}$ and a diagram $Q : \mathcal{C}^{\triangleright} \times \Lambda^2_0 \to \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ with the following properties:

(1) *Let $p : \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$ be the forgetful functor. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}^{\triangleright} \times \Lambda^2_0 & \xrightarrow{Q} & \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{C}^{\triangleright} & \xrightarrow{\overline{q}} & \mathcal{X}, \end{array}$$

where $\overline{q}|\,\mathcal{C} = q$ and $\overline{q}$ carries the cone point of $\mathcal{C}^{\triangleright}$ to $\mathbf{1} \in \mathcal{X}$.

(2) *For each object $C \in \mathcal{C}$, restriction of Q to $\{C\} \times \Lambda^2_0$ determines an elementary diagram in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/q(C)})$.*

(3) *The fibers of q are essentially small sifted ∞-categories (in fact, we can assume that the fibers of q are nonempty and admit coproducts for pairs of objects; see Remark 2.4).*

(4) *For each vertex $i \in \Lambda^2_0$, the restriction $Q|\,\mathcal{C} \times \{i\}$ carries q-Cartesian morphisms in $\mathcal{C}$ to p-Cartesian morphisms in $\underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$.*

(5) *For each vertex $i \in \Lambda^2_0$, the restriction $Q|\,\mathcal{C}^{\triangleright} \times \{i\}$ is a p-colimit diagram in $\underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$.*

(6) *The map Q carries the cone point of $\mathcal{C}^{\triangleright}$ to the diagram τ.*

Remark 3.6. Proposition 2.11 is an immediate consequence of Proposition 3.5, applied to the case where τ is a constant diagram; see Remark 3.3.

Assuming Proposition 3.5 for the moment, we can complete the proof of our main result.

Proof of Proposition 1.8. Assume that $\mathcal{T}$ is unramified and that we are given a map $\tau : \Lambda^2_0 \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/\mathbf{1}})$, corresponding to a diagram

$$\overline{\mathcal{O}}_0 \leftarrow \overline{\mathcal{O}} \xrightarrow{\alpha} \mathcal{O}$$

where α is an effective epimorphism. Choose a diagram

$$\begin{array}{ccc} \mathcal{C}^{\triangleright} \times \Lambda^2_0 & \xrightarrow{Q} & \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{C}^{\triangleright} & \xrightarrow{\overline{q}} & \mathcal{X} \end{array}$$

satisfying the conclusions of Proposition 3.5. Let v denote the cone point of $\mathcal{C}^{\triangleright}$, $q = \overline{q}|\,\mathcal{C}$, and let $\mathcal{D}$ denote the full subcategory of $\mathcal{C}^{\triangleright} \times \Delta^1 \times \Delta^1$ obtained by omitting the final object. We first claim that Q admits a p-left Kan extension $Q' : \mathcal{D} \to \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ (compatible with $\overline{q}$). In view of Lemma T.4.3.2.13, it will suffice to show that for every object $C \in \mathcal{C}$, the restriction $Q|(\mathcal{C}_{/C} \times \Lambda^2_0)$ admits a p-colimit compatible with $\overline{q}$. Since the inclusion $\{\mathrm{id}_C\} \hookrightarrow \mathcal{C}_{/C}$ is left cofinal and p is a Cartesian fibration, it suffices to show that $Q|\{C\} \times \Lambda^2_0$ admits a colimit in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/\overline{q}(C)})$. Since $Q|\{C\} \times \Lambda^2_0$ is an elementary diagram and $\mathcal{T}$ is unramified, this follows from Lemma 3.1. Moreover, we learn that $Q'_0 = Q'|\,\mathcal{C} \times \{(1,1)\}$ carries q-Cartesian morphisms of $\mathcal{C}$ to p-Cartesian morphisms of $\underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Furthermore, for every geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$, the composition $f^* \circ Q'$ is a p'-left Kan extension of $f^* \circ Q$, where p' denotes the projection $\underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y}) \to \mathcal{Y}$.

We next claim that Q' can be extended to a p-colimit diagram $Q'' : \mathcal{C}^{\triangleright} \times \Delta^1 \times \Delta^1 \to \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ (compatible with $\overline{q}$). Let $Q'_1 = Q'|(\mathcal{C} \times \Delta^1 \times \Delta^1)$. Our assumptions guarantee that Q' is a p-left Kan extension of Q'_1. Using Lemma T.4.3.2.7, it suffices to show that Q'_1 can be extended to a p-colimit diagram (compatible with $\overline{q}$). Since the inclusion $\mathcal{C} \times \{(1,1)\} \hookrightarrow \mathcal{D}$ is left cofinal, it suffices to show that Q'_0 can be extended to a p-colimit diagram compatible with $\overline{q}$, which follows immediately from Proposition 2.2. Using Proposition T.4.3.2.8, we conclude that Q'' is a p-left Kan extension of Q. Moreover, the same arguments show that $f^* \circ Q''$ is a p'-left Kan extension of $f^* \circ Q$, for every geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$.

Let $Q''' = Q''|(\mathcal{C}^{\triangleright} \times \Lambda^2_0)^{\triangleright}$. For every geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$, $f^* \circ Q''$ is a p'-colimit diagram. Since the inclusion $\{v\} \times \Lambda^2_0 \subseteq \mathcal{C}^{\triangleright} \times \Lambda^2_0$, it follows that $f^* \circ (Q''|\{v\} \times \Delta^1 \times \Delta^1)$ is p'-colimit diagram in $\underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$. In particular, setting $\mathcal{O}_0 = Q''(v,1,1)$, we deduce that the diagram

$$\begin{array}{ccc} f^*\overline{\mathcal{O}} & \longrightarrow & f^*\,\mathcal{O} \\ \downarrow & & \downarrow \\ f^*\overline{\mathcal{O}}_0 & \longrightarrow & f^*\,\mathcal{O}_0 \end{array}$$

is a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y})$. This completes the proofs of (1) and (3).

To complete the proof, we must show that the map $\mathcal{O} \to \mathcal{O}_0$ is an effective epimorphism of $\mathcal{T}$-structures on $\mathcal{X}$. Fix $X \in \mathcal{T}$; we wish to show that the morphism $u : Q''(v,0,1)(X) \to Q''(v,1,1)(X)$ is an effective epimorphism in $\mathcal{X}$. We note that u is the colimit of a diagram of morphisms $u_C : Q''(C,0,1)(X) \to Q''(C,1,1)(X)$ indexed by $C \in \mathcal{C}$. It therefore suffices to show that each u_C is an effective epimorphism. This is clear, since u_C admits a section (by virtue of our assumption that $Q|(\{C\} \times \Lambda^2_0)$ is elementary) □

The remainder of this section is devoted to the proof of Proposition 3.5.

Construction 3.7. Let NFin denote the category of nonempty finite sets, and let $* \in \mathrm{NFin}$ correspond to a set with one element. Fix a pregeometry $\mathcal{T}$, an ∞-topos $\mathcal{X}$ containing a final object $\mathbf{1}$, and an object $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Let $p : \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1},\mathcal{O})} \to \mathcal{X}$ be the projection map and let $\mathcal{E} \subseteq \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1},\mathcal{O})}$ be as in Lemma 2.12. Note that p is a Cartesian fibration and restricts to a Cartesian fibration $p_0 : \mathcal{E} \to \mathcal{X}$. It follows from Remark 2.9 that for every object $U \in \mathcal{X}$, the fiber $\mathcal{E}_U = p_0^{-1}\{U\}$ admits finite coproducts (which are also coproducts in the larger ∞-category $p^{-1}\{U\}$). Since p_0 is a Cartesian fibration, Corollary T.4.3.1.15 implies that finite coproducts in $\mathcal{E}_U$ are also p_0-coproduct diagrams (and even p-coproduct diagrams). Let $v : \mathcal{E} \times \mathrm{N}(\mathrm{NFin}) \to \mathcal{E}$ be the projection onto the first factor. Using Lemma T.4.3.2.13, we deduce that there exists another functor $u : \mathcal{E} \times \mathrm{N}(\mathrm{NFin}) \to \mathcal{E}$ such that $p_0 \circ u = p_0 \circ v$, such that $u|(\mathcal{E} \times \{*\})$ is the identity map and u is a p-left Kan extension of $u|(\mathcal{E} \times \{*\})$. Since $u|(\mathcal{E} \times \{*\}) = v|(\mathcal{E} \times \{*\})$, there is an essentially unique natural transformation $\gamma : u \to v$ such that $p(\alpha) = \mathrm{id}$ and γ restricts to the identity on $\mathcal{E} \times \{[0]\}$.

Let $\alpha : (\mathbf{1}, \overline{\mathcal{O}}) \to (\overline{\mathbf{1}}, \mathcal{O})$ be a morphism in $\underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. We let $\overline{\mathcal{E}}$ denote the fiber product

$$(\mathcal{E} \times \mathrm{N}(\mathrm{NFin})) \times_{\underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1},\mathcal{O})}} \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\alpha},$$

where $\mathcal{E} \times \mathrm{N}(\boldsymbol{\Delta})$ maps to $\underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1},\mathcal{O})}$ via u. Amalgamating the projection map $\overline{\mathcal{E}} \to \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\alpha}$ with γ, we obtain a functor $\overline{\gamma} : \overline{\mathcal{E}} \to \mathrm{Fun}(\Delta^1, \underline{\mathit{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}))_{/\alpha}$.

Remark 3.8. In the situation of Construction 3.7, the forgetful functor $\overline{p} : \overline{\mathcal{E}} \to \mathcal{X}$ factors as a composition

$$\overline{\mathcal{E}} \stackrel{p''}{\to} \mathcal{E} \times \mathrm{N}(\mathrm{NFin}) \stackrel{p'}{\to} \mathcal{E} \stackrel{p}{\to} \mathcal{X}\,.$$

The map p'' is a right fibration, the map p' obviously a Cartesian fibration, and the map p is a Cartesian fibration by Lemma 2.12; it follows that $\overline{p}$ is a Cartesian fibration. It is easy to see that the fibers of $\overline{p}$ are nonempty (for each $U \in \mathcal{X}$, we can choose an object of $\mathcal{E}$ corresponding to a map $\eta : U \to \mathcal{O}(X)$, where X is a final object of $\mathcal{T}$; for every finite set S, the object $(U, \mathcal{O}_\eta, S)$ lifts to $\overline{\mathcal{E}}$ in an essentially unique way). Since NFin admits finite coproducts and p'' is a right fibration, Proposition T.1.2.13.8 and Lemma 2.12 guarantee that the fibers of $\overline{p}$ admit finite coproducts. It follows from Remark 2.4 that the fibers of p are sifted.

Lemma 3.9. *Let $\mathcal{X}$ be an ∞-topos, $\mathcal{T}$ a pregeometry, and $p : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$ the forgetful functor. Let $\overline{q} : K^{\triangleright} \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ be a diagram such that $\overline{q}$ carries every morphism in $K^{\triangleright}$ to a p-Cartesian morphism in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, and $p \circ \overline{q}$ is a colimit diagram in $\mathcal{X}$. Then $\overline{q}$ is a p-colimit diagram.*

Proof. In view of Lemma VII.5.17, it will suffice to show that the Cartesian fibration p is classified by a functor $\chi : \mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which preserves small limits. This is an easy consequence of Theorem T.6.1.3.9. □

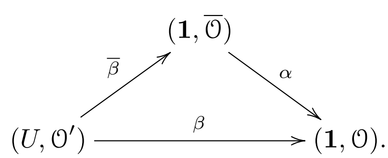
Lemma 3.10. *Let $\mathcal{T}$ be a pregeometry and $\mathcal{X}$ an ∞-topos, let $\alpha : \overline{\mathcal{O}} \to \mathcal{O}$ be a morphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}^{/\mathbf{1}}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, and let $\overline{\mathcal{E}}$ and $\overline{\gamma} : \overline{\mathcal{E}} \to \mathrm{Fun}(\Delta^1, \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}))_{/\alpha}$ be as in Construction 3.7. Let $p : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathcal{X}$ be the projection map.*

We can identify $\overline{\gamma}$ with a map $\overline{\gamma}' : \overline{\mathcal{E}}^{\triangleright} \times \Delta^1 \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Assume that α is an effective epimorphism. Then:

- $(*)$ *For $i \in \{0, 1\}$, the restriction $\overline{\gamma}' | (\overline{\mathcal{E}}^{\triangleright} \times \{i\})$ determines a p-colimit diagram in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$.*

Proof. We first prove that $\overline{\gamma}'_0$ is a p-colimit diagram. Let $\overline{\mathcal{E}}' \subseteq \overline{\mathcal{E}}$ denote the inverse image of $\mathcal{E} \times \{*\}$ in $\overline{\mathcal{E}}$. Note that $\overline{\gamma}'_0 | \overline{\mathcal{E}}$ is a p-left Kan extension of $\overline{\gamma}'_0 | \overline{\mathcal{E}}'$. Using Lemma T.4.3.2.7, we are reduced to proving that $\overline{\gamma}'_0 | \overline{\mathcal{E}}'^{\triangleright}$ is a p-colimit diagram. Note $\overline{\gamma}'_0$ induces a fully faithful embedding $\overline{\mathcal{E}}' \to \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1}, \overline{\mathcal{O}})}$, whose essential image is the collection of maps $(U, \mathcal{O}') \to (\mathbf{1}, \overline{\mathcal{O}})$ where $\mathcal{O}'$ is elementary. The desired result now follows from Lemma 2.13.

We now prove that $\overline{\gamma}'_1$ is a p-colimit diagram. Let $\mathcal{E}_0$ denote the essential image of the forgetful functor $\overline{\mathcal{E}} \to \mathcal{E}$. More concretely, an object $\beta : (U, \mathcal{O}') \to (\mathbf{1}, \mathcal{O})$ of $\mathcal{E}$ belongs to $\mathcal{E}_0$ if β can be lifted to a commutative diagram



We observe that $\overline{\gamma}'_1$ factors as a composition

$$\overline{\mathcal{E}}^{\triangleright} \stackrel{\delta'}{\to} \mathcal{E}_0^{\triangleright} \stackrel{\delta}{\to} \underline{Str}_{\mathcal{T}}(\mathcal{X}).$$

We will prove:

- (a) The forgetful functor $\overline{\mathcal{E}} \to \mathcal{E}_0$ is left cofinal.
- (b) The diagram $\delta | \mathcal{E}$ is a p-left Kan extension of $\delta | \mathcal{E}_0$.

Note that δ is a p-colimit diagram (Lemma 2.13); combining this with (b) and Lemma T.4.3.2.7, we conclude that $\delta | \mathcal{E}_0^{\triangleright}$ is a p-colimit diagram. Using (a) we conclude that $\delta \circ \delta' = \overline{\gamma}'_1$ is a p-colimit diagram as desired.

It remains to prove (a) and (b). We first prove (a). We have seen above that the forgetful functor $\overline{\mathcal{E}} \to \mathcal{E}$ is a Cartesian fibration (so that $\mathcal{E}_0 \subseteq \mathcal{E}$ is a sieve). According to Lemma T.4.1.3.2, it will suffice to show that for each object $E \in \mathcal{E}_0$, the fiber $\overline{\mathcal{E}} \times_{\mathcal{E}} \{E\}$ is weakly contractible. This follows from Remark 2.4 and Proposition T.5.5.8.7, since $\overline{\mathcal{E}} \times_{\mathcal{E}} \{E\}$ is a nonempty ∞-category which admits pairwise coproducts.

We now prove (b). Fix an object $E \in \mathcal{E}$, corresponding to a map $(U, \mathcal{O}') \to (\mathbf{1}, \mathcal{O})$ in $\underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})$, and let $(\mathcal{E}_0)_{/E}$ denote the fiber product $\mathcal{E}_0 \times_{\mathcal{E}} \mathcal{E}_{/E}$. We wish to prove that δ exhibits $\delta(E) = (U, \mathcal{O}')$ as a p-colimit of the diagram $\delta|(\mathcal{E}_0)_{/E}$. Since α is an effective epimorphism, there exists an effective epimorphism $f : V \to U$ such that the composite map $(V, \mathcal{O}'|V) \to (U, \mathcal{O}') \to (\mathbf{1}, \mathcal{O})$ factors through $(\mathbf{1}, \overline{\mathcal{O}})$. Let $\mathcal{D} \subseteq \mathcal{E}_{/E}$ denote the full subcategory of $\mathcal{E}_{/E}$ spanned by those objects which determine maps $(\widetilde{U}, \widetilde{\mathcal{O}}') \to (U, \mathcal{O}')$ such that $\widetilde{U} \to U$ factors through f. Note that $\mathcal{D} \subseteq (\mathcal{E}_0)_{/E}$. We will prove:

(c) The forgetful functor $\theta : \mathcal{E}_{/E} \to \mathcal{E} \subseteq \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})$ is a p-left Kan extension of its restriction to $\mathcal{D}$.

Assertion (c) implies that E is a p-colimit of the diagram $\theta|\mathcal{D}$, so that E is a p-colimit of $\theta|(\mathcal{E}_0)_{/E}$ by Lemma T.4.3.2.7 and the proof is complete. To prove (c), choose an object $\widetilde{E} \in \mathcal{E}_{/E}$, corresponding to a map $(\widetilde{U}, \widetilde{\mathcal{O}}) \to (U, \mathcal{O})$ in $\underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})$. We wish to prove that θ exhibits $\widetilde{E}$ as a p-colimit of $\theta|(\mathcal{D} \times_{\mathcal{E}_{/E}} \mathcal{E}_{/\widetilde{E}})$. Replacing E by $\widetilde{E}$ (and V by $V \times_U \widetilde{U}$), we are reduced to proving that θ exhibits E as a colimit of $\theta|\mathcal{D}$.

Let $V_\bullet$ be the Čech nerve of the map $f : V \to U$, and let $\mathcal{O}'_\bullet$ be the pullback of $\mathcal{O}'$ to $V_\bullet$. The construction $[n] \mapsto (V_n, \mathcal{O}'_n)$ determines a left cofinal map $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{D}$; it will therefore suffice to show that θ exhibits $E = (U, \mathcal{O}')$ as a p-colimit of the simplicial object $(V_\bullet, \mathcal{O}'_\bullet)$ of $\underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})$. Since f is an effective epimorphism, we have $U \simeq |V_\bullet|$ and the desired result follows from Lemma 3.9. □

Proof of Proposition 3.5. Fix an ∞-topos $\mathcal{X}$ containing a final object $\mathbf{1}$, a pregeometry $\mathcal{T}$, and a diagram

$$\overline{\mathcal{O}}_0 \stackrel{\beta}{\leftarrow} \overline{\mathcal{O}} \stackrel{\alpha}{\to} \mathcal{O}$$

in $\mathrm{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X}^{/\mathbf{1}}) \simeq \mathrm{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})$. Let $\overline{\mathcal{E}}$ be defined as in Construction 3.7. Note that evaluation at $\{0\} \subseteq \Delta^1$ induces a functor $\overline{\mathcal{E}} \to \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1}, \overline{\mathcal{O}})}$. We let $\mathcal{C}$ denote the full subcategory of the fiber product

$$\overline{\mathcal{E}} \times_{\mathrm{Fun}_{\mathcal{X}}(\mathcal{X} \times \{0\}, \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1}, \overline{\mathcal{O}})})} \mathrm{Fun}_{\mathcal{X}}(\mathcal{X} \times \Delta^1, \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})_{/\beta})$$

spanned by those objects whose image in $\mathrm{Fun}_{\mathcal{X}}(\mathcal{X} \times \Delta^1, \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})_{/\beta})$ corresponds to a commutative diagram

$$\begin{array}{ccc} (U, \overline{\mathcal{O}}') & \longrightarrow & (U, \overline{\mathcal{O}}'_0) \\ \downarrow & & \downarrow \\ (\mathbf{1}, \overline{\mathcal{O}}) & \stackrel{\beta}{\longrightarrow} & (\mathbf{1}, \overline{\mathcal{O}}_0) \end{array}$$

with the following property: there exists an object $X \in \mathcal{T}$ and a morphism $\eta : U \to \overline{\mathcal{O}}_0(X)$ in $\mathcal{X}$ such that $\overline{\mathcal{O}}'_0$ is a coproduct of $\overline{\mathcal{O}}'$ with $\mathcal{O}_\eta$ in $\mathrm{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X}^{/U})_{/\overline{\mathcal{O}}_0|U}$. By construction, the maps $\mathcal{C} \to \overline{\mathcal{E}} \to \mathrm{Fun}(\Delta^1, \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X}))_{/\alpha}$ and $\mathcal{C} \to \mathrm{Fun}(\Delta^1, \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X}))_{/\beta}$ amalgamate to determine a map $\mathcal{C}^{\triangleright} \times \Lambda^2_0 \to \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})$ satisfying conditions (1), (2), (3), (4), and (6).

It remains to verify (5). Let $\mathcal{D}$ be the full subcategory of $\underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1}, \overline{\mathcal{O}}_0)}$ spanned by those morphisms $(U, \overline{\mathcal{O}}'_0) \to (\mathbf{1}, \overline{\mathcal{O}}_0)$ such that $(U, \overline{\mathcal{O}}'_0)$ is elementary. We have an evident pair of forgetful functors

$$\mathcal{D} \stackrel{u}{\leftarrow} \mathcal{C} \stackrel{v}{\to} \overline{\mathcal{E}}.$$

In view of Lemmas 3.10 and 2.13, it will suffice to show that the maps u and v are left cofinal.

The map v is left cofinal because it has a right adjoint: in fact, v has a fully faithful right adjoint, whose essential image is the full subcategory of $\mathcal{C}$ spanned by those objects whose image in $\mathrm{Fun}_{\mathcal{X}}(\mathcal{X} \times \Delta^1, \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})_{/\beta})$

corresponds to a commutative diagram

$$\begin{array}{ccc} (U, \overline{\mathcal{O}}') & \xrightarrow{\beta'} & (U, \overline{\mathcal{O}}'_0) \\ \downarrow & & \downarrow \\ (\mathbf{1}, \overline{\mathcal{O}}) & \xrightarrow{\beta} & (\mathbf{1}, \overline{\mathcal{O}}_0) \end{array}$$

such that β' is an equivalence.

We will prove that u is left cofinal using Theorem T.4.1.3.1. Fix an object of $D \in \mathcal{D}$, corresponding to a map $(U, \overline{\mathcal{O}}'_0) \to (\mathbf{1}, \overline{\mathcal{O}}_0)$. We wish to prove that the ∞-category $\mathcal{C}_{D/} = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}_{D/}$ spanned by those objects C whose images in $\mathcal{D}_{D/}$ and $\mathcal{E} \subseteq \underline{Str}^{\text{loc}}_{\mathcal{T}}(\mathcal{X})_{/(\mathbf{1}, \mathcal{O})}$. It is not difficult to show that the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}_{D/}$ admits a right adjoint, and is therefore a weak homotopy equivalence. We now observe that the forgetful functor $\mathcal{C}' \to \mathrm{N}(\mathrm{NFin})$ is a trivial Kan fibration, so that $\mathcal{C}'$ is sifted (Remark 2.4) and therefore weakly contractible (Proposition T.5.5.8.7). □

4 Closed Immersions in Spectral Algebraic Geometry

In §1, we introduced the notion of an *unramified* pregeometry. Roughly speaking, a pregeometry $\mathcal{T}$ is unramified if the there is a good theory of closed immersions between $\mathcal{T}$-structured ∞-topoi. Our goal in this section is to give some examples of unramified pregeometries which arise in the study of spectral algebraic geometry.

Proposition 4.1. *Let k be a connective $\mathbb{E}_\infty$-ring. Then the pregeometry $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ of Definition VII.2.18 and the pregeometry and $\mathcal{T}^{\mathrm{Sp}}_{\acute{e}t}(k)$ of Definition VII.8.26 are unramified.*

Proof. We will give the proof for $\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k)$; the proof for the pregeometry $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ is similar (and easier). Unwinding the definitions, we are reduced to proving the following assertion:

(∗) Let A, B, and C be $\mathbb{E}_\infty$-algebras over k which belong to $\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k)$, and let $\phi : B \to A$ be a morphism in CAlg_k. In the diagram

$$\begin{array}{ccc} A \otimes_k B & \longrightarrow & A \\ \downarrow & & \downarrow \\ A \otimes_k B \otimes_k C & & A \otimes_k C \end{array}$$

in CAlg_k determines a pullback square of ∞-topoi

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}((\mathrm{CAlg}^{\text{ét}}_{A \otimes_k B})^{op}) & \xleftarrow{\psi_*} & \mathcal{S}\mathrm{hv}((\mathrm{CAlg}^{\text{ét}}_{A})^{op}) \\ \uparrow & & \uparrow \\ \mathcal{S}\mathrm{hv}((\mathrm{CAlg}^{\text{ét}}_{A \otimes_k B \otimes_k C})^{op}) & \longleftarrow & \mathcal{S}\mathrm{hv}((\mathrm{CAlg}^{\text{ét}}_{A \otimes_k C})^{op}) \end{array}$$

To prove (∗), let $U \in \mathcal{X} = \mathcal{S}\mathrm{hv}((\mathrm{CAlg}^{\text{ét}}_{A \otimes_k B})^{op})$ be defined by the formula

$$U(R) = \begin{cases} \Delta^0 & \text{if } R \otimes_{A \otimes_k B} A \simeq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

and let U' denote the inverse image of U in $\mathcal{X}' = \mathcal{S}\mathrm{hv}((\mathrm{CAlg}^{\text{ét}}_{A \otimes_k B \otimes_k C})^{op})$. The proof of Proposition VIII.1.2.7 allows us to identify σ with the diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{X} & \longleftarrow & \mathcal{X}_{/U} \\ \uparrow & & \uparrow \\ \mathcal{X}' & \longleftarrow & \mathcal{X}'_{/U'}, \end{array}$$

which is pullback square by Proposition T.7.3.2.13. □

Remark 4.2. Let k be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{T}$ be one of pregeometries $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$, $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$, or $\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k)$. Let $\mathcal{X}$ be an ∞-topos, and let $\alpha : \mathcal{O} \to \mathcal{O}'$ be a local morphism of sheaves of $\mathcal{T}$-structures on $\mathcal{X}$. The following conditions are equivalent:

(1) The map α is an effective epimorphism.

(2) The map α induces an effective epimorphism $\mathcal{O}(k\{x\}) \to \mathcal{O}'(k\{x\})$, where $k\{x\}$ denotes the free $\mathbb{E}_\infty$-algebra over k on one generator.

The implication (1) $\Rightarrow$ (2) is obvious, and the implication (2) $\Rightarrow$ (1) follows from the observation that every object $U \in \mathcal{T}$ admits an admissible morphism $U \to X$, where X is a product of finitely many copies of $k\{x\}$.

Remark 4.3. Let $f : (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ be a morphism of spectrally ringed ∞-topoi. Assume that $\mathcal{O}_\mathcal{X}$ and $\mathcal{O}_\mathcal{Y}$ are connective, so that we can identify $\mathcal{O}_\mathcal{X}$ and $\mathcal{O}_\mathcal{Y}$ with $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}$-structures on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then f is a closed immersion in $\mathcal{T}\mathrm{op}(\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}})$ if and only if the following conditions are satisfied:

(a) The underlying geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$ is a closed immersion.

(b) The map of structure sheaves $f^* \mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$ is is connective: that is, $\mathrm{fib}(f^* \mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X})$ is a connective sheaf of spectra on $\mathcal{X}$.

If the morphism f belongs to the subcategory $\mathrm{RingTop}_{\mathrm{Zar}} \subseteq \mathrm{RingTop}$, then f is a closed immersion in $\mathcal{T}\mathrm{op}(\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}})$ if and only if conditions (a) and (b) are satisfied. If f belongs to the subcategory $\mathrm{RingTop}_{\text{ét}} \subseteq \mathrm{RingTop}_{\mathrm{Zar}}$, then f is closed immersion in $\mathcal{T}\mathrm{op}(\mathcal{T}^{\mathrm{Sp}}_{\text{ét}})$ if and only (a) and (b) are satisfied.

It follows from Propositions 4.1 that there is a good theory of closed immersions in the setting of spectral algebraic geometry. Our other goal in this section is to give an explicit description of the local structure of a closed immersion. To simplify the exposition, we will restrict our attention to the case of spectral Deligne-Mumford stacks.

Theorem 4.4. *Let A be a connective $\mathbb{E}_\infty$-ring and let $f : (\mathcal{X}, \mathcal{O}) \to \operatorname{Spec} A$ be a map of spectral Deligne-Mumford stacks. Then f is a closed immersion if and only if there exists an equivalence $(\mathcal{X}, \mathcal{O}) \simeq \operatorname{Spec} B$ for which the induced map $A \to B$ induces a surjection $\pi_0 A \to \pi_0 B$.*

Remark 4.5. Theorem 4.4 has an evident analogue in the setting of spectral schemes, which can be proven by the same methods.

Remark 4.6. The condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be a closed immersion is local on the target with respect to the étale topology (see Definition VIII.3.1.1). Together with Theorem 4.4, this observation completely determines the class of closed immersions between spectral Deligne-Mumford stacks.

The proof of Theorem 4.4 will require a number of preliminaries.

Lemma 4.7. *Let $\mathcal{X}$ be an ∞-topos and let $f_* : \mathcal{Y} \to \mathcal{X}$ be a closed immersion of ∞-topoi. If $\mathcal{X}$ is n-coherent (for some $n \geq 0$), then $\mathcal{Y}$ is n-coherent.*

Proof. The proof proceeds by induction on n. In the case $n = 0$, we must show that if $\mathcal{X}$ is quasi-compact then $\mathcal{Y}$ is also quasi-compact. Assume that $\mathcal{X}$ is quasi-compact and choose an effective epimorphism $\theta : \coprod_{i \in I} Y_i \to \mathbf{1}$ in $\mathcal{Y}$, where $\mathbf{1}$ denotes the final object of $\mathcal{Y}$. We wish to show that there exists a finite subset $I_0 \subseteq I$ such that the induced map $\coprod_{i \in I_0} Y_i \to \mathbf{1}$ is an effective epimorphism. If I is empty, we can take $I_0 = I$. Otherwise, we can use the fact that f_* is a closed immersion and Lemma 5.4 to see that the induced map $\coprod_{i \in I} f_*(Y_i) \to f_* \mathbf{1}$ is effective epimorphism in $\mathcal{X}$. Since $\mathcal{X}$ is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the map $\coprod_{i \in I_0} f_*(Y_i) \to f_* \mathbf{1}$ is an effective epimorphism, so that $\coprod_{i \in I_0} Y_i \to \mathbf{1}$ is an effective epimorphism in $\mathcal{Y}$.

Now suppose that $n > 0$. The inductive hypothesis guarantees that the pullback functor f^* carries $(n-1)$-coherent objects of $\mathcal{X}$ to $(n-1)$-coherent objects of $\mathcal{Y}$. Assume that $\mathcal{X}$ is $(n-1)$-coherent. Let $\mathcal{Y}_0 \subseteq \mathcal{Y}$ be the collection of all objects of the form $f^* X$, where $X \in \mathcal{X}$ is $(n-1)$-coherent. Then $\mathcal{Y}_0$ is stable under products in $\mathcal{Y}$ and every object $Y \in \mathcal{Y}$ admits an effective epimorphism $\coprod Y_i \to Y$, where each $Y_i \in \mathcal{Y}_0$. Applying Corollary VII.3.10, we deduce that $\mathcal{Y}$ is n-coherent. □

Lemma 4.8. *Let $f_* : \mathcal{Y} \to \mathcal{X}$ be a closed immersion of ∞-topoi. Then:*

(1) *The functor f_* carries n-connective objects of $\mathcal{Y}$ to n-connective objects of $\mathcal{X}$.*

(2) *The functor f_* carries n-connective morphisms of $\mathcal{Y}$ to n-connective morphisms of $\mathcal{X}$.*

(3) *The functor f_* induces a left t-exact functor $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$.*

Proof. Since f_* carries final objects of $\mathcal{Y}$ to final objects of $\mathcal{X}$, assertion (1) follows from (2). Assertion (3) is an immediate consequence of (1). It will therefore suffice to prove (2). We proceed by induction on n: when $n = 0$, the desired result follows from Lemma 5.4. Assume that $n > 0$, and let $u : X \to Y$ be an n-connective morphism in $\mathcal{Y}$. We wish to show that $f_*(u)$ is n-connective. We have already seen that $f_*(u)$ is an effective epimorphism; it will therefore suffice to show that the diagonal map $f_*(X) \to f_*(X) \times_{f_*(Y)} f_*(X)$ is $(n-1)$-connective. This follows from the inductive hypothesis, since the diagonal map $X \to X \times_Y X$ is $(n-1)$-connective. □

Lemma 4.9. *Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of spectral Deligne-Mumford stacks, and assume that the underlying geometric morphism $\mathcal{X} \to \mathcal{Y}$ is a closed immersion in $\mathcal{T}\mathrm{op}$. Then the pushforward functor $f_* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ carries quasi-coherent sheaves to quasi-coherent sheaves.*

Proof. Note that f_* is left t-exact and commutes with small limits. It follows that if $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$ is quasi-coherent, then

$$f_* \mathcal{F} \simeq f_* \varprojlim (\tau_{\leq n} \mathcal{F}) \simeq \varprojlim f_* \tau_{\leq n} \mathcal{F}$$

is the limit of a tower of truncated objects of $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$, and therefore hypercomplete. According to Proposition VIII.2.3.21, it will suffice to show that each homotopy group $\pi_n f_* \mathcal{F}$ is quasi-coherent. Since f_* is right t-exact (Lemma 4.8), we can replace $\mathcal{F}$ by $\tau_{\leq n} \mathcal{F}$. The desired result now follows from Corollary VIII.2.5.22, since Lemma 4.7 implies that f is k-quasi-separated for every integer $k \geq 0$. □

Lemma 4.10. *Suppose we are given a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f_*} & \mathcal{Y} \\ & {\scriptstyle h_*} \searrow \quad \swarrow {\scriptstyle g_*} & \\ & \mathcal{Z} & \end{array}$$

in $\mathcal{T}\mathrm{op}$, where g_* and h_* are closed immersions of ∞-topoi. Then f_* is a closed immersion of ∞-topoi.

Proof. Without loss of generality, we may assume that $\mathcal{X} = \mathcal{Z}_{/U}$ and $\mathcal{Y} = \mathcal{Z}_{/V}$ for some (-1)-truncated objects $U, V \in \mathcal{Z}$. The commutativity of the diagram implies $\mathcal{Z}_{/U} \subseteq \mathcal{Z}_{/V}$. An object $Z \in \mathcal{Z}$ belongs to $\mathcal{Z}_{/U}$ if and only if the projection map $p : Z \times U \to U$ is an equivalence. Using Lemma A.A.5.11, we see that this is equivalent to the assertion that $Z \in \mathcal{Y}_{/V}$ and that $g^*(p)$ is an equivalence. Thus f_* induces an equivalence from $\mathcal{X}$ to $\mathcal{Y}_{/g^*U}$ and is therefore a closed immersion. □

Proof of Theorem 4.4. We must prove two things:

(1) If $\phi : A \to B$ is a morphism of connective $\mathbb{E}_\infty$-rings which induces a surjective ring homomorphism $\pi_0 A \to \pi_0 B$, then the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is a closed immersion of spectral Deligne-Mumford stacks.

(2) Every closed immersion $(\mathcal{X}, \mathcal{O}) \to \operatorname{Spec} A$ arises in this way.

We first prove (1). For every connective $\mathbb{E}_\infty$-ring R, let $\operatorname{Spec} R = (\mathcal{X}_R, \mathcal{O}_R)$. It follows from Proposition VIII.1.2.7 that if $\phi : A \to B$ induces a surjection $\pi_0 A \to \pi_0 B$, then the corresponding map $f : (\mathcal{X}_B, \mathcal{O}_B) \to (\mathcal{X}_A, \mathcal{O}_A)$ induces a closed immersion of ∞-topoi $\mathcal{X}_B \to \mathcal{X}_A$. It remains to show that the map $f^* \mathcal{O}_A \to \mathcal{O}_B$ has a connective fiber. Since f_* is a closed immersion, it suffices to show that the adjoint map $\mathcal{O}_A \simeq f_* f^* \mathcal{O}_A \to f_* \mathcal{O}_B$ has a connective fiber. It follows from Lemma 4.9 that $f_* \mathcal{O}_B$ is a quasi-coherent sheaf on $(\mathcal{X}_A, \mathcal{O}_A)$. Since the equivalence $\operatorname{QCoh}(\mathcal{X}_A) \simeq \operatorname{Mod}_A$ is t-exact, it suffices to show that the map

$$A \simeq \Gamma(\mathcal{X}_A; \mathcal{O}_A) \to \Gamma(\mathcal{X}_A; f_* \mathcal{O}_B) \simeq \Gamma(\mathcal{X}_B; \mathcal{O}_B) \simeq B$$

has a connective fiber, which follows from our assumption on ϕ.

We now prove (2). Suppose we are given a closed immersion of spectral Deligne-Mumford stacks $f : (\mathcal{X}, \mathcal{O}) \to (\mathcal{X}_A, \mathcal{O}_A)$. Let $B = \Gamma(\mathcal{X}; \mathcal{O}) \simeq \Gamma(\mathcal{X}_A; f_* \mathcal{O})$. Using Lemmas 4.8 and 4.9, we deduce that $f_* \mathcal{O}$ is a connective quasi-coherent sheaf on $(\mathcal{X}_A, \mathcal{O}_A)$, so that B is a connective A-algebra. Moreover, since f is a closed immersion, the map $f^* \mathcal{O}_A \to \mathcal{O}$ is an epimorphism on π_0. Since f is a closed immersion, the adjoint map $\mathcal{O}_A \simeq f_* f^* \mathcal{O}_A \to f_* \mathcal{O}$ is a map of quasi-coherent sheaves with connective fiber. Since the equivalence $\operatorname{Mod}_A \simeq \operatorname{QCoh}(\mathcal{X}_A)$ is t-exact, we conclude that the map $A \to B$ is surjective on π_0. The universal property of $\operatorname{Spec}^{\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}}(B)$ gives a commutative diagram

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{O}) & \xrightarrow{f'} & \operatorname{Spec} B \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f''} & \\ & \operatorname{Spec} A. & \end{array}$$

Part (1) shows that f'' is a closed immersion of spectral Deligne-Mumford stacks. Using Lemma 4.10, we see that f' induces a closed immersion of ∞-topoi $\mathcal{X} \to \mathcal{X}_B$. Moreover, f' induces a map of quasi-coherent sheaves $\mathcal{O}_B \to f'_* \mathcal{O}$ which induces an equivalence on global sections; it follows that $\mathcal{O}_B \simeq f'_* \mathcal{O}$. We will complete the proof by showing that f' induces an equivalence of ∞-topoi $\mathcal{X} \simeq \mathcal{X}_B$. We have an equivalence $\mathcal{X} \simeq \mathcal{X}_{B/U}$ for some (-1)-truncated object $U \in \mathcal{X}_B$; we wish to show that U is an initial object of $\mathcal{X}_B$. The results of §V.2.2 show that we can identify $\mathcal{X}_B$ with the ∞-topos $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}^{\text{ét}}_{B/})^{op})$. If U is not an initial object, then $U(B')$ is nonempty for some nonzero étale B-algebra B'. Then

$$B' \simeq \mathcal{O}_B(B') \simeq (f'_* \mathcal{O})(B') \simeq 0$$

and we obtain a contradiction. □

Definition 4.11. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is *strongly separated* if the diagonal map $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is a closed immersion. We will say that $\mathfrak{X}$ is a *separated spectral algebraic space* if the absolute diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is a closed immersion.

Remark 4.12. Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a closed immersion of spectral Deligne-Mumford stacks. If R is a discrete commutative ring, then the induced map

$$\theta : \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{Y}) \to \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{X})$$

is (-1)-truncated. It follows that if f is strongly separated, then the map θ is 0-truncated. In particular, if $\mathfrak{X}$ is a separated spectral algebraic space, then the mapping space $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{X})$ is 0-truncated for any commutative ring R: that is, $\mathfrak{X}$ is a spectral algebraic space.

Remark 4.13. It follows from Theorem 4.4 that if $\mathfrak{X}$ is a separated spectral algebraic space, then the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is affine. In particular, if f is quasi-compact, then f is geometric (see Definition VIII.3.4.15).

Remark 4.14. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks, and let $\delta : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ be the diagonal map. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} = (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$. The map $\delta^* \mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{X}}$ admits a right inverse, and therefore induces an epimorphism $\pi_0 \delta^* \mathcal{O}_{\mathcal{Z}} \to \pi_0 \mathcal{O}_{\mathcal{X}}$. It follows that δ is a closed immersion if and only if the underlying geometric morphism of ∞-topoi $\delta_* : \mathcal{X} \to \mathcal{Z}$ is a closed immersion.

Remark 4.15. If $j : \mathfrak{U} \to \mathfrak{X}$ is an open immersion, then the diagonal map $\mathfrak{U} \to \mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}$ is an equivalence. It follows that every open immersion between spectral Deligne-Mumford stacks is strongly separated. In particular, if $\mathfrak{X}$ is a separated spectral algebraic space, then $\mathfrak{U}$ is also a separated spectral algebraic space.

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Remark 4.16. Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{X} \\ \downarrow & & \downarrow f \\ \mathfrak{Y}' & \longrightarrow & \mathfrak{Y}. \end{array}$$

If f is strongly separated, then so is f'; this follows immediately from Corollary 1.7.

Remark 4.17. Remark 4.16 has a partial converse. If $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a map of spectral Deligne-Mumford stacks and there exists an effective epimorphism $\coprod U_\alpha \to \mathbf{1}$ in $\mathcal{Y}$ such that each of the induced maps $(\mathcal{X}_{/f^* U_\alpha}, \mathcal{O}_{\mathcal{X}} | f^* U_\alpha) \to (\mathcal{Y}_{/U_\alpha}, \mathcal{O}_{\mathcal{Y}} | U_\alpha)$ is strongly separated, then f is strongly separated.

Remark 4.18. Suppose we are given a commutative diagram

$$\begin{array}{ccc} & \mathfrak{X} & \\ {}^{f}\swarrow & & \searrow^{h} \\ \mathfrak{Y} & \xrightarrow{g} & \mathfrak{Z} \end{array}$$

of spectral Deligne-Mumford stacks. If g is strongly separated, then f is strongly separated if and only if f is strongly separated. In particular, the collection of strongly separated morphisms is closed under composition. To see this, consider the diagram

$$\begin{array}{ccccc} \mathfrak{X} & \xrightarrow{\delta} & \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} & \xrightarrow{\delta'} & \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{X} \\ & & \downarrow & & \downarrow \\ & & \mathfrak{Y} & \xrightarrow{\delta''} & \mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{Y}. \end{array}$$

Since g is strongly separated, δ'' is a closed immersion. It follows from Corollary 1.7 that δ' is a closed immersion, so that δ is a closed immersion if and only if $\delta' \circ \delta$ is a closed immersion (Remark 1.2).

Remark 4.19. Suppose we are given morphisms of spectral Deligne-Mumford stacks $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xleftarrow{g} \mathfrak{Z}$. If f and g are strongly separated, then the induced map $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z} \to \mathfrak{Y}$ is also strongly separated. This follows immediately from Remarks 4.18 and 4.16.

Remark 4.20. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a closed immersion of spectral Deligne-Mumford stacks, then for every commutative ring R the map $\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X}) \to \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{Y})$ has (-1)-truncated homotopy fibers. It follows that if $f : \mathfrak{X} \to \mathfrak{Y}$ is strongly separated, then the map $\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X}) \to \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{Y})$ has 0-truncated homotopy fibers. In particular, if f is strongly separated and $\mathfrak{Y}$ is a spectral algebraic space, then $\mathfrak{X}$ is also a spectral algebraic space.

Remark 4.21. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of spectral Deligne-Mumford stacks. The condition that f be strongly separated depends only the map of underlying 0-truncated spectral Deligne-Mumford stacks $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$.

Definition 4.22. We say that a spectral algebraic space $\mathfrak{X}$ is *separated* if it the morphism $\mathfrak{X} \to \operatorname{Spec} S$ is strongly separated; here S denotes the sphere spectrum.

Example 4.23. Let R be a connective $\mathbb{E}_\infty$-ring. Then $\operatorname{Spec} R \in \mathrm{Stk}$ is a separated spectral algebraic space. To prove this, it suffices to show that the multiplication $R \otimes_S R \to R$ induces a closed immersion $\operatorname{Spec} R \to \operatorname{Spec} R \otimes_S R$, which follows from Theorem 4.4.

Example 4.24. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Suppose that, for every commutative ring R, the induced map $\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X}) \to \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{Y})$ is (-1)-truncated. It follows that

$$\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X}) \simeq \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X})$$

is an equivalence for every discrete commutative ring R. The map $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ induces an equivalence of 0-truncated spectral Deligne-Mumford stacks, and is therefore a closed immersion (Remark 4.21. It follows that f is separated.

Remark 4.25. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Assume that $\mathfrak{Y}$ is a separated spectral algebraic space. Then $\mathfrak{X}$ is a separated spectral algebraic space if and only if f is strongly separated: this follows immediately from Remark 4.18. In particular, if we are given a map $f : \mathfrak{X} \to \operatorname{Spec} R$ for some connective $\mathbb{E}_\infty$-ring R, then $\mathfrak{X}$ is a separated spectral algebraic space if and only if f is strongly separated: that is, if and only if the diagonal map $\mathfrak{X} \to \mathfrak{X} \times_{\operatorname{Spec} R} \mathfrak{X}$ is a closed immersion.

Remark 4.26. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an affine map between spectral Deligne-Mumford stacks. Then f is strongly separated: this follows from Remark 4.17. In particular, any closed immersion of spectral Deligne-Mumford stacks is strongly separated.

We close this section with a few remarks about the relationship between closed and open immersions. If X is a topological space, then there is a bijection from the set of closed subsets of X to the set of open subsets of X, given by $Y \mapsto X - Y$. In algebraic geometry, the situation is more subtle: every closed subscheme $Y \subseteq X$ has an open complement $U = X - Y$. However, this construction is not bijective: a closed subset of X generally admits many different scheme structures. However, we can recover a bijective correspondence by restricting our attention to *reduced* closed subschemes of X. There is an entirely analogous discussion in the setting of spectral algebraic geometry.

Definition 4.27. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. We will say that $\mathfrak{X}$ is *reduced* if, for every étale map $\operatorname{Spec} R \to \mathfrak{X}$, the $\mathbb{E}_\infty$-ring R is discrete and the underlying commutative ring $\pi_0 R$ is reduced.

Remark 4.28. The condition that a spectral Deligne-Mumford stack $\mathfrak{X}$ be reduced is local with respect to the étale topology.

Proposition 4.29. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. Let* $\mathrm{Sub}^{\mathrm{red}}(\mathfrak{X})$ *denote the full subcategory of* $\mathrm{Stk}_{/\mathfrak{X}}$ *spanned by the closed immersions $i : \mathfrak{X}_0 \to \mathfrak{X}$, where $\mathfrak{X}_0$ is reduced. For every object $(i : \mathfrak{X}_0 \to \mathfrak{X})$ of* $\mathrm{Stk}_{/\mathfrak{X}}$*, let $\mathfrak{X}_0^c$ denote the object $i_*(\emptyset)$, where $\emptyset$ denotes an initial object in the underlying ∞-topos of $\mathfrak{X}_0$. Then the construction $(i : \mathfrak{X}_0 \to \mathfrak{X}) \mapsto \mathfrak{X}_0^c$ determines an equivalence of ∞-categories* $\mathrm{Sub}^{\mathrm{red}}(\mathfrak{X}) \simeq (\tau_{\leq -1}\mathcal{X})^{op}$.

Remark 4.30. We can state Proposition 4.29 more informally as follows: there is an order-reversing bijection between equivalence classes of reduced closed substacks of $\mathfrak{X}$ and open substacks of $\mathfrak{X}$.

Proof. The assertion is local on $\mathfrak{X}$. We may therefore assume without loss of generality that $\mathfrak{X} = \mathrm{Spec}\, R$ is affine. Using Theorem 4.4, we see that every closed immersion $i : \mathfrak{X}_0 \to \mathfrak{X}$ is induced by a map of connective $\mathbb{E}_\infty$-rings $R \to R'$ which induces a surjection $\pi_0 R \to \pi_0 R'$. Moreover, $\mathfrak{X}_0$ is reduced if and only if R' is a discrete commutative ring of the form $(\pi_0 R)/I$ for some radical ideal $I \subseteq \pi_0 R$. It follows that $\mathrm{Sub}^{\mathrm{red}}(\mathfrak{X})$ is equivalent to the the nerve of the partially ordered set of closed subsets of the Zariski spectrum $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$. The desired equivalence now follows from Lemma VII.9.7. □

5 Gluing along Closed Immersions

Suppose we are given a diagram of schemes

$$X \stackrel{f}{\hookleftarrow} K \stackrel{g}{\hookrightarrow} Y,$$

where f and g are closed immersions. Then there exists a pushout diagram σ :

$$\begin{array}{ccc} K & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow{f'} & Z \end{array}$$

in the category of schemes. The underlying topological space of Z is given by gluing Y to X along the common closed subset X_0, and the structure sheaf $\mathcal{O}_Z$ can be described as the fiber product $g'_* \mathcal{O}_X \times_{(g' \circ f)_* \mathcal{O}_{X_0}} f'_* \mathcal{O}_Y$. Moreover, the diagram σ is a pushout diagram not only in the category of schemes, but also in the larger category of locally ringed spaces.

In this section, we will study the problem of gluing along closed immersions in the setting of $\mathcal{T}$-structured ∞-topoi, where $\mathcal{T}$ is an arbitrary pregeometry. Our main result can be stated as follows:

Theorem 5.1. *Let $\mathcal{T}$ be a pregeometry. Suppose we are given morphisms*

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \leftarrow (\mathcal{K}, \mathcal{O}_{\mathcal{K}}) \rightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$$

in $\mathcal{T}\mathrm{op}(\mathcal{T})$ *which induce closed immersions of ∞-topoi $\mathcal{X} \leftarrow \mathcal{K} \rightarrow \mathcal{Y}$. Let $\mathcal{Z}$ be the pushout $\mathcal{X} \coprod_{\mathcal{K}} \mathcal{Y}$ in* $\mathcal{T}\mathrm{op}$*, so that we have a commutative diagram of geometric morphisms*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{g'_*} & \mathcal{X} \\ \downarrow{\scriptstyle f'_*} & \searrow{\scriptstyle h_*} & \downarrow{\scriptstyle f_*} \\ \mathcal{Y} & \xrightarrow{g_*} & \mathcal{Z}. \end{array}$$

Let $\mathcal{O}_{\mathcal{Z}} : \mathcal{T} \to \mathcal{Z}$ be the functor given by the formula

$$\mathcal{O}_{\mathcal{Z}}(X) = f_* \mathcal{O}_{\mathcal{X}}(X) \times_{h_* \mathcal{O}_{\mathcal{K}}(X)} g_* \mathcal{O}_{\mathcal{Y}}(X).$$

Then $\mathcal{O}_{\mathcal{Z}}$ is a $\mathcal{T}$-structure on $\mathcal{Z}$, and the diagram σ :

$$\begin{array}{ccc} (\mathcal{K}, \mathcal{O}_{\mathcal{K}}) & \xrightarrow{g'} & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \\ \downarrow{\scriptstyle f'} & \searrow^{h} & \downarrow{\scriptstyle f} \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) & \xrightarrow{g} & (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \end{array}$$

is a pushout square in $\mathcal{T}\mathrm{op}(\mathcal{T})$. In particular, the evident natural transformations $g^* \mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{Y}}$ and $f^* \mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{X}}$ are local.

Remark 5.2. In the situation of Theorem 5.1, suppose that $f' : (\mathcal{K}, \mathcal{O}_{\mathcal{K}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a closed immersion in $\mathcal{T}\mathrm{op}(\mathcal{T})$, in the sense of Definition 1.1. Then $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ is also a closed immersion: in other words, f induces an effective epimorphism $f^* \mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{Y}}$. To prove this, let us fix an object $M \in \mathcal{T}$; we wish to show that the map $f^* \mathcal{O}_{\mathcal{Z}}(M) \to \mathcal{O}_{\mathcal{Y}}(M)$ is an effective epimorphism. Since f_* is a closed immersion of ∞-topoi, this is equivalent to the assertion that the adjoint map $\mathcal{O}_{\mathcal{Z}}(M) \to f_* \mathcal{O}_{\mathcal{Y}}(M)$ is an effective epimorphism in $\mathcal{Z}$ (Lemma 5.4). This map is a pullback of $\theta : g_* \mathcal{O}_{\mathcal{X}}(M) \to h_* \mathcal{O}_{\mathcal{K}}(M)$; it therefore suffices to show that θ is an effective epimorphism. Applying Lemma 5.4 to the closed immersion h_*, we are reduced to showing that the map $f'^* \mathcal{O}_{\mathcal{X}}(M) \to \mathcal{O}_{\mathcal{K}}(M)$ is an effective epimorphism in $\mathcal{K}$, which follows from our assumption that f' is a closed immersion in $\mathcal{T}\mathrm{op}(\mathcal{T})$.

We will give the proof of Theorem 5.1 at the end of this section. We begin by studying the purely topological aspects of the problem: that is, by considering the problem of gluing ∞-topoi along closed immersions.

Proposition 5.3. *Let $\mathcal{X}$ be an ∞-topos, let U be a subobject of the final object of $\mathcal{X}$ (that is, U is a (-1)-truncated object of $\mathcal{X}$), and let $\mathcal{X}/U$ be the corresponding closed subtopos of $\mathcal{X}$ (see §T.7.3.2). Suppose we are given a geometric morphim $f_* : \mathcal{X}/U \to \mathcal{Y}$, and form a pushout diagram*

$$\begin{array}{ccc} \mathcal{X}/U & \xrightarrow{g_*} & \mathcal{X} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f'_*} \\ \mathcal{Y} & \xrightarrow{g'_*} & \mathcal{Z} \end{array}$$

in $\mathcal{T}\mathrm{op}$. Let $V = f'_*(U) \in \mathcal{Z}$. Then:

(1) *The functor f'_* induces an equivalence $\mathcal{X}_{/U} \simeq \mathcal{Z}_{/V}$.*

(2) *The functor g'_* is fully faithful and its essential image is $\mathcal{Z}/V$. In particular, g'_* is a closed immersion.*

Proof. According to Proposition T.6.3.2.3, we can identify $\mathcal{Z}$ with the homotopy fiber product of $\mathcal{X}$ with $\mathcal{Y}$ over $\mathcal{X}/U$ (along the functors $g^* : \mathcal{X} \to \mathcal{X}/U$ and $f^* : \mathcal{Y} \to \mathcal{X}/U$ adjoint to g_* and f_*). Since g^*U is an initial object of $\mathcal{X}/U$, there exists an object $V' \in \mathcal{Z}$ such that $f'^*V' \simeq U$ and g'^*V' is an initial object of $\mathcal{Y}$. For each $Z \in \mathcal{Z}$, we have a homotopy pullback diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{Z}}(Z, V') & \longrightarrow & \mathrm{Map}_{\mathcal{X}}(f'^*Z, U) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{Y}}(g'^*Z, g'^*V') & \longrightarrow & \mathrm{Map}_{\mathcal{X}/U}(g^*f'^*Z, g^*U). \end{array}$$

It follows that $\mathrm{Map}_{\mathcal{Z}}(Z, V')$ is contractible if g'^*Z is an initial object of $\mathcal{Y}$, and empty otherwise. Consequently, the equivalence $f'^*V' \simeq U$ is adjoint to an equivalence $V' \to V$. We may therefore assume without loss of generality that $V' = V$.

We now prove (1). Note that the projection map $\mathcal{Z}_{/V} \to \mathcal{Z}$ is a fully faithful embedding, whose essential image is the full subcategory $\mathcal{Z}_0 \subseteq \mathcal{Z}$ spanned by those objects $Z \in \mathcal{Z}$ such that $g'^* Z$ is an initial object of $\mathcal{Y}$. Similarly, the projection $\mathcal{X}_{/U} \to \mathcal{X}$ is a fully faithful embedding whose essential image is the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ spanned by those objects X such that $g^* X$ is an initial object of $\mathcal{X}/U$. It will therefore suffice to prove that the functor f'^* induces an equivalence of $\mathcal{Z}_0$ with $\mathcal{X}_0$, which follows immediately from Proposition T.6.3.2.3.

We now prove (2). We first claim that g'_* carries $\mathcal{Y}$ into $\mathcal{Z}/V$. That is, we claim that if $Y \in \mathcal{Y}$, then the projection map $(g'_* Y) \times V \to V$ is an equivalence. Let $Z \in \mathcal{Z}$ be an object; we wish to show that the map

$$\mathrm{Map}_{\mathcal{Z}}(Z, g'_* Y \times V) \simeq \mathrm{Map}_{\mathcal{Y}}(g'^* Z, Y) \times \mathrm{Map}_{\mathcal{Z}}(Z, V) \to \mathrm{Map}_{\mathcal{Z}}(Z, V)$$

is a homotopy equivalence. If $Z \notin \mathcal{Z}_0$, then both sides are empty and the result is obvious. If $Z \in \mathcal{Z}_0$, then the result follows since $g' * Z$ is an initial object of $\mathcal{Y}$.

Now suppose $Z \in \mathcal{Z}/V$. We claim that the unit map $u : Z \to g'_* g'^* Z$ is an equivalence in $\mathcal{Z}$. To prove this, we show that for each $Z' \in \mathcal{Z}$, composition with u induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{Z}}(Z', Z) \to \mathrm{Map}_{\mathcal{Z}}(Z', g'_* g'^* Z) \simeq \mathrm{Map}_{\mathcal{Y}}(g'^* Z', g'^* Z).$$

This map fits into a homotopy pullback diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{Z}}(Z', Z) & \longrightarrow & \mathrm{Map}_{\mathcal{Y}}(g'^* Z', g'^* Z) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{X}}(f'^* Z', f'^* Z) & \longrightarrow & \mathrm{Map}_{\mathcal{X}/U}(g^* f'^* Z', g^* f'^* Z). \end{array}$$

It therefore suffices to show that the bottom horizontal map is a homotopy equivalence. In other words, we are reduced to showing that the unit map $f'^* Z \to g_* g^* f'^* Z$ is an equivalence in $\mathcal{X}$, which follows from the observation that $f'^* \in \mathcal{X}/U$.

The argument of the preceding paragraph shows that g'^* induces a fully faithful embedding $\mathcal{Z}/V \to \mathcal{Y}$. To complete the proof, we show the functor $g'^* | (\mathcal{Z}/V)$ is essentially surjective. Fix an object $Y \in \mathcal{Y}$, and let $X = f^* Y \in \mathcal{X}/U$. Since g_* is fully faithful, the counit map $g^* g_* X \to X$ is an equivalence. It follows that there exists an object $Z \in \mathcal{Z}$ such that $g'^* Z \simeq Y$ and $f'^* Z \simeq g_* X$. It remains only to verify that $Z \in \mathcal{Z}/V$: that is, that the projection map $Z \times V \to V$ is an equivalence. Using (1), we are reduced to proving that the map $f'^*(Z \times V) \to f'^* V \simeq U$ is an equivalence. In other words, we are reduced to proving that $f'^* Z \simeq g_* X$ belongs to $\mathcal{X}/U$, which is clear. □

Lemma 5.4. *Let $\mathcal{X}$ be an ∞-topos containing a (-1)-truncated object U, and let $i^* : \mathcal{X} \to \mathcal{X}/U$ and $j^* : \mathcal{X} \to \mathcal{X}_{/U}$ be the associated geometric morphisms. Suppose we are given a family of morphisms $\{\alpha_i : X_i \to X\}$ in $\mathcal{X}$. Then the composite map $\coprod X_i \to X$ is an effective epimorphism if and only if the corresponding maps*

$$\coprod i^* X_i \to f^* X \qquad \coprod j^* X_i \to g^* X$$

are effective epimorphisms in $\mathcal{X}/U$ and $\mathcal{X}_{/U}$, respectively.

Proof. Since i^* and j^* commute with coproducts, we can replace the family $\{X_i \to X\}$ by the single map $\alpha : X_0 \to X$, where $X_0 = \coprod_i X_i$. Let $X_\bullet$ be the Čech nerve of α and let X' be its geometric realization. Then α is an effective epimorphism if and only if it induces an equivalence $\beta : X' \to X$. Similarly, $f^*(\alpha)$ and $g^*(\alpha)$ are effective epimorphisms if and only if $i^*(\beta)$ and $j^*(\beta)$ are equivalences. The desired result now follows from Lemma A.A.5.11. □

Lemma 5.5. *Let $\mathcal{T}$ be a pregeometry, let $\mathcal{X}$ be an ∞-topos containing a (-1)-truncated object U, and let $i^* : \mathcal{X} \to \mathcal{X}/U$ and $j^* : \mathcal{X} \to \mathcal{X}_{/U}$ be the associated geometric morphisms. Then:*

(1) *Let* $\mathcal{O} : \mathcal{T} \to \mathcal{X}$ *be an arbitrary functor. Then* $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ *if and only if* $i^* \mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}/U)$ *and* $j^* \mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}_{/U})$.

(2) *Let* $\alpha : \mathcal{O} \to \mathcal{O}'$ *be a natural transformation between functors* $\mathcal{O}, \mathcal{O}' \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. *Then* α *is a morphism in* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ *if and only if* $i^*(\alpha)$ *and* $j^*(\alpha)$ *are morphisms of* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}/U)$ *and* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}_{/U})$*, respectively.*

Proof. Combine Lemmas A.A.5.11 and 5.4. □

Lemma 5.6. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be* ∞*-topoi containing* (-1)*-truncated objects* $U \in \mathcal{X}$ *and* $V \in \mathcal{Y}$*, and suppose we are given an equivalence of* ∞*-topoi*

$$\mathcal{X}/U \simeq \mathcal{K} \simeq \mathcal{Y}/V.$$

Form a pushout diagram

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle f_*} \\ \mathcal{Y} & \xrightarrow{g_*} & \mathcal{Z} \end{array}$$

in $\mathcal{T}\mathrm{op}$*. Then:*

(1) *The geometric morphisms* f_* *and* g_* *are closed immersions.*

(2) *Let* $\mathcal{T}$ *be a pregeometry and* $\mathcal{O} : \mathcal{T} \to \mathcal{Z}$ *a functor. Then* $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$ *if and only if the following conditions are satisfied:*

   (a) *The pullback map* $h^* : \mathcal{Z} \to \mathcal{K}$ *carries* $\mathcal{O}$ *to* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{K})$.

   (b) *The pullback map* $f'^* : \mathcal{Z} \xrightarrow{f^*} \mathcal{X} \to \mathcal{X}_{/U}$ *carries* $\mathcal{O}$ *to* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}_{/U})$.

   (c) *The pullback map* $g'^* : \mathcal{Z} \xrightarrow{g^*} \mathcal{Y} \to \mathcal{Y}_{/V}$ *carries* $\mathcal{O}$ *to* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y}_{/V})$.

(3) *Let* $\mathcal{T}$ *be a pregeometry and* $\alpha : \mathcal{O} \to \mathcal{O}'$ *a natural transformation between functors* $\mathcal{O}, \mathcal{O}' \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. *Then* α *is a morphism in* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ *if and only if* $h^*(\alpha)$, $f'^*(\alpha)$*, and* $g'^*(\alpha)$ *are morphisms of* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{K})$, $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}_{/U})$*, and* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y}_{/V})$*, respectively.*

Proof. Assertion (1) follows from Proposition 5.3. Assertions (2) and (3) follow by combining Proposition 5.3 with Lemma 5.5. □

Lemma 5.7. *Let* $\mathcal{X}$ *be an* ∞*-topos,* $\mathcal{T}$ *a pregeometry and suppose we are given morphisms* $\mathcal{O}' \xrightarrow{\alpha} \mathcal{O} \xleftarrow{\beta} \mathcal{O}''$ *in* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. *Then the fiber product* $\mathcal{O}' \times_{\mathcal{O}} \mathcal{O}''$ *(formed in the* ∞*-category* $\mathrm{Fun}(\mathcal{T}, \mathcal{X})$*) belongs to* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. *Moreover, the projection maps* $\mathcal{O}' \leftarrow \mathcal{O}' \times_{\mathcal{O}} \mathcal{O}'' \to \mathcal{O}''$ *are morphisms in* $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$.

Proof. The only nontrivial point to verify is condition (3) of Definition V.3.1.4. Suppose we are given an admissible covering $\{U_i \to X\}$ in $\mathcal{T}$; we wish to show that the induced map

$$\coprod \mathcal{O}'(U_i) \times_{\mathcal{O}(U_i)} \mathcal{O}''(U_i) \to \mathcal{O}'(X) \times_{\mathcal{O}(X)} \mathcal{O}''(X)$$

is an effective epimorphism in $\mathcal{X}$. Since colimits in $\mathcal{X}$ are universal, the hypothesis that α and β are morphisms in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ guarantees the existence of a pullback diagram

$$\begin{array}{ccc} \coprod \mathcal{O}'(U_i) \times_{\mathcal{O}(U_i)} \mathcal{O}''(U_i) & \longrightarrow & \mathcal{O}'(X) \times_{\mathcal{O}(X)} \mathcal{O}''(X) \\ \downarrow & & \downarrow \\ \coprod \mathcal{O}(U_i) & \longrightarrow & \mathcal{O}(X). \end{array}$$

It therefore suffices to show that the map $\coprod \mathcal{O}(U_i) \to \mathcal{O}(X)$ is an effective epimorphism, which follows from our assumption that $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. □

Lemma 5.8. *Let $\mathcal{T}$ be a pregeometry. Suppose we are given morphisms*

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \leftarrow (\mathcal{K}, \mathcal{O}_{\mathcal{K}}) \rightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$$

in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T})^{op}$ which induce closed immersions of ∞-topoi $\mathcal{X} \leftarrow \mathcal{K} \rightarrow \mathcal{Y}$. Form a pushout diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{g'_*} & \mathcal{X} \\ \downarrow{\scriptstyle f'_*} & \searrow^{h_*} & \downarrow{\scriptstyle f_*} \\ \mathcal{Y} & \xrightarrow{g_*} & \mathcal{Z} \end{array}$$

in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}^{op}$. Then the fiber product $\mathcal{O}_{\mathcal{Z}} = f_* \mathcal{O}_{\mathcal{X}} \times_{h_* \mathcal{O}_{\mathcal{K}}} g_* \mathcal{O}_{\mathcal{Y}}$ (formed in the ∞-category $\mathrm{Fun}(\mathcal{T}, \mathcal{Z})$*) is a $\mathcal{T}$-structure on $\mathcal{Z}$.*

Proof. The pullback

$$h^* \mathcal{O}_{\mathcal{Z}} \simeq g'^* \mathcal{O}_{\mathcal{X}} \times_{\mathcal{O}_{\mathcal{K}}} f'^* \mathcal{O}_{\mathcal{Y}}$$

is a $\mathcal{T}$-structure on $\mathcal{K}$ by Lemma 5.7. Choose (-1)-truncated objects $U \in \mathcal{X}$, $V \in \mathcal{Y}$ such that $\mathcal{X}/U \simeq \mathcal{K} \simeq \mathcal{Y}/V$. Let $i^* : \mathcal{X} \rightarrow \mathcal{X}_{/U}$ and $j^* : \mathcal{Y} \rightarrow \mathcal{Y}_{/V}$ be the associated pullback functors. Using Proposition 5.3, we see that $i^* f^*$ carries $h_* \mathcal{O}_{\mathcal{K}}$ and $g_* \mathcal{O}_{\mathcal{Y}}$ to the constant functor $\mathcal{T} \rightarrow \mathcal{X}_{/U}$ taking the value U. Consequently, we get an equivalence

$$(i^* f^*)(\mathcal{O}_{\mathcal{Z}}) \simeq i^* \mathcal{O}_{\mathcal{X}} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}_{/U}).$$

A similar argument shows that

$$(j^* g^*)(\mathcal{O}_{\mathcal{Z}}) \simeq j^* \mathcal{O}_{\mathcal{Y}} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y}_{/V}).$$

Applying Lemma 5.6, we conclude that $\mathcal{O}_{\mathcal{Z}} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$. ☐

Proof of Theorem 5.1. Let $\mathcal{C} = \mathrm{Fun}(\mathcal{T}, \overline{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}}) \times_{\mathrm{Fun}(\mathcal{K}, {}^{\mathrm{L}}\mathcal{T}\mathrm{op})} {}^{\mathrm{L}}\mathcal{T}\mathrm{op}$ be defined as in Definition V.3.1.9, so that ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T})$ is a subcategory of $\mathcal{C}$. The projection map $p : \mathcal{T}\mathrm{op}(\mathcal{T}) \rightarrow \mathcal{T}\mathrm{op}$ extends to a coCartesian fibration $\overline{p} : \mathcal{C} \rightarrow \mathcal{T}\mathrm{op}$. We note that the fiber of $\overline{p}$ over an ∞-topos $\mathcal{W}$ can be identified with $\mathrm{Fun}(\mathcal{T}, \mathcal{W})^{op}$. Moreover, to each geometric morphism $\phi_* : \mathcal{W} \rightarrow \mathcal{W}'$, the associated functor $\mathrm{Fun}(\mathcal{T}, \mathcal{W})^{op} \rightarrow \mathrm{Fun}(\mathcal{T}, \mathcal{W}')^{op}$ is given by composition with ϕ_* (and therefore preserves small limits). Using Propositions T.4.3.1.9 and T.4.3.1.10, we deduce that σ is an $\overline{p}$-colimit diagram in $\mathcal{C}^{op}$. Note that $\overline{p}(\sigma)$ is a pushout diagram in $\mathcal{T}\mathrm{op}$. It follows that σ is a pushout diagram in $\mathcal{C}^{op}$. We wish to show that σ remains a pushout diagram in the subcategory $\mathcal{T}\mathrm{op}(\mathcal{T}) \subseteq \mathcal{C}^{op}$ (so that, in particular, σ factors through $\mathcal{T}\mathrm{op}(\mathcal{T})$). Unwinding the definitions, this amounts to the following assertion: given an object $(\mathcal{W}, \mathcal{O}_{\mathcal{W}}) \in \mathcal{T}\mathrm{op}(\mathcal{T})$ and a morphism $\psi : (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \rightarrow (\mathcal{W}, \mathcal{O}_{\mathcal{W}})$ in $\mathcal{C}^{op}$, the map ψ is a morphism in $\mathcal{T}\mathrm{op}(\mathcal{T})$ if and only if each of the composite maps

$$\psi' : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \rightarrow (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \rightarrow (\mathcal{W}, \mathcal{O}_{\mathcal{W}})$$

$$\psi'' : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \rightarrow (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \rightarrow (\mathcal{W}, \mathcal{O}_{\mathcal{W}})$$

is a morphism in $\mathcal{T}\mathrm{op}(\mathcal{T})$. Let $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$ denote the pullback of $\mathcal{O}_{\mathcal{W}}$ to $\mathcal{Z}$ (under the geometric morphism determined by ψ), so that ψ determines a natural transformation $\alpha : \mathcal{O} \rightarrow \mathcal{O}_{\mathcal{W}}$ of functors $\mathcal{T} \rightarrow \mathcal{Z}$.

Choose (-1)-truncated objects $U \in \mathcal{X}$, $V \in \mathcal{Y}$ such that $\mathcal{X}/U \simeq \mathcal{Z} \simeq \mathcal{Y}/V$. Let $i^* : \mathcal{X} \rightarrow \mathcal{X}_{/U}$ and $j^* : \mathcal{Y} \rightarrow \mathcal{Y}_{/V}$ be the associated pullback functors. According to Lemma 5.6, the morphism α belongs to $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$ if and only if the following three conditions are satisfied:

(*a*) The pullback $h^*(\alpha) : h^* \mathcal{O} \rightarrow h^* \mathcal{O}_{\mathcal{Z}}$ is a morphism of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{K})$.

(*b*) The pullback $i^* f^*(\alpha) : i^* f^* \mathcal{O} \rightarrow i^* f^* \mathcal{O}_{\mathcal{Z}} \simeq i^* \mathcal{O}_{\mathcal{X}}$ is a morphism of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}_{/U})$.

(*c*) The pullback $j^* g^*(\alpha) : j^* g^* \mathcal{O} \rightarrow j^* g^* \mathcal{O}_{\mathcal{Y}} \simeq j^* \mathcal{O}_{\mathcal{Y}}$ is a morphism of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Y}_{/V})$.

By the same reasoning, we can use Lemma 5.5 to deduce that ψ' is a morphism in $\mathcal{T}\mathrm{op}(\mathcal{T})$ if and only if ψ satisfies condition (b) and the following version of (a):

(a') The composite map $h^* \mathcal{O} \to h^* \mathcal{O}_{\mathcal{Z}} \to g'^* \mathcal{O}_{\mathcal{X}}$ belongs to $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$.

Similarly, ψ'' is a morphism in $\mathcal{T}\mathrm{op}(\mathcal{T})$ if and only if ψ satisfies (c) together with the following version of (a):

(a'') The composite map $h^* \mathcal{O} \to h^* \mathcal{O}_{\mathcal{Z}} \to f'^* \mathcal{O}_{\mathcal{Y}}$ belongs to $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$.

We will complete the proof by showing that conditions (a), (a'), and (a'') are equivalent. By symmetry, it will suffice to show that (a) and (a') are equivalent. Fix an admissible morphism $X' \to X$ in $\mathcal{T}$ and consider the diagram

$$\begin{array}{ccccc} h^* \mathcal{O}(X') & \longrightarrow & h^* \mathcal{O}_{\mathcal{Z}}(X') & \longrightarrow & g'^* \mathcal{O}_{\mathcal{X}}(X') \\ \downarrow & & \downarrow & & \downarrow \\ h^* \mathcal{O}(X) & \longrightarrow & h^* \mathcal{O}_{\mathcal{Z}}(X) & \longrightarrow & g'^* \mathcal{O}_{\mathcal{X}}(X'); \end{array}$$

we will show that the outer rectangle is a pullback diagram if and only if the left square is a pullback diagram. For this, it suffices to show that the right square is a pullback diagram. Since g'^* is left exact, we are reduced to proving that the diagram τ :

$$\begin{array}{ccc} f^* \mathcal{O}_{\mathcal{Z}}(X') & \longrightarrow & \mathcal{O}_{\mathcal{X}}(X') \\ \downarrow & & \downarrow \\ f^* \mathcal{O}_{\mathcal{Z}}(X) & \longrightarrow & \mathcal{O}_{\mathcal{X}}(X) \end{array}$$

is a pullback square in $\mathcal{X}$. Using Lemma A.A.5.11, we are reduced to proving that $g'^*(\tau)$ and $i^*(\tau)$ are pullback diagrams. In the case of $g'^*(\tau)$ this is obvious (the horizontal maps in $g'^*(\tau)$ are equivalences); in the case of $i^*(\tau)$, it follows from Lemma 5.7. □

6 Gluing Spectral Deligne-Mumford Stacks

Let $\mathcal{T}$ be a pregeometry. In §5, we discussed the operation of gluing $\mathcal{T}$-structured ∞-topoi along closed immersions. In good cases, one can show that the collection of $\mathcal{T}$-schemes is closed under this operation. In this section, we will prove this when $\mathcal{T} = \mathcal{T}^{\mathrm{Sp}}_{\text{ét}}$ is the pregeometry which controls the theory of spectral Deligne-Mumford stacks. Our main result can be stated as follows:

Theorem 6.1. *Suppose given a pair of closed immersions of spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{X} \to \mathfrak{X}'$. Then:*

(1) *There exists a pushout diagram σ :*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} \\ \mathfrak{X}' & \xrightarrow{g'} & \mathfrak{Y}' \end{array}$$

in $\mathrm{RingTop}_{\acute{e}t}$.

(2) *The image of σ in $\mathcal{T}\mathrm{op}$ is a pushout diagram of ∞-topoi.*

(3) *The maps f' and g' are closed immersions.*

(4) *The pushout $\mathfrak{Y}'$ is a spectral Deligne-Mumford stack.*

(5) *If $\mathfrak{X}'$ and $\mathfrak{Y}$ are affine, then $\mathfrak{Y}'$ is also affine.*

Remark 6.2. Theorem 6.1 has an evident analogue in the setting of spectral schemes, which can be proven in the same way.

The proof of Theorem 6.1 will require a number of preliminaries.

Lemma 6.3. *Let $\mathcal{X}$ be an ∞-topos containing a pair of (-1)-truncated objects U and V. Suppose that $U \times V$ is an initial object of $\mathcal{X}$. Then the coproduct $U \amalg V$ is also a (-1)-truncated object of $\mathcal{X}$.*

Proof. We must prove that the diagonal map

$$U \amalg V \to (U \amalg V) \times (U \amalg V) \simeq (U \times U) \amalg (U \times V) \amalg (V \times U) \amalg (V \times V)$$

is an equivalence. This map factors as as a composition

$$U \amalg V \to (U \times U) \amalg (V \times V) \to (U \times U) \amalg (U \times V) \amalg (V \times U) \amalg (V \times V)$$

where the first map is an equivalence since U and V are (-1)-truncated, and the second is an equivalence since $U \times V \simeq V \times U$ is an initial object of $\mathcal{X}$. □

Proposition 6.4. *Let $\mathcal{X}$ be an ∞-topos containing a pair of (-1)-truncated object U and V whose product $U \times V$ is an initial object of $\mathcal{X}$. Then the diagram*

$$\begin{array}{ccc} \mathcal{X}_{/(U \amalg V)} & \longrightarrow & \mathcal{X}_{/U} \\ \downarrow & & \downarrow \\ \mathcal{X}_{/V} & \longrightarrow & \mathcal{X} \end{array}$$

is a pushout square in $\mathcal{T}\mathrm{op}$.

Proof. Let $f^* : \mathcal{X} \to \mathcal{X}_{/U}$, $g^* : \mathcal{X} \to \mathcal{X}_{/V}$ and $h^* : \mathcal{X} \to \mathcal{X}_{/(U \amalg V)}$ be the associated pullback functors. Form a pushout diagram

$$\begin{array}{ccc} \mathcal{X}_{/(U \amalg V)} & \longrightarrow & \mathcal{X}_{/U} \\ \downarrow & & \downarrow \\ \mathcal{X}_{/V} & \longrightarrow & \mathcal{Y} \end{array}$$

in $\mathcal{T}\mathrm{op}$; we wish to prove that the induced geometric morphism $\phi_* : \mathcal{Y} \to \mathcal{X}$ is an equivalence of ∞-topoi. We first claim that the pullback functor ϕ^* is conservative. In other words, we claim that a morphism $u : X \to X'$ in $\mathcal{X}$ is an equivalence provided that both $f^*(u)$ and $g^*(u)$ are equivalences. Indeed, either of these conditions guarantees that $h^*(u)$ is an equivalence, so (by Lemma A.A.5.11) it will suffice to show that u induces an equivalence $X \times (U \amalg V) \to X' \times (U \amalg V)$. Since colimits in $\mathcal{X}$ are universal, it suffices to show that $u_U : X \times U \to X' \times U$ and $u_V : X \times V \to X' \times V$ are equivalences. We prove that u_U is an equivalence; the proof that u_V is an equivalence is similar. We have a commutative diagram

$$\begin{array}{ccc} X \times U & \xrightarrow{v} & (g_* g^* X) \times U \\ \downarrow & & \downarrow \\ X' \times U & \xrightarrow{v'} & (g_* g^* X') \times U. \end{array}$$

Since the right vertical map is an equivalence, it will suffice to show that v and v' are equivalences. We will prove that v is an equivalence; the roof that v' is an equivalence is similar. Since g_* is fully faithfully, the map $g^*(v)$ is an equivalence. Using Lemma A.A.5.11 again, we are reduced to proving that the map $X \times U \times V \to (g_* g^* X) \times U \times V$ is an equivalence. This is clear, since both $X \times U \times V$ and $(g_* g^* X) \times U \times V$ are initial objects of $\mathcal{X}$.

To complete the proof that ϕ_* is an equivalence, it suffices to show that the counit map $\phi^* \phi_* Y \to Y$ is an equivalence for each object $Y \in \mathcal{Y}$. Using Proposition T.6.3.2.3, we can identify the ∞-category $\mathcal{Y}$ with the fiber product $\mathcal{X}_{/U} \times_{\mathcal{X}_{/(U \amalg V)}} \mathcal{X}_{/V}$. Thus $Y \in \mathcal{Y}$ corresponds to a pair of objects $Y_0 \in \mathcal{X}_{/U}$, $Y_1 \in \mathcal{X}_{/V}$ together with an equivalence $g^* Y_0 \simeq f^* Y_1 = Y_{01} \in \mathcal{X}_{/(U \amalg V)}$. Unwinding the definitions, we see that the pushforward functor ϕ_* is given by the formula $\phi_* Y = f_* Y_0 \times_{h_* Y_{01}} g_* Y_1$. Since a morphism in $\mathcal{Y}$ is an equivalence if and only if its pullbacks to $\mathcal{X}_{/U}$ and $\mathcal{X}_{/V}$ are equivalences, we are reduced to proving that the projection maps

$$\alpha : f^*(f_* Y_0 \times_{h_* Y_{01}} g_* Y_1) \to Y_0$$
$$\beta : g^*(f_* Y_0 \times_{h_* Y_{01}} g_* Y_1) \to Y_1$$

are equivalences in $\mathcal{X}_{/U}$ and $\mathcal{X}_{/V}$, respectively. We prove that α is an equivalence; the proof for β is similar. The map α factors as a composition

$$f^*(f_* Y_0 \times_{h_* Y_{01}} g_* Y_1) \xrightarrow{\alpha'} f^* f_* Y_0 \xrightarrow{\alpha''} Y_0.$$

Since f_* is fully faithful, the counit map α'' is an equivalence; it will therefore suffice to show that α is an equivalence. Since f^* is left exact, the map α' is a pullback of $\gamma : f^* g_* Y_1 \to f^* h_* Y_{01}$. It will therefore suffice to show that γ is an equivalence in $\mathcal{X}_{/U}$, which is an immediate translation of the condition that $Y_{01} \simeq f^* Y_1 \in \mathcal{X}_{/(U \amalg V)} \subseteq \mathcal{X}_{/U}$. □

Corollary 6.5. *Suppose we are given a pullback diagram of $\mathbb{E}_\infty$-rings*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{g} & B \end{array}$$

where A, B, and B' are connective and the maps f and g induce surjective ring homomorphisms $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$. Then:

(1) *The $\mathbb{E}_\infty$-ring A' is connective.*

(2) *The maps $\pi_0 A' \to \pi_0 A$ and $\pi_0 B' \to \pi_0 B$ are surjective.*

(3) *The induced diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} B' \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A' \end{array}$$

is a pushout diagram in $\mathrm{RingTop}_{ét}$.

Proof. Assertions (1) and (2) are obvious. For every $\mathbb{E}_\infty$-ring R, let $\mathcal{X}_R$ denote the underlying ∞-topos of $\operatorname{Spec} R$ (that is, the ∞-topos of étale sheaves on $\mathrm{CAlg}_R^{ét}$), and let $\mathcal{O}_R$ denote the structure sheaf of $\operatorname{Spec} R$. In view of Theorem 5.1, it will suffice to show the following:

(3′) The diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{X}_B & \longrightarrow & \mathcal{X}_{B'} \\ \downarrow & \searrow^{h_*} & \downarrow f_* \\ \mathcal{X}_A & \xrightarrow{g_*} & \mathcal{X}_{A'} \end{array}$$

is a pushout square in $\mathcal{T}\mathrm{op}$.

(3″) The diagram

$$\begin{array}{ccc} \mathcal{O}_{A'} & \longrightarrow & g_* \mathcal{O}_A \\ \downarrow & & \downarrow \\ f_* \mathcal{O}_{B'} & \longrightarrow & h_* \mathcal{O}_B \end{array}$$

is a pullback square in $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathfrak{X}_{A'})$.

We first prove (3′). Using (2) and Proposition VIII.1.2.7, we deduce the existence of equivalences

$$\mathfrak{X}_{B'} \simeq \mathfrak{X}_{A'} /U \qquad \mathfrak{X}_A \simeq \mathfrak{X}_{A'} /V \qquad \mathfrak{X}_B \simeq \mathfrak{X}_{A'} /W$$

for (-1)-truncated objects $U, V, W \in \mathfrak{X}_{A'}$. Under the equivalence $\mathfrak{X}_{A'} \simeq \mathcal{S}\mathrm{hv}((\mathrm{CAlg}^{\text{ét}}_{A'})^{op}) \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_{A'}, \mathcal{S})$, the objects U, V, and W can be described by the formulas

$$U(R) = \begin{cases} \Delta^0 & \text{if } R \otimes_{A'} B' \simeq 0 \\ \emptyset & \text{otherwise} \end{cases} \qquad V(R) = \begin{cases} \Delta^0 & \text{if } R \otimes_{A'} A \simeq 0 \\ \emptyset & \text{otherwise} \end{cases} \qquad W(R) = \begin{cases} \Delta^0 & \text{if } R \otimes_{A'} B \simeq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

For any A-algebra R, we have a pullback diagram τ_R :

$$\begin{array}{ccc} R & \longrightarrow & R \otimes_{A'} A \\ \downarrow & & \downarrow \\ R \otimes_{A'} B' & \longrightarrow & R \otimes_{A'} B. \end{array}$$

It follows that if $R \otimes_{A'} A \simeq R \otimes_{A'} B' \simeq 0$, then $R \simeq 0$; this proves that $U \times V$ is an initial object of $\mathfrak{X}_{A'}$. According to Lemma 6.3, the coproduct $U \amalg V$ in $\mathfrak{X}_{A'}$ is (-1)-truncated. There is an evident map $j : U \amalg V \to W$. We will prove that j is an equivalence, so that assertion (3′) follows from Proposition 6.4. We must show that if R is an étale A'-algebra such that $W(R)$ is nonempty, then $(U \amalg V)(R)$ is nonempty. Indeed, if $W(R)$ is nonempty then we have $R \otimes_{A'} B \simeq 0$, so that the pullback diagram $\tau_{R'}$ shows that $R \simeq R_0 \times R_1$ with $R_0 = R \otimes_{A'} A$ and $R_1 = R \otimes_{A'} B'$). Then R_0 and R_1 are also étale A'-algebras, and we have $(U \amalg V)(R) \simeq (U \amalg V)(R_0) \times (U \amalg V)(R_1)$. It will therefore suffice to show that both $(U \amalg V)(R_0)$ and $(U \amalg V)(R_1)$ are nonempty. We will prove that $(U \amalg V)(R_0)$ is nonempty; the proof in the other case is similar. Since we have a map $U(R_0) \to (U \amalg V)(R_0)$, it is sufficient to show that $U(R_0)$ is nonempty: that is, that $R_0 \otimes_{A'} B' \simeq 0$. Let $I \subseteq \pi_0 A'$ be the kernel of the map $\pi_0 A' \to \pi_0 B'$, and $J \subseteq \pi_0 A'$ the kernel of the map $\pi_0 A' \to \pi_0 A$, so that the ideal $I + J$ is the kernel of the map $\pi_0 A' \to \pi_0 B$. We now compute

$$\pi_0(R_0 \otimes_{A'} B') \simeq (\pi_0 R_0)/(I\pi_0 R_0) \simeq (\pi_0 R/J\pi_0 R)/I\pi_0 R_0 \simeq (\pi_0 R)/(I + J)\pi_0 R \simeq \pi_0(R \otimes_{A'} B) \simeq 0.$$

It remains to prove (3″). Unwinding the definitions, we are reduced to the evident observation that if $R \in \mathrm{CAlg}^{\text{ét}}_{A'}$, then the diagram

$$\begin{array}{ccc} R & \longrightarrow & R \otimes_{A'} A \\ \downarrow & & \downarrow \\ R \otimes_{A'} B' & \longrightarrow & R \otimes_{A'} B \end{array}$$

is a pullback square of $\mathbb{E}_\infty$-rings. □

Lemma 6.6. *Suppose we are given a closed immersion $f : (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ of spectral Deligne-Mumford stacks, where $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is affine. Let $U \in \mathcal{X}$ be an object such that $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X} |U)$ is also an affine Deligne-Mumford stack. Then there exists an equivalence $U \simeq f^*V$ for some object $V \in \mathcal{Y}$ such that $(\mathcal{Y}_{/V}, \mathcal{O}_\mathcal{Y} |V)$ is affine.*

Proof. Let $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \simeq \operatorname{Spec} A$ for some connective $\mathbb{E}_\infty$-ring A. Then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \operatorname{Spec} B$ for some connective A-algebra B such that $\pi_0 A \to \pi_0 B$ is surjective. Using Theorem VIII.1.2.1, we see that the object $U \in \mathcal{X}$ corresponds to an étale B-algebra B'. Using Proposition VII.8.10, we can write $B' \simeq A' \otimes_A B$ for some étale A-algebra A', which determines an object $V \in \mathcal{Y}$ with the desired properties. $\square$

Proof of Theorem 6.1. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, and $\mathfrak{X}' = (\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. Assertions (1), (2) and (3) follow from Theorem 5.1, so that we can define the pushout $\mathfrak{Y}' = (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$. We now prove (5). Assume that $(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) = \operatorname{Spec} B'$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \simeq \operatorname{Spec} A$ are affine. Since f and g are closed immersions, Theorem 4.4 implies that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \operatorname{Spec} B$ is affine, and that f and g induce surjective ring homomorphisms $\pi_0 B' \to \pi_0 B \leftarrow \pi_0 A$. Form a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B. \end{array}$$

Using Corollary 6.5, we obtain a pushout diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} \\ \mathfrak{X}' & \xrightarrow{g'} & \operatorname{Spec} A' \end{array}$$

in $\mathrm{RingTop}_{\text{ét}}$. It follows that $\mathfrak{Y}' \simeq \operatorname{Spec} A'$ is an affine spectral Deligne-Mumford stack.

It remains to prove (4). The assertion that $(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$ is a spectral Deligne-Mumford stack is local on $\mathcal{Y}'$. Let $\mathcal{Y}'_0$ be the full subcategory of $\mathcal{Y}'$ spanned by those objects Y for which $(\mathcal{Y}'_{/Y}, \mathcal{O}_{\mathcal{Y}'}|Y)$ is a spectral Deligne-Mumford stack; it will suffice to show that there exists an effective epimorphism $\coprod Y_i \to \mathbf{1}_{\mathcal{Y}'}$, where each $Y_i \in \mathcal{Y}'_0$ and $\mathbf{1}_{\mathcal{Y}'}$ is the final object of $\mathcal{Y}'$. Since f' and g' are closed immersions, we can write $\mathcal{X}' \simeq \mathcal{Y}'/U$ and $\mathcal{Y} \simeq \mathcal{Y}'/V$ for some (-1)-truncated objects $U, V \in \mathcal{Y}'$. Then

$$(\mathcal{Y}'_{/U}, \mathcal{O}_{\mathcal{Y}'}|U) \simeq (\mathcal{Y}_{/f'^*U}, \mathcal{O}_{\mathcal{Y}}|f'^*U) \qquad (\mathcal{Y}'_{/V}, \mathcal{O}_{\mathcal{Y}'}|V) \simeq (\mathcal{X}'_{/g'^*V}, \mathcal{O}_{\mathcal{X}'}|g'^*V)$$

are spectral Deligne-Mumford stacks, so that $U, V \in \mathcal{Y}'_0$. Let $h^* = f^* \circ g'^* : \mathcal{Y}' \to \mathcal{X}$. Using Lemma 5.4, we are reduced to proving the existence of an effective epimorphism $\coprod X_i \to \mathbf{1}_{\mathcal{X}}$, where each X_i belongs to the essential image $h^* \mathcal{Y}'_0 \subseteq \mathcal{X}$ and $\mathbf{1}_{\mathcal{X}}$ denotes the final object of $\mathcal{X}$.

Let $\mathcal{X}_0$ be the full subcategory of $\mathcal{X}$ spanned by those objects $X \in \mathcal{X}$ with the following properties:

- (a) There exists an object $Y \in \mathcal{Y}$ such that $f^*Y \simeq X$ and $(\mathcal{Y}_{/Y}, \mathcal{O}_{\mathcal{Y}}|Y)$ is affine.
- (b) There exists an object $X' \in \mathcal{X}'$ such that $g^*X' \simeq X$ and $(\mathcal{X}'_{/X'}, \mathcal{O}_{\mathcal{X}'}|X')$ is affine.

Using Proposition T.6.3.2.3, we see that every object $X \in \mathcal{X}_0$ has the form h^*Y' for some $Y' \in \mathcal{Y}$ such that $(\mathcal{X}'_{/g'^*Y'}, \mathcal{O}_{\mathcal{X}'}|g'^*Y')$ and $(\mathcal{Y}_{/f'^*Y'}, \mathcal{O}_{\mathcal{Y}}|f'^*Y')$ are affine. It then follows from (5) that $Y' \in \mathcal{Y}_0$, so that $X \in h^* \mathcal{Y}'_0$. It will therefore suffice to show that there exists an effective epimorphism $\coprod X_i \to \mathbf{1}_{\mathcal{X}}$, where each $X_i \in \mathcal{X}_0$.

Since $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a spectral Deligne-Mumford stack, there exists an effective epimorphism $\coprod_i X_i \to \mathbf{1}_{\mathcal{X}}$ where each $X_i \in \mathcal{X}$ satisfies condition (i). Using our assumption that $(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a spectral Deligne-Mumford stack, we can choose for each index i an effective epimorphism $\theta_i : \coprod_j X'_{i,j} \to g_* X_i$ in $\mathcal{X}'$. Set $X_{i,j} = g^* X'_j$, so that the maps θ_i induce effective epimorphisms

$$\coprod_{i,j} X_{i,j} \to \coprod_i X_i \to \mathbf{1}_{\mathcal{X}}.$$

It will therefore suffice to show that each $X_{i,j}$ satisfies conditions (a) and (b). Condition (b) is evident from the construction, and condition (a) follows from Lemma 6.6 (together with our assumption that X_i satisfies (a)). $\square$

7 Clutching of Quasi-Coherent Sheaves

Suppose we are given a diagram of spectral Deligne-Mumford stacks σ :

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ \mathfrak{X}_1 & \xrightarrow{i'} & \mathfrak{X}. \end{array}$$

Any quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ determines quasi-coherent sheaves $\mathcal{F}_0 = j'^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_0)$ and $\mathcal{F}_1 = i'^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_1)$, together with an equivalence $\alpha : i^* \mathcal{F}_0 \simeq j^* \mathcal{F}_1$. Our goal in this section is to study conditions under which $\mathcal{F}$ can be recovered from the triple $(\mathcal{F}_0, \mathcal{F}_1, \alpha)$. Our main result can be stated as follows:

Theorem 7.1. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks* σ :

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ \mathfrak{X}_1 & \xrightarrow{i'} & \mathfrak{X}, \end{array}$$

where i *and* j *are closed immersions. Then the induced diagram of* ∞*-categories*

$$\begin{array}{ccc} \mathrm{QCoh}(\mathfrak{X}_{01}) & \longleftarrow & \mathrm{QCoh}(\mathfrak{X}_0) \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(\mathfrak{X}_1) & \longleftarrow & \mathrm{QCoh}(\mathfrak{X}) \end{array}$$

induces a fully faithful embedding

$$\theta : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{X}_0) \times_{\mathrm{QCoh}(\mathfrak{X}_{01})} \mathrm{QCoh}(\mathfrak{X}_1).$$

Moreover, θ *restricts to an equivalence of* ∞*-categories*

$$\mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}} \to \mathrm{QCoh}(\mathfrak{X}_0)^{\mathrm{cn}} \times_{\mathrm{QCoh}(\mathfrak{X}_{01})^{\mathrm{cn}}} \mathrm{QCoh}(\mathfrak{X}_1)^{\mathrm{cn}}.$$

To prove Theorem 7.1, we can work locally on $\mathfrak{X}$, and thereby reduce to the case where $\mathfrak{X} = \operatorname{Spec} A$ is affine. In this case, Theorem 4.4 guarantees that $\mathfrak{X}_0$, $\mathfrak{X}_1$, and $\mathfrak{X}_{01}$ are also affine, so that the diagram σ is induced by a diagram of connective $\mathbb{E}_\infty$-rings τ :

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01}. \end{array}$$

Since σ is a pushout diagram of closed immersions, τ is a pullback diagram of $\mathbb{E}_\infty$-rings, and the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjections. We are therefore reduced to proving the following:

Theorem 7.2. *Suppose we are given a pullback diagram of* $\mathbb{E}_\infty$*-rings* τ :

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01}. \end{array}$$

Then the induced functor

$$\theta : \mathrm{Mod}_A \to \mathrm{Mod}_{A_0} \times_{\mathrm{Mod}_{A_{01}}} \mathrm{Mod}_{A_1}$$

is fully faithful. If τ is a pullback diagram of connective $\mathbb{E}_\infty$-rings and the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, then θ restricts to an equivalence of ∞-categories $\mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{Mod}_{A_0}^{\mathrm{cn}} \times_{\mathrm{Mod}_{A_{01}}^{\mathrm{cn}}} \mathrm{Mod}_{A_1}^{\mathrm{cn}}$.

Warning 7.3. The the situation of Theorem 7.2, the functor θ need not be an equivalence, even when τ is a diagram of connective $\mathbb{E}_\infty$-rings which induces surjections $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$. For example, let k be a field and let $A_0 = k[x]$, $A_{01} = k$, and $A_1 = k[y]$ (regarded as discrete $\mathbb{E}_\infty$-rings). Let M denote the A_0-module which is the direct sum of copies of $A_{01}[i]$ for i odd, and let N be the A_1-module which is the direct sum of copies of $A_{01}[i]$ for i even. Then $A_{01} \otimes_{A_0} M$ and $A_{01} \otimes_{A_1} N$ are both equivalent to the A_{01}-module $P = \bigoplus_{i \in \mathbf{Z}} A_{01}[i]$. We may therefore choose an equivalence $\alpha : A_{01} \otimes_{A_0} M \simeq A_{01} \otimes_{A_1} N$. The triple (M, N, α) can be regarded as an object of the fiber product $\mathrm{Mod}_{A_0} \times_{\mathrm{Mod}_{A_{01}}} \mathrm{Mod}_{A_1}$ which does not lie in the essential image of θ (since $M \times_P N \simeq 0$).

The first assertion of Theorem 7.2 is a consequence of the following more general claim:

Proposition 7.4. *Let k be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a k-linear ∞-category. Suppose we are given a pullback diagram τ :*

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

in Alg_k. Then the induced functor

$$F : \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_{A_0}(\mathcal{C}) \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})} \mathrm{LMod}_{A_1}(\mathcal{C})$$

is fully faithful.

Proof. Let G denote a right adjoint to F, and fix $M \in \mathrm{LMod}_A(\mathcal{C})$; we wish to show that the unit map $M \to (G \circ F)(M)$ is an equivalence. Unwinding the definitions, this is equivalent to showing that the map

$$M \to (A_0 \otimes_A M) \times_{A_{01} \otimes_A M} (A_1 \otimes_A M)$$

is an equivalence; that is, we must show that the diagram σ :

$$\begin{array}{ccc} A \otimes_A M & \longrightarrow & A_0 \otimes_A M \\ \downarrow & & \downarrow \\ A_1 \otimes_A M & \longrightarrow & A_{01} \otimes_A M \end{array}$$

is a pullback diagram in $\mathcal{C}$. Since $\mathcal{C}$ is stable, this is equivalent to showing that σ is a pushout diagram in $\mathcal{C}$. Since the relative tensor product over A preserves colimits, we need only verify that τ is a pushout diagram in RMod_A, which is equivalent to the requirement that τ be a pushout diagram of spectra. Since the ∞-category of spectra is stable, we reduce to our assumption that τ is a pullback diagram (in either Sp or Alg_k). □

To complete the proof of Theorem 7.2, we need a refinement of Proposition 7.4 which applies when the ∞-category $\mathcal{C}$ is equipped with a t-structure.

Remark 7.5. Let k be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a k-linear ∞-category equipped with an accessible t-structure. Let $\mathcal{X} \subseteq \mathrm{LMod}_k$ denote the full subcategory spanned by those k-modules M such that the operation of tensor product with M carries $\mathcal{C}_{\geq 0}$ to itself. Then $\mathcal{X}$ is closed under colimits and contains k, and therefore contains all connective k-modules. It follows that if $A \in \mathrm{Alg}_k$ is a connective k-algebra, then the ∞-category $\mathrm{LMod}_A(\mathcal{C})$ inherits an accessible t-structure from $\mathcal{C}$ (Proposition VII.6.20). Moreover, if the t-structure on $\mathcal{C}$ is left complete (excellent), then the induced t-structure on $\mathrm{LMod}_A(\mathcal{C})$ is also left complete (excellent).

Proposition 7.6. *Let k be a connective $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a k-linear ∞-category equipped with a left complete accessible t-structure. Suppose we are given a pullback diagram*

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow{\scriptstyle f} \\ A_1 & \xrightarrow{g} & A_{01} \end{array}$$

in $\mathrm{Alg}_k^{\mathrm{cn}}$, where f and g induce surjections $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$. Then the induced functor

$$F : \mathrm{LMod}_A(\mathcal{C}) \to \mathrm{LMod}_{A_0}(\mathcal{C}) \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})} \mathrm{LMod}_{A_1}(\mathcal{C})$$

determines an equivalence of ∞-categories

$$F_{\geq 0} : \mathrm{LMod}_A(\mathcal{C})_{\geq 0} \to \mathrm{LMod}_{A_0}(\mathcal{C})_{\geq 0} \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})_{\geq 0}} \mathrm{LMod}_{A_1}(\mathcal{C})_{\geq 0}.$$

Proof. Let G be a right adjoint to F. We first claim that G carries $\mathrm{LMod}_{A_0}(\mathcal{C})_{\geq 0} \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})_{\geq 0}} \mathrm{LMod}_{A_1}(\mathcal{C})_{\geq 0}$ into $\mathrm{LMod}_A(\mathcal{C})_{\geq 0}$. To this end, suppose we are given $M \in \mathrm{LMod}_{A_0}(\mathcal{C})_{\geq 0}$ and $N \in \mathrm{LMod}_{A_1}(\mathcal{C})_{\geq 0}$ and equivalences $A_{01} \otimes_{A_0} M \simeq P \simeq A_{01} \otimes_{A_1} N$ in the ∞-category $\mathrm{LMod}_{A_{01}}(\mathcal{C})_{\geq 0}$. We must show that $M \times_P N \in \mathrm{LMod}_A(\mathcal{C})_{\geq 0}$. We have a fiber sequence in $\mathcal{C}$

$$M \times_P N \to M \times N \to P.$$

Since M, N, and P belong to $\mathcal{C}_{\geq 0}$, it will suffice to show that the map $\pi_0(M \times N) \to \pi_0 P$ is an epimorphism in the abelian category $\mathcal{C}^{\heartsuit}$. In fact, we claim that $\pi_0 M \to \pi_0 P$ is an epimorphism. Choose a cofiber sequence

$$I \to A_0 \xrightarrow{f} A_{01}$$

in RMod_{A_0}. Since f induces a surjection $\pi_0 A_0 \to \pi_0 A_{01}$, we have $I \in (\mathrm{RMod}_{A_0})_{\geq 0}$. We have an induced cofiber sequence

$$I \otimes_{A_0} M \to M \to P,$$

so that the cokernel of $\pi_0 M \to \pi_0 P$ in $\mathcal{C}^{\heartsuit}$ is a subobject of $\pi_{-1}(I \otimes_{A_0} M) \simeq 0$.

It follows that G restricts to a functor

$$G_{\geq 0} : \mathrm{LMod}_{A_0}(\mathcal{C})_{\geq 0} \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})_{\geq 0}} \mathrm{LMod}_{A_1}(\mathcal{C})_{\geq 0} \to \mathrm{LMod}_A(\mathcal{C})_{\geq 0}$$

which is right adjoint to $F_{\geq 0}$. Proposition 7.4 implies that $F_{\geq 0}$ is fully faithful; it will therefore suffice to show that $G_{\geq 0}$ is conservative. Let $\alpha : X \to Y$ be a morphism in $\mathrm{LMod}_{A_0}(\mathcal{C})_{\geq 0} \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})_{\geq 0}} \mathrm{LMod}_{A_1}(\mathcal{C})_{\geq 0}$ such that $G_{\geq 0}(\alpha)$ is an equivalence. Then $G_{\geq 0}(\mathrm{cofib}(\alpha)) \simeq 0$. We will complete the proof by showing that $\mathrm{cofib}(\alpha) \simeq 0$, so that α is an equivalence.

We can identify $\mathrm{cofib}(\alpha)$ with a triple of objects $M \in \mathrm{LMod}_{A_0}(\mathcal{C})_{\geq 0}$, $N \in \mathrm{LMod}_{A_1}(\mathcal{C})_{\geq 0}$, and $P \simeq A_{01} \otimes_A M \simeq A_{01} \otimes_{A_1} N \in \mathrm{LMod}_{A_{01}}(\mathcal{C})_{\geq 0}$ as above. We will prove by induction on n that $M \in \mathrm{LMod}_A(\mathcal{C})_{\geq n}$ and $N \in \mathrm{LMod}_{B'}(\mathcal{C})_{\geq n}$. Provided that this is true for all n, the assumption that $\mathcal{C}$ is left complete will imply that $M \simeq N \simeq 0$ so that $\mathrm{cofib}(\alpha) \simeq 0$ as desired.

In the case $n = 0$, there is nothing to prove. Assume therefore that $M \in \mathrm{LMod}_A(\mathcal{C})_{\geq n}$ and $N \in \mathrm{LMod}_A(\mathcal{C})_{\geq n}$. Since $G(\mathrm{cofib}(\alpha)) = M \times_P N \simeq 0$, we have an isomorphism $\pi_n M \oplus \pi_n N \to \pi_n P$ in the abelian category $\mathcal{C}^{\heartsuit}$. The first part of the proof shows that the map $\pi_n M \to \pi_n P$ is an epimorphism, so that $\pi_n N \simeq 0$. A similar argument shows that $\pi_n M \simeq 0$, so that $M \in \mathrm{LMod}_{A_0}(\mathcal{C})_{\geq n+1})$ and $N \in \mathrm{LMod}_{A_1}(\mathcal{C})_{\geq n+1}$ as desired. □

Proposition 7.7. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks σ :*

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ \mathfrak{X}_1 & \xrightarrow{i'} & \mathfrak{X}, \end{array}$$

where i and j are closed immersions. Let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, let $\mathcal{F}_0 = j'^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_0)$, and let $\mathcal{F}_1 = i'^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_1)$. Then:

(1) *The quasi-coherent sheaf $\mathcal{F}$ is n-connective if and only if $\mathcal{F}_0$ and $\mathcal{F}_1$ are n-connective.*

(2) *The quasi-coherent sheaf $\mathcal{F}$ is almost connective if and only if $\mathcal{F}_0$ and $\mathcal{F}_1$ are almost connective.*

(3) *The quasi-coherent sheaf $\mathcal{F}$ has* Tor*-amplitude $\leq n$ if and only if $\mathcal{F}_0$ and $\mathcal{F}_1$ have* Tor*-amplitude $\leq n$.*

(4) *The quasi-coherent sheaf $\mathcal{F}$ is flat if and only if $\mathcal{F}_0$ and $\mathcal{F}_1$ are flat.*

(5) *The quasi-coherent sheaf is perfect to order n if and only if $\mathcal{F}_0$ and $\mathcal{F}_1$ are perfect to order n.*

(6) *The quasi-coherent sheaf $\mathcal{F}$ is almost perfect if and only if $\mathcal{F}_0$ and $\mathcal{F}_1$ are almost perfect.*

(7) *The quasi-coherent sheaf $\mathcal{F}$ is perfect if and only if $\mathcal{F}_0$ and $\mathcal{F}_1$ are perfect.*

(8) *The quasi-coherent sheaf $\mathcal{F}$ is locally free of finite rank if and only if $\mathcal{F}_0$ and $\mathcal{F}_1$ are locally free of finite rank.*

Proof. The "only if" directions follow immediately from Propositions VIII.2.7.20 and VIII.2.7.31. To prove the reverse directions, we may work locally on $\mathfrak{X}$ and thereby reduce to the case where $\mathfrak{X} = \operatorname{Spec} A$ is affine. Then σ is determined by a pullback square of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

for which the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective. The quasi-coherent sheaf $\mathcal{F}$ corresponds to an A-module M. Set

$$M_0 = A_0 \otimes_A M \qquad M_{01} = A_{01} \otimes_A M \qquad M_1 = A_1 \otimes_A M.$$

To prove (1), we may assume without loss of generality that $n = 0$. The desired result then follows from the first step in the proof of Proposition 7.6. Assertion (2) is a consequence of (1). Assertion (4) follows from (1) and (3), assertion (6) follows from (5), assertion (7) follows from (3), (6), and Proposition A.7.2.5.23, and assertion (8) follows from (4), (6), and Proposition A.7.2.5.20.It will therefore suffice to prove (3) and (5).

We now prove (3). Assume that M_0 and M_1 have Tor-amplitude $\leq n$ over A_0 and A_1, respectively. We wish to show that M has Tor-amplitude $\leq n$ over A. Let N be a discrete A-module; we wish to show that $M \otimes_A N$ is n-truncated. Let $I_0 \subseteq \pi_0 A$ be the kernel of the map $\pi_0 A \to \pi_0 A_0$, and define $I_1 \subseteq \pi_0 A$ similarly. We have an exact sequence of discrete A-modules

$$0 \to I_0 N \to N \to N/I_0 N \to 0$$

Note that $N/I_0 N$ admits the structure of an A_0-module, so that $M \otimes_A (N/I_0 N) \simeq M_0 \otimes_{A_0} (N/I_0 N)$ is n-truncated. We are therefore reduced to proving that $M \otimes_A I_0 N$ is n-truncated. We have a short exact sequence

$$0 \to I_0 I_1 N \to I_0 N \to I_0 N / I_0 I_1 N \to 0.$$

Since the quotient $I_0 N / I_0 I_1 N$ admits the structure of an A_1-module, we deduce that $M \otimes_A (I_0 N / I_0 I_1 N) \simeq M_1 \otimes_{A_1} (I_0 N / I_0 I_1 N)$ is n-truncated. We are therefore reduced to proving that $M \otimes_A I_0 I_1 N$ is n-truncated. Note that the ideal $I_0 I_1$ belongs to the kernel of the map $\pi_0 A \to \pi_0 A_0 \oplus \pi_0 A_1$, and therefore to the image of the map $\pi_1 A_{01} \to \pi_0 A$. It follows that the ideal $I_0 I_1$ has the structure of a module over $\pi_0 A_{01}$, and is therefore annihilated by I_0. We conclude that $I_0 I_1 N$ admits the structure of an A_0-module, so that $M \otimes_A I_0 I_1 N \simeq M_0 \otimes_{A_0} I_0 I_1 N$ is n-truncated as desired.

It remains to prove (5). Assume that M_0 and M_1 are perfect to order n; we wish to show that M is perfect to order n. Using (2), we deduce that M is almost connective. Replacing M by a shift, we may assume that M is connective. We now proceed by induction on n, the case $n < 0$ being trivial. If $n = 0$, we must show that $\pi_0 M$ is finitely generated as a module over $\pi_0 A$ (Proposition VIII.2.6.12). Since M_0 and M_1 are perfect to order 0 and the maps $\pi_0 M \to \pi_0 M_0$ and $\pi_0 M \to \pi_0 M_1$, we can choose finitely many elements of $\pi_0 M$ whose images generate $\pi_0 M_0$ and $\pi_0 M_1$. This choice of elements determines a fiber sequence

$$M' \to A^m \to M.$$

Then $A_0 \otimes_A M'$ and $A_1 \otimes_A M'$ are connective, so that (1) implies that M' is connective. It follows that the map $\pi_0 A^m \to \pi_0 M$ is surjective, so that M is perfect to order 0 as desired.

Now suppose that $n > 0$. The argument above shows that $\pi_0 M$ is finitely generated, so we can choose a fiber sequence

$$M' \to A^m \to M$$

where M' is connective. Using Proposition VIII.2.6.12, we deduce that $A_0 \otimes_A M'$ and $A_1 \otimes_A M'$ are perfect to order $n-1$ as modules over A_0 and A_1, respectively. Invoking the inductive hypothesis, we conclude that M' is perfect to order $n-1$, so that M is perfect to order n by Proposition VIII.2.6.12. □

For later use, we record the following consequence of Proposition 7.6.

Proposition 7.8. *Let*

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow{\scriptstyle f} \\ A_1 & \xrightarrow{g} & A_{01} \end{array}$$

be a pullback diagram of connective $\mathbb{E}_2$*-rings, and suppose that* f *and* g *induce surjections* $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$*. Then the induced functor*

$$F : \mathrm{LMod}_{\mathrm{LMod}_A}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}) \to \mathrm{LMod}_{\mathrm{LMod}_{A_0}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}) \times_{\mathrm{LMod}_{\mathrm{LMod}_{A_{01}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm})} \mathrm{LMod}_{\mathrm{LMod}_{A_1}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm})$$

is fully faithful.

Proof. We note that F is the restriction of functors

$$F' : \mathrm{LMod}_{\mathrm{LMod}_A}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}) \to \mathrm{LMod}_{\mathrm{LMod}_{A_0}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}) \times_{\mathrm{LMod}_{\mathrm{LMod}_{A_{01}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+})} \mathrm{LMod}_{\mathrm{LMod}_{A_1}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+})$$

$$F'' : \mathrm{LMod}_{\mathrm{LMod}_A}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{LMod}_{\mathrm{LMod}_{A_0}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \times_{\mathrm{LMod}_{\mathrm{LMod}_{A_{01}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \mathrm{LMod}_{\mathrm{LMod}_{A_1}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}).$$

Let G, G', and G'' denote right adjoints to F, F', and F''. Lemmas VIII.4.6.13 and VIII.4.6.14 imply that

$$G \simeq G' | \mathrm{LMod}_{\mathrm{LMod}_{A_0}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}) \times_{\mathrm{LMod}_{\mathrm{LMod}_{A_{01}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm})} \mathrm{LMod}_{\mathrm{LMod}_{A_1}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}).$$

Proposition VIII.4.6.11 and Remark VIII.4.6.10 imply G' is given by composing G'' with the functor

$$U : \mathrm{LMod}_{\mathrm{LMod}_A}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{LMod}_{\mathrm{LMod}_A}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+})$$

induced by right completion.

Fix an object $\mathcal{C} \in \mathrm{LMod}_{\mathrm{LMod}_A}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm})$. We wish to show that the unit map $\mathcal{C} \to (G \circ F)(\mathcal{C})$ is an equivalence in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}$. Unwinding the definitions, we are required to show that the induced map

$$\theta : \mathcal{C} \to \mathrm{LMod}_{A_0}(\mathcal{C}) \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})} \mathrm{LMod}_{A_1}(\mathcal{C})$$

exhibits $\mathcal{C}$ as a right completion of $\mathrm{LMod}_{A_0}(\mathcal{C}) \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})} \mathrm{LMod}_{A_1}(\mathcal{C})$. This is equivalent to the assertion that θ induces an equivalence

$$\mathcal{C}_{\geq 0} \to \mathrm{LMod}_{A_0}(\mathcal{C})_{\geq 0} \times_{\mathrm{LMod}_{A_{01}}(\mathcal{C})_{\geq 0}} \mathrm{LMod}_{A_1}(\mathcal{C})_{\geq 0},$$

which follows from Proposition 7.6. □

8 Digression: Local Finite Presentation

Let R be a commutative ring and let X be an R-scheme. Recall that X is said to be *locally of finite type* over R if, for every affine open subscheme $\operatorname{Spec} A \subseteq X$, the commutative ring A is finitely generated as an R-algebra. The scheme X is said to be *locally of finite presentation* over R if, for every open affine subscheme $\operatorname{Spec} A \subseteq X$, the commutative ring A is finitely presented as an R-algebra. Our goal in this section is to study the analogous finiteness conditions in the setting of spectral algebraic geometry. We begin by considering the affine case.

Definition 8.1. Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings and let $n \geq 0$. We will say that f is *of finite presentation to order* n if the following conditions are satisfied:

(1) For every filtered diagram $\{C_\alpha\}$ of n-truncated connective $\mathbb{E}_\infty$-algebras over A having colimit C, the canonical map
$$\varinjlim_\alpha \operatorname{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \operatorname{Map}_{\mathrm{CAlg}_A}(B, C)$$
has (-1)-truncated homotopy fibers.

(2) If $n > 0$, then for every filtered diagram $\{C_\alpha\}$ of $(n-1)$-truncated connective $\mathbb{E}_\infty$-algebras over A having colimit C, the canonical map
$$\varinjlim_\alpha \operatorname{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \operatorname{Map}_{\mathrm{CAlg}_A}(B, C)$$
is a homotopy equivalence.

Remark 8.2. In the situation of Definition 8.1, the map f is almost of finite presentation if and only if it is of finite presentation to order n, for each $n \geq 0$.

Remark 8.3. If $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings which is of finite presentation to order $n > 0$, then f exhibits $\tau_{\leq n-1} B$ as a compact object of $\tau_{\leq n-1} \mathrm{CAlg}^{\mathrm{cn}}_A$. In particular, if f is of finite presentation to order $n > 0$, then $\pi_0 B$ is finitely presented as a $\pi_0 A$-algebra (in the sense of classical commutative algebra).

Example 8.4. Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Then f is of finite presentation to order 0 if and only if, for every filtered diagram of discrete commutative rings $\{C_\alpha\}$ having colimit C, the canonical map
$$\varinjlim_\alpha \operatorname{Hom}(\pi_0 B, C_\alpha) \to \operatorname{Hom}(\pi_0 B, C)$$
is injective (here the Hom-sets are computed in the category of commutative $\pi_0 A$-algebras). This is equivalent to requirement that $\pi_0 B$ is finitely generated as an algebra over $\pi_0 A$.

Remark 8.5. Suppose we are given a commutative diagram of connective $\mathbb{E}_1$-rings
$$\begin{array}{ccccc} & & B & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ A & & \xrightarrow{f'} & & B'. \end{array}$$
Let $n \geq 0$, and assume that the map g induces a surjection $\pi_n B \to \pi_n B'$ and a bijection $\pi_i B \to \pi_i B'$ for $i < n$. Let C be a connective $\mathbb{E}_\infty$-algebra over A. If C is $(n-1)$-truncated, then the map $\operatorname{Map}_{\mathrm{CAlg}_A}(B', C) \to \operatorname{Map}_{\mathrm{CAlg}_A}(B, C)$ is a homotopy equivalence. If C is n-truncated, then $\operatorname{Map}_{\mathrm{CAlg}_A}(B', C) \to \operatorname{Map}_{\mathrm{CAlg}_A}(B, C)$ has (-1)-truncated homotopy fibers. It follows that if f is of finite presentation to order n, then so is f'.

Remark 8.6. Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings which is of finite presentation to order $n > 0$. Then the induced map $A \to \tau_{\leq n-1} B$ is also of finite presentation to order n (this is a special case of Remark 8.5).

Remark 8.7. Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings which exhibits B as a compact object of $\tau_{\leq n-1}\,\mathrm{CAlg}_A^{\mathrm{cn}}$ (for some $n > 0$). Then B is a retract of $\tau_{\leq n-1} R$ for some compact object $B' \in \mathrm{CAlg}_A^{\mathrm{cn}}$. Since the map $A \to B'$ is of finite presentation to order n, so is the map $A \to B$ (by Remark 8.6). It follows that an object $B \in \tau_{\leq n-1}\,\mathrm{CAlg}_A^{\mathrm{cn}}$ is compact if and only if the map $A \to B$ is of finite presentation to order n.

The following result can be viewed as a refinement of Theorem A.7.4.3.18.

Proposition 8.8. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings and let $n > 0$. Then f is of finite presentation to order n if and only if the following conditions are satisfied:*

(1) *The commutative ring $\pi_0 B$ is of finite presentation over $\pi_0 A$.*

(2) *The relative cotangent complex $L_{B/A}$ is perfect to order n as an A-module.*

Proof. First suppose that f is of finite presentation to order n. Since $n > 0$, condition (1) follows from Remark 8.3. To prove (2), choose a filtered diagram $\{N_\alpha\}$ in $(\mathrm{Mod}_B)_{\leq 0}$ having colimit N. We wish to show that the canonical map

$$\varinjlim_\alpha \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, N_\alpha[n]) \to \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, N[n])$$

has (-1)-truncated fibers. Replacing $\{N_\alpha\}$ by $\{\tau_{\geq -n} N_\alpha\}$ if necessary, we may assume that each $N_\alpha[n]$ is connective. Using the definition of $L_{B/A}$, we are reduced to proving that the map

$$\varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, B \oplus N_\alpha[n]) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, B \oplus N[n])$$

has (-1)-truncated fibers, which follows immediately from the assumption that f is of finite presentation to order n.

Now suppose that conditions (1) and (2) are satisfied and choose a filtered diagram $\{C_\alpha\}$ in $\mathrm{CAlg}_A^{\mathrm{cn}}$ having colimit C For $0 \leq i \leq n$, let θ_i denote the canonical map

$$\varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, C).$$

We will prove that θ_i is a homotopy equivalence for $0 \leq i < n$ and (-1)-truncated when $i = n$. The proof proceeds by induction on i. When $i = 0$, the desired result follows from (1). Assume that $0 < i \leq n$ and that θ_i is a homotopy equivalence. Using the results of §A.7.4.1, we obtain a map of fiber sequences

$$\begin{array}{ccccc}
\varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leq i} C_\alpha) & \longrightarrow & \varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leq i-1} C_\alpha) & \longrightarrow & \varinjlim_\alpha \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, (\pi_i C_\alpha)[i+1]) \\
\downarrow{\scriptstyle \theta_i} & & \downarrow{\scriptstyle \theta_{i-1}} & & \downarrow{\scriptstyle \phi} \\
\mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leq i} C) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}_A}(B, \tau_{\leq i-1} C) & \longrightarrow & \mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, (\pi_i C)[i+1]).
\end{array}$$

Using assumption (2) we deduce that ϕ is an equivalence if $i \leq n-2$ has (-1)-truncated fibers if $i = n-1$, and has 0-truncated fibers if $i = n$. It follows that θ_i is an equivalence if $i \leq n-1$ and has (-1)-truncated fibers if $i = n$. □

Remark 8.9. The proof of Proposition 8.8 yields the following slightly stronger assertion: if $f : A \to B$ is a map satisfying conditions (1) and (2) and $\{C_\alpha\}$ is a filtered diagram of m-truncated objects of $\mathrm{CAlg}_A^{\mathrm{cn}}$, then the canonical map

$$\varinjlim_\alpha \mathrm{Map}_{\mathrm{CAlg}_A}(B, C_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_A}(B, \varinjlim_\alpha C_\alpha)$$

has $(m - n - 1)$-truncated homotopy fibers.

Proposition 8.10. *Suppose we are given maps $f : A \to B$ and $g : B \to C$ of connective $\mathbb{E}_\infty$-rings. Assume that f is of finite presentation to order n. Then g is of finite presentation to order n if and only if $g \circ f$ is of finite presentation to order n.*

Proof. If $n = 0$, then the desired result follows immediately from Example 8.4. Let us therefore assume that $n > 0$. Then $\pi_0 B$ is finitely presented as an algebra over $\pi_0 A$. It follows that $\pi_0 C$ is finitely presented over $\pi_0 B$ if and only if it is finitely presented over $\pi_0 A$. Using Proposition 8.8, we are reduced to proving that $L_{C/A}$ is perfect to order n if and only if $L_{C/B}$ is perfect to order n. This follows by applying Remark VIII.2.6.8 to the fiber sequence

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B},$$

since $C \otimes_B L_{B/A}$ is perfect to order n by Proposition 8.8 and Proposition VIII.2.6.13. □

Example 8.11. If $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings which exhibits B as a compact object of CAlg_A, then f is of finite presentation to order n. In particular, any étale map of of finite presentation to order n (Corollary A.7.5.4.4).

Proposition 8.12. *Suppose we are given a pushout diagram of connective $\mathbb{E}_\infty$-rings σ :*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B'. \end{array}$$

If f is of finite presentation to order n, then f' is of finite presentation to order n.

Proof. The diagram σ determines an isomorphism of commutative rings $\pi_0 B' \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 A')$ (Proposition A.7.2.2.13). It follows that if $\pi_0 B$ is finitely generated (finitely presented) as a commutative ring over $\pi_0 A$, then $\pi_0 B'$ is finitely generated (finitely presented) as a commutative ring over $\pi_0 A'$. This completes the proof when $n = 0$ (Example 8.4). If $n > 0$, we must also show that $L_{B'/A'} \simeq B' \otimes_B L_{B/A}$ is perfect to order n as a B'-module (Proposition 8.8), which follows from Proposition VIII.2.6.13 (since $L_{B/A}$ is perfect to order n as a B-module, by Proposition 8.8). □

Corollary 8.13. *Let R be a connective $\mathbb{E}_\infty$-ring, and suppose we are given a pushout diagram of connective $\mathbb{E}_\infty$-algebras over R:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B'. \end{array}$$

If A, B, and A' are of finite presentation to order n over R, then B' is of finite presentation to order n over R.

Proof. Since A' is of finite presentation to order n over R, it will suffice to show that f' is of finite presentation to order n (Proposition 8.10). Using Proposition 8.12, we are reduced to proving that f is of finite presentation to order n. This follows from Proposition 8.10, since A and B are both of finite presentation to order n over R. □

Proposition 8.14. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Then:*

(1) *For every étale map $B \to B'$, if f is of finite presentation to order n, then so is the composite map $A \to B \to B'$.*

(2) *Suppose we are given a finite collection of étale maps $B \to B_\alpha$ such that the composition $B \to \prod_\alpha B_\alpha$ is faithfully flat. If each of the composite maps $A \to B \to B_\alpha$ is of finite presentation to order n, then f is of finite presentation to order n.*

Proof. Assertion (1) is immediate from Proposition 8.10 and Example 8.11. Let us prove (2). We first treat the case $n = 0$. Let $B' = \prod_\alpha B_\alpha$. Since each $\pi_0 B_\alpha$ is finitely generated as an algebra over $\pi_0 A$, we deduce that $\pi_0 B'$ is finitely generated over $\pi_0 A$. The map $\pi_0 B \to \pi_0 B'$ is étale . Using the structure theory of étale morphisms, we can choose a subalgebra $R \subseteq \pi_0 B$ which is finitely generated over $\pi_0 A$ and a faithfully flat étale R-algebra R' fitting into a pushout diagram

$$\begin{array}{ccc} \pi_0 B & \longrightarrow & \pi_0 B' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R'. \end{array}$$

Since $\pi_0 B'$ is finitely generated, we may assume (after enlarging R if necessary) that the map $R' \to \pi_0 B'$ is surjective. Then $R' \otimes_R (\pi_0 B/R) \simeq 0$, so (by faithful flatness) we deduce that $\pi_0 B = R$ is finitely generated as a $\pi_0 A$-algebra.

Assume now that $n > 0$. We will show that the map $A \to B$ satisfies conditions (1) and (2) of Proposition 8.8. We first show that $\pi_0 B$ is finitely presented as a $\pi_0 A$-algebra. We have already seen that $\pi_0 B$ is finitely generated as a $\pi_0 A$-algebra, so we can choose an isomorphism $\pi_0 B \simeq R/I$ where R is a polynomial ring over $\pi_0 A$. Write I as a filtered colimit of finitely generated ideals $I_\alpha \subseteq I$. Using the structure theory of étale morphisms, we can choose a finitely generated ideal $J \subseteq I$ and a faithfully flat étale map $R/J \to S$ such that $\pi_0 B' \simeq S \otimes_{R/J} R/I$. Then $\pi_0 B'$ is the quotient of S by the ideal IS. Since $\pi_0 B'$ is finitely presented as an $\pi_0 A$-algebra, we deduce that $\pi_0 B' \simeq S/I_0 S$ for some finitely generated ideal $I_0 \subseteq I$ containing J. Since S is faithfully flat over R/J, the map $I/I_0 \to IS/I_0 S \simeq 0$ is injective. Thus $I = I_0$ is a finitely generated ideal and $\pi_0 B$ is finitely presented over $\pi_0 A$ as desired.

It remains to verify condition (2) of Proposition 8.8. We have a fiber sequence

$$B' \otimes_B L_{B/A} \to L_{B'/A} \to L_{B'/B}.$$

Since B' is étale over B, the relative cotangent complex $L_{B'/B}$ vanishes. It follows that $L_{B'/A} \simeq B' \otimes_B L_{B/A}$. Proposition 8.8 implies that $L_{B'/A}$ is perfect to order n as a B'-module, so that $L_{B/A}$ is perfect to order n as a B-module by Proposition VIII.2.6.13. □

Remark 8.15. In the situation of part (2) of Proposition 8.14, suppose that each B_α is locally of finite presentation over A. Then B is locally of finite presentation over A. To prove this, we note that it follows from Proposition 8.14 that B is almost of finite presentation over A. It will therefore suffice to show that $L_{B/A}$ is perfect as a module over B. Using Proposition VIII.2.6.15, we are reduced to proving that each $B_\alpha \otimes_B L_{B/A}$ is perfect as a module over B_α. This is clear, since the vanishing of $L_{B_\alpha/B}$ implies that the canonical map $B_\alpha \otimes_B L_{B/A} \to L_{B_\alpha/A}$ is an equivalence.

Definition 8.16. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is *locally of finite presentation to order n* (*locally almost of finite presentation*, *locally of finite presentation*) if the following condition is satisfied: for every commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spec} R & \longrightarrow & \mathfrak{Y} \end{array}$$

where the horizontal maps are étale, the $\mathbb{E}_\infty$-ring A is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R.

Example 8.17. A map of spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \mathfrak{Y}$ is locally of finite presentation to order 0 if and only if the underlying map of ordinary Deligne-Mumford stacks is locally of finite type (in the sense of classical algebraic geometry).

If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of spectral Deligne-Mumford stacks which is of finite presentation to order 1, then the underlying map of ordinary Deligne-Mumford stacks is locally of finite presentation (in the sense of classical algebraic geometry). The converse holds if the structure sheaf of $\mathfrak{X}$ is 0-truncated.

Proposition 8.18. *The condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation) is local on the source with respect to the étale topology (see Definition VIII.1.5.7).*

Proof. It is clear that if f is locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentatiion) and $g : \mathfrak{U} \to \mathfrak{X}$ is étale, $f \circ g$ is locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation). To complete the proof, let us suppose that $g : \mathfrak{Y} \to \mathfrak{Z}$ is arbitrary and that we are given a jointly surjective collection of étale maps $\{f_\alpha : \mathfrak{X}_\alpha \to \mathfrak{Y}\}$ such that each composition $g \circ f_\alpha$ is locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation). We wish to show that g has the same property. Choose a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A & \longrightarrow & \mathfrak{Y} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \mathfrak{Z} \end{array}$$

where the horizontal maps are étale. We wish to show that A is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R. Since the maps f_α are jointly surjective, we can choose an étale covering $\{A \to A_i\}$ such that each of the composite maps $\operatorname{Spec} A_i \to \operatorname{Spec} A \to \mathfrak{Y}$ factors through some $\mathfrak{X}_\alpha$. Using our assumption on $g \circ f_\alpha$, we deduce that each A_i is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R. The desired result now follows from Proposition 8.14 and Remark 8.15. □

Remark 8.19. Let $f : \mathfrak{X} \to \operatorname{Spec} R$ be a map of spectral Deligne-Mumford stacks. Then f is locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation) if and only if the following condition is satisfied:

(∗) For every étale map $\operatorname{Spec} A \to \mathfrak{X}$, the $\mathbb{E}_\infty$-ring A is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R.

The "only if" direction is obvious. Conversely, suppose that (∗) is satisfied and we are given a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \end{array}$$

where the horizontal maps are étale. Using condition (∗), we deduce that A is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R. It follows from Theorem VIII.1.2.1 that R' is étale over R, and therefore locally of finite presentation over R. The desired result now follows from Proposition 8.10 and Remark A.7.2.5.29.

Remark 8.20. Let $f : \operatorname{Spec} A \to \operatorname{Spec} R$ be a map of affine spectral Deligne-Mumford stacks. Then f is locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation) if and only if A is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R. The "only if" direction follows immediately from the definitions. To prove the converse, it suffices to observe that if A is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R and A' is an étale A-algebra, then A' is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R.

Remark 8.21. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Then f is locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation) if and only if, for every étale map $\operatorname{Spec} R \to \mathfrak{Y}$, the induced map $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec} R \to \operatorname{Spec} R$ is locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation).

Remark 8.22. Suppose we are given a finite collection $\{f_\alpha : A_\alpha \to B_\alpha\}$ of morphisms between connective $\mathbb{E}_\infty$-rings. Let $f : \prod_\alpha A_\alpha \to \prod_\alpha B_\alpha$ be the induced map. Then f is of finite presentation to order n (almost of finite presentation, locally of finite presentation) if and only if each of the morphisms f_α is of finite presentation to order n (almost of finite presentation, locally of finite presentation).

Proposition 8.23. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. Suppose that there exists a finite collection of flat morphisms $A \to A_\alpha$ with the following properties:*

1. *The map $A \to \prod_\alpha A_\alpha$ is faithfully flat.*
2. *Each of the maps $f_\alpha : A_\alpha \to A_\alpha \otimes_A B$ is of finite presentation to order n (almost of finite presentation, locally of finite presentation).*

Then f is of finite presentation to order n (almost of finite presentation, locally of finite presentation).

Proof. Let $A' = \prod_\alpha A_\alpha$, let $B' = A' \otimes_A B$, and let $f' : A' \to B'$ be the induced map. Let us first suppose that each f_α is of finite presentation to order 0. Then f' is of finite presentation to order 0 (Remark 8.22), so that $\pi_0 B' = \pi_0 A' \otimes_{\pi_0 A} \pi_0 B$ is finitely generated as a commutative algebra over $\pi_0 A'$. We may therefore choose a finite collection of elements $x_1, \ldots, x_n \in \pi_0 B$ which generate $\pi_0 B'$ as an algebra over $\pi_0 A'$. Let R denote the polynomial ring $(\pi_0 A)[x_1, \ldots, x_n]$, so we have a map of commutative rings $\phi : R \to \pi_0 B$ which induces a surjection $\pi_0 A' \otimes_{\pi_0 A} R \to \pi_0 A' \otimes_{\pi_0 A} \pi_0 B$. Since $\pi_0 A'$ is faithfully flat over $\pi_0 A$, we deduce that ϕ is surjective, so that $\pi_0 B$ is finitely generated as a commutative ring over $\pi_0 A$ and therefore f is of finite presentation to order 0 (Example 8.4).

Now suppose that each f_α is of finite presentation to order 1. Using Remark 8.22 and Proposition 8.8, we deduce that $\pi_0 B'$ is finitely presented as an algebra over $\pi_0 A'$. Define $\phi : R \to \pi_0 B$ be the surjection defined above. Let $I = \ker(\phi)$. Since A' is flat over A, we can identify $I \otimes_{\pi_0 A} \pi_0 A'$ with the kernel of the surjection $\pi_0 A' \otimes_{\pi_0 A} R \to \pi_0 B'$. The assumption that $\pi_0 B'$ is of finite presentation over $\pi_0 A'$ implies that this kernel is finitely generated. We may therefore choose a finitely generated submodule $J \subseteq I$ such that the induced map $J \otimes_{\pi_0 A} \pi_0 A' \to I \otimes_{\pi_0 A} \pi_0 A'$ is surjective. Since A' is faithfully flat over A, we deduce that $I = J$ is finitely generated, so that $\pi_0 B \simeq R/I$ is finitely presented as a commutative ring over $\pi_0 A$. Suppose that we wish to prove that f is of finite presentation to order $n \geq 1$ (almost of finite presentation, locally of finite presentation). Using Proposition 8.8 and Theorem A.7.4.3.18, we are reduced to proving that the relative cotangent complex $L_{B/A}$ is perfect to order n (almost perfect, perfect) as a module over B. Using Remark 8.22 we deduce that $B' \otimes_B L_{B/A} \simeq L_{B'/A'}$ is perfect to order n (almost perfect, perfect) as a module over B'. Since B' is faithfully flat over B, the desired result follows from Proposition VIII.2.6.15. $\square$

Proposition 8.24. *The condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation) is local on the target with respect to the fpqc topology. That is, if there exists a flat covering $\{\coprod \mathfrak{Y}_\alpha \to \mathfrak{Y}\}$ such that each of the induced maps $f_\alpha : \mathfrak{Y}_\alpha \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{Y}_\alpha$ is locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation), then f has the same property.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R & \longrightarrow & \mathfrak{Y} \end{array}$$

where the horizontal maps are étale; we wish to show that A is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R. Since the map $\coprod \mathfrak{Y}_\alpha \to \mathfrak{Y}$ is a flat covering, there exists a faithfully flat map $R \to \prod_{1 \leq i \leq n} R_i$ such that each of the maps $\operatorname{Spec} R_i \to \operatorname{Spec} R \to \mathfrak{Y}$ factors through an étale map $\operatorname{Spec} R_i \to \mathfrak{Y}_\alpha$. Using our hypothesis on $\mathfrak{Y}_\alpha$, we deduce that $R_i \otimes_R A$ is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R_i. We now conclude the proof by invoking Proposition 8.23. ☐

Proposition 8.25. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}' & \longrightarrow & \mathfrak{Y} \end{array}$$

of spectral Deligne-Mumford stacks. If f is locally of finite presentation to order n (locally almost of finite presentation, locally of finite presentation), then so is f'.

Proof. Proposition 8.24 implies that the assertion is local on $\mathfrak{Y}'$; we may therefore assume without loss of generality that $\mathfrak{Y}' = \operatorname{Spec} R'$ is affine and that the map $\mathfrak{Y}' \to \mathfrak{Y}$ factors as a composition $\operatorname{Spec} R' \to \operatorname{Spec} R \xrightarrow{u} \mathfrak{Y}$, where u is étale. Replacing $\mathfrak{Y}$ by $\operatorname{Spec} R$, we may assume that $\mathfrak{Y}$ is also affine. Proposition 8.18 implies that the assertion is local on $\mathfrak{X}'$ and therefore local on $\mathfrak{X}$; we may therefore suppose that $\mathfrak{X} = \operatorname{Spec} A$ is affine. Then $\mathfrak{X}' = \operatorname{Spec} A'$, where $A' = R' \otimes_R A$. Using Remark 8.20, we are reduced to proving that A' is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R'. This follows from Proposition 8.12, since A is of finite presentation to order n (almost of finite presentation, locally of finite presentation) over R. ☐

9 Clutching of Spectral Deligne-Mumford Stacks

Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}, \end{array}$$

where i and j are closed immersions. In §7, we proved that a quasi-coherent sheaf on $\mathfrak{X}$ is determined by its restriction to $\mathfrak{X}_0$ and $\mathfrak{X}_1$ (Theorem 7.1). In this section, we will prove a "nonlinear" version of this result: rather than considering quasi-coherent sheaves on $\mathfrak{X}$, we consider morphisms of spectral Deligne-Mumford stacks $\mathfrak{Y} \to \mathfrak{X}$. Our main result can be stated as follows:

Theorem 9.1. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}, \end{array}$$

where i and j are closed immersions. Then the induced diagram

$$\begin{array}{ccc} \mathrm{Stk}_{/\mathfrak{X}} & \longrightarrow & \mathrm{Stk}_{/\mathfrak{X}_1} \\ \downarrow & & \downarrow \\ \mathrm{Stk}_{/\mathfrak{X}_0} & \longrightarrow & \mathrm{Stk}_{/\mathfrak{X}_{01}} \end{array}$$

is a pullback square of ∞-categories.

The proof of Theorem 9.1 relies on the following observation:

Proposition 9.2. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks* σ :

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}, \end{array}$$

where i *and* j *are closed immersions. Let* $f : \mathfrak{Y} \to \mathfrak{X}$ *be a map of spectral Deligne-Mumford stacks. Then the diagram* σ' :

$$\begin{array}{ccc} \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}_{01} & \xrightarrow{i'} & \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}_0 \\ \downarrow{\scriptstyle j'} & & \downarrow \\ \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}_1 & \longrightarrow & \mathfrak{Y} \end{array}$$

is also a pushout square of spectral Deligne-Mumford stacks (note that the morphisms i' *and* j' *are also closed immersions).*

Proof. The assertion is local on both $\mathfrak{X}$ and $\mathfrak{Y}$. We may therefore assume without loss of generality that $\mathfrak{X} = \operatorname{Spec} A$, so that σ is determined by a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & A_{01} \end{array}$$

where the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective. We may also assume that $\mathfrak{Y} = \operatorname{Spec} R$ is affine. In this case, σ' is determined by the diagram of connective $\mathbb{E}_\infty$-rings τ :

$$\begin{array}{ccc} R & \longrightarrow & R \otimes_A A_1 \\ \downarrow & & \downarrow \\ R \otimes_A A_0 & \longrightarrow & R \otimes_A A_{01}. \end{array}$$

Since the operation of tensor product with R is exact, τ is a pullback diagram. The desired result now follows from Corollary 6.5. $\square$

Proof of Theorem 9.1. We wish to prove that the canonical map

$$G : \mathrm{Stk}_{/\mathfrak{X}} \to \mathrm{Stk}_{/\mathfrak{X}_0} \times_{\mathrm{Stk}_{/\mathfrak{X}_{01}}} \mathrm{Stk}_{/\mathfrak{X}_1}$$

is an equivalence of ∞-categories. We can identify objects of the fiber product $\mathrm{Stk}_{/\mathfrak{X}_0} \times_{\mathrm{Stk}_{/\mathfrak{X}_{01}}} \mathrm{Stk}_{/\mathfrak{X}_1}$ with triples $(\mathfrak{Y}_0, \mathfrak{Y}_1, \alpha)$, where $\mathfrak{Y}_0 \in \mathrm{Stk}_{/\mathfrak{X}_0}$, $\mathfrak{Y}_1 \in \mathrm{Stk}_{/\mathfrak{X}_1}$, and $\alpha : \mathfrak{X}_{01} \times_{\mathfrak{X}_0} \mathfrak{Y}_0 \simeq \mathfrak{X}_{01} \times_{\mathfrak{X}_1} \mathfrak{Y}_1$ is an equivalence in $\mathrm{Stk}_{/\mathfrak{X}_{01}}$. Given such a triple, let $\mathfrak{Y}_{01} = \mathfrak{X}_{01} \times_{\mathfrak{X}_0} \mathfrak{Y}_0$. We have a commutative diagram

$$\begin{array}{ccccc} \mathfrak{Y}_0 & \xleftarrow[i']{} & \mathfrak{Y}_{01} & \xrightarrow{j'} & \mathfrak{Y}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{X}_0 & \xleftarrow[i]{} & \mathfrak{X}_{01} & \xrightarrow{j} & \mathfrak{X}_1 \end{array}$$

where both squares are pullbacks. It follows that i' and j' are closed immersions, so that there exists a pushout diagram

$$\begin{array}{ccc} \mathfrak{Y}_{01} & \xrightarrow{i'} & \mathfrak{Y}_0 \\ \downarrow{\scriptstyle j'} & & \downarrow \\ \mathfrak{Y}_1 & \longrightarrow & \mathfrak{Y} \end{array}$$

of spectral Deligne-Mumford stacks (Theorem 6.1). The construction $(\mathfrak{Y}_0, \mathfrak{Y}_1, \alpha) \mapsto \mathfrak{Y}$ determines a functor $F : \mathrm{Stk}_{/\mathfrak{X}_0} \times_{\mathrm{Stk}_{/\mathfrak{X}_{01}}} \mathrm{Stk}_{/\mathfrak{X}_1} \to \mathrm{Stk}_{/\mathfrak{X}}$ which is left adjoint to G. It follows from Proposition 9.2 that the counit map $v : F \circ G \to \mathrm{id}$ is an equivalence.

We now prove that u is an equivalence. Fix an object $(\mathfrak{Y}_0, \mathfrak{Y}_1, \alpha) \in \mathrm{Stk}_{/\mathfrak{X}_0} \times_{\mathrm{Stk}_{/\mathfrak{X}_{01}}} \mathrm{Stk}_{/\mathfrak{X}_1}$, so that we have a pushout diagram τ :

$$\begin{array}{ccc} \mathfrak{Y}_{01} & \xrightarrow{i'} & \mathfrak{Y}_0 \\ \downarrow{\scriptstyle j'} & & \downarrow \\ \mathfrak{Y}_1 & \longrightarrow & \mathfrak{Y} \end{array}$$

as above. We wish to prove that the induced maps

$$\phi : \mathfrak{Y}_0 \to \mathfrak{X}_0 \times_{\mathfrak{X}} \mathfrak{Y} \qquad \psi : \mathfrak{Y}_1 \to \mathfrak{X}_1 \times_{\mathfrak{X}} \mathfrak{Y}$$

are equivalences. We will prove that ϕ is an equivalence; the proof that ψ is an equivalence is similar. The assertion is local on $\mathfrak{X}$ and $\mathfrak{Y}$; we may therefore assume that $\mathfrak{X} \simeq \mathrm{Spec}\, A$ and $\mathfrak{Y} \simeq \mathrm{Spec}\, R$ are affine. It follows that $\mathfrak{X}_0$, $\mathfrak{X}_1$, and $\mathfrak{X}_{01}$ are affine; write

$$\mathfrak{X}_0 \simeq \mathrm{Spec}\, A_0 \qquad \mathfrak{X}_{01} \simeq \mathrm{Spec}\, A_{01} \qquad \mathfrak{X}_1 \simeq \mathrm{Spec}\, A_1$$

$$\mathfrak{Y}_0 \simeq \mathrm{Spec}\, R_0 \qquad \mathfrak{Y}_{01} \simeq \mathrm{Spec}\, R_{01} \qquad \mathfrak{Y}_1 \simeq \mathrm{Spec}\, R_1.$$

Then α determines an equivalence $\beta : A_{01} \otimes_{A_0} R_0 \simeq A_{01} \otimes_{A_1} R_1$, and we can identify the triple (R_0, R_1, β) with an object of $\mathrm{Mod}^{\mathrm{cn}}_{A_0} \times_{\mathrm{Mod}^{\mathrm{cn}}_{A_{01}}} \mathrm{Mod}^{\mathrm{cn}}_{A_1}$. Using Proposition 7.6, we deduce that (R_0, R_1, β) is determined by an object $M \in \mathrm{Mod}^{\mathrm{cn}}_A$. Then Corollary 6.5 supplies an equivalence $\theta : R \simeq R_0 \times_{R_{01}} R_1 \simeq (A_0 \times_{A_{01}} A_1) \otimes_A M \simeq M$. To prove that ϕ is an equivalence, it will suffice to show that it induces an equivalence of $\mathbb{E}_\infty$-rings $R_0 \otimes_R A \to A_0$. It now suffices to observe that the underlying map of spectra is given by the composition

$$R_0 \otimes_R A \overset{\theta}{\simeq} R_0 \otimes_R M \simeq A_0.$$

□

In the situation of Proposition 9.2, many important properties of morphism $f : \mathfrak{Y} \to \mathfrak{X}$ can be tested after pullback along the closed immersions $\mathfrak{X}_0 \hookrightarrow \mathfrak{X} \hookleftarrow \mathfrak{X}_1$.

Proposition 9.3. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks σ :*

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}, \end{array}$$

where i and j are closed immersions. Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a map of spectral Deligne-Mumford stacks. Set $\mathfrak{Y}_0 = \mathfrak{X}_0 \times_{\mathfrak{X}} \mathfrak{Y}$, $\mathfrak{Y}_1 = \mathfrak{X}_1 \times_{\mathfrak{X}} \mathfrak{Y}$, and let $f_0 : \mathfrak{Y}_0 \to \mathfrak{X}_0$ and $f_1 : \mathfrak{Y}_1 \to \mathfrak{X}_0$ be the projection maps. Then:

(1) *The map f is locally of finite presentation to order n if and only if both f_0 and f_1 are locally of finite presentation to order n (for any $n \geq 0$).*

(2) *The map f is locally almost of finite presentation if and only if both f_0 and f_1 are locally almost of finite presentation.*

(3) *The map f is locally of finite presentation if and only if f_0 and f_1 are locally of finite presentation.*

(4) *The map f is étale if and only if both f_0 and f_1 are étale.*

(5) *The map f is an equivalence if and only if both f_0 and f_1 are equivalences.*

(6) *The map f is an open immersion of and only if both f_0 and f_1 are open immersions.*

(7) *The map f is flat if and only if both f_0 and f_1 are flat.*

(8) *The map f is affine if and only if both f_0 and f_1 are affine.*

(9) *The map f is a closed immersion if and only if both f_0 and f_1 are closed immersions.*

(10) *The map f is strongly separated if and only f_0 and f_1 are strongly separated.*

(11) *The map f is n-quasi-compact if and only if f_0 and f_1 are n-quasi-compact, for $0 \leq n \leq \infty$.*

Proof. The "only if" directions are obvious. The converse assertions are all local on $\mathfrak{X}$, so we may assume without loss of generality that $\mathfrak{X} = \operatorname{Spec} A$ is affine. In this case, the diagram σ is determined by a pullback square of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & A_{01}, \end{array}$$

where the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective.

We first prove (1). The assertion is local on $\mathfrak{Y}$, so we may assume that $\mathfrak{Y} = \operatorname{Spec} R$ is affine. By assumption, the $\mathbb{E}_\infty$-algebras $R_0 = A_0 \otimes_A R$ and $R_1 = A_1 \otimes_A R$ are of finite presentation to order n over A_0 and A_1, respectively. We wish to show that R is of finite presentation to order n over A. We first treat the case $n = 0$. Then $\pi_0 R_0$ and $\pi_0 R_1$ are finitely generated as commutative rings over $\pi_0 A$. Since the maps $\pi_0 R \to \pi_0 R_0$ and $pi_0 R \to \pi_0 R_1$ are surjective, we can choose a finite collection $x_1, \ldots, x_n \in \pi_0 R$ whose images generate $\pi_0 R_0$ and $\pi_0 R_1$ over $\pi_0 A$. Let $B = A\{X_1, \ldots, X_n\}$ denote the $\mathbb{E}_\infty$-algebra over A freely generated by a collection of elements $X_1, \ldots, X_n$, so that there is a map of $\mathbb{E}_\infty$-algebras $\phi : B \to R$ which is determined uniquely up to homotopy by the requirement that $\phi(X_i) = x_i \in \pi_0 R$ for $1 \leq i \leq k$. Let I denote the fiber of ϕ, and regard I as a R-module. Then $A_0 \otimes_A I$ and $A_1 \otimes_A I$ can be identified with the fibers of the induced maps

$$A_0\{X_1, \ldots, X_k\} \to R_0 \qquad A_1\{X_1, \ldots, X_k\} \to R_1,$$

and are therefore connective. It follows that I is connective, so that ϕ induces a surjection $\pi_0 B \simeq \pi_0 A[X_1, \ldots, X_k] \to \pi_0 R$. This proves that $\pi_0 R$ is finitely generated as a commutative ring over $\pi_0 A$, so that R is of finite presentation to order 0 over A (Example 8.4).

We now prove (1) in the case $n \geq 1$. Assume that R_0 and R_1 are of finite presentation to order n over A_0 and A_1, respectively. Then the relative cotangent complexes

$$L_{R_0/A_0} \simeq R_0 \otimes_R L_{R/A} \qquad L_{R_1/A_1} \simeq R_1 \otimes_R L_{R/A}$$

are perfect to order n over R_0 and R_1, respectively. Using Proposition 7.7, we deduce that $L_{R/A}$ is perfect to order n as an R-module. Consequently, to show that R is of finite presentation to order n over A, it will suffice to show that $\pi_0 R$ is finitely presented as a commutative ring over $\pi_0 A$ (Proposition 8.8). Let $\phi : B \to R$ and $I = \mathrm{fib}(\phi)$ be defined as above. We have surjective maps

$$\pi_0 I \to \pi_0(A_0 \otimes_A I) \to \ker((\pi_0 A_0)[X_1, \ldots, X_k] \to \pi_0 R_0)$$

$$\pi_0 I \to \pi_0(A_1 \otimes_A I) \to \ker((\pi_0 A_1)[X_1, \ldots, X_k] \to \pi_0 R_1).$$

Since $\pi_0 R_0$ and $\pi_0 R_1$ are finitely presented as commutative rings over $\pi_0 A_0$ and $\pi_0 A_1$, respectively, we can choose a finite collection of elements $y_1, \ldots, y_m \in \pi_0 I$ whose images generate the ideals

$$\ker((\pi_0 A_0)[X_1, \ldots, X_k] \to \pi_0 R_0) \qquad \ker((\pi_0 A_1)[X_1, \ldots, X_k] \to \pi_0 R_1).$$

Let $A\{Y_1, \ldots, Y_m\}$ be the corresponding free $\mathbb{E}_\infty$-algebra over A, so that the choice of elements y_i determine a commutative diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A\{Y_1, \ldots, Y_m\} & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & R. \end{array}$$

Let B' denote the pushout $A \otimes_{A\{Y_1, \ldots, Y_m\}} B$, so that we obtain a map of $\mathbb{E}_\infty$-rings $\psi : B' \to R$. Then B' is of finite presentation over A, so that $\pi_0 B'$ is finitely presented as a commutative ring over $\pi_0 A$. By construction, ψ induces a surjection $\pi_0 B' \to \pi_0 R$. Let $J \subseteq \pi_0 B'$ denote the kernel of this surjection. To complete the proof, it will suffice to show that J is a finitely generated ideal in $\pi_0 B'$.

Let $B'_0 = A_0 \otimes_A B'$, and define B'_1 and B'_{01} similarly. We have a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} B' & \longrightarrow & B'_0 \\ \downarrow & & \downarrow \\ B'_1 & \longrightarrow & B'_{01}. \end{array}$$

By construction ψ induces isomorphisms $\pi_0 B'_0 \simeq \pi_0 R_0$ and $\pi_0 B'_1 \simeq \pi_0 R_1$. It follows that J is contained in the kernel of the map $\pi_0 B \to \pi_0 B'_0 \times \pi_0 B'_1$, which is the image of the boundary map $\pi_1 B'_{01} \to \pi_0 B'$. It follows that the action of $\pi_0 B'$ on J factors through the acton of $\pi_0 B'_{01}$. Since J belongs to the kernel of the map $\pi_0 B' \to \pi_0 B'_{01}$, we deduce that $J^2 = 0$. Consequently, the map $J \to \mathrm{Tor}_0^{\pi_0 B'}(\pi_0 R, J) \simeq \pi_0(R \otimes_{B'} \mathrm{fib}(\psi))$ is bijective. To prove that J is a finitely generated ideal in $\pi_0 B'$, it suffices to show that $\pi_0(R \otimes_{B'} \mathrm{fib}(\psi))$ is finitely generated as a module over $\pi_0 R$. Applying Theorem A.7.4.3.1, we obtain a canonical isomorphism $\pi_0(R \otimes_{B'} \mathrm{fib}(\psi)) \simeq \pi_1 L_{R/B'}$. Since $L_{R/B'}$ is 1-connective, it will suffice to show that $L_{R/B'}$ is perfect to order 1 as an R-module. We have a fiber seqence

$$R \otimes_{B'} L_{B'/A} \to L_{R/A} \to L_{R/B'}.$$

By construction, B' is finitely presented over A, so that $L_{B'/A}$ is perfect as a B'-module. Using Remark VIII.2.6.8, we are reduced to showing that $L_{R/A}$ is perfect to order 1 as an R-module, which was already proven above. This completes the proof of (1).

Assertion (2) follows immediately from (1). We now prove (3). As before, we may suppose that $\mathfrak{Y} = \operatorname{Spec} R$ is affine. Then R_0 and R_1 are locally of finite presentation over A_0 and A_1, respectively, so that the relative cotangent complexes $L_{R_0/A_0} \simeq R_0 \otimes_R L_{R/A}$ and $L_{R_1/A_1} \simeq R_1 \otimes_R L_{R/A}$ are perfect. Using Proposition 7.7, we deduce that $L_{R/A}$ is a perfect R-module. Since R is almost of finite presentation over A (by (2)), we deduce from Theorem A.7.4.3.18 that R is locally of finite presentation over A.

We now prove (4). Once again we may suppose that $\mathfrak{Y} = \operatorname{Spec} R$ is affine. If R_0 and R_1 are étale over A_0 and A_1, then we obtain

$$R_0 \otimes_R L_{R/A} \simeq L_{R_0/A_0} \simeq 0 \simeq L_{R_1/A_1} \simeq R_1 \otimes_R L_{R/A}.$$

Using Theorem 7.2 we deduce that $L_{R/A} \simeq 0$. Since R is almost of finite presentation over A (by (2)), it follows from Lemma VII.8.9 that R is étale over A.

Assertion (5) follows immediately from Theorem 9.1. Assertion (6) follows from (4) and (5), since f is an open immersion if and only if f is étale and the diagonal map $\mathfrak{Y} \to \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ is an equivalence. To prove (7), we may assume without loss of generality that $\mathfrak{Y} = \operatorname{Spec} R$ is affine. In this case, the desired result follows from Proposition 7.7. Assertion (8) follows from Theorem 6.1.

We now prove (9). Assume that f_0 and f_1 are closed immersions. It follows from (8) that f is affine, so that $\mathfrak{Y} \simeq \operatorname{Spec} R$ for some connective $\mathbb{E}_\infty$-ring R. We have a fiber sequence of A-modules $I \to A \to R$. The map f is a closed immersion if and only if the map $\pi_0 A \to \pi_0 R$ is surjective, which is equivalent to the requirement that I is connective. Since f_0 and f_1 are closed immersions, the tensor products $A_0 \otimes_A I$ and $A_1 \otimes_A I$ are connective. It follows from Proposition 7.7 that I is connective, as desired.

Assertion (10) follows by applying (9) to the diagonal map $\mathfrak{Y} \to \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$. To prove (11), it suffices to treat the case where $n < \infty$. We proceed by induction on n. If $n > 0$, it suffices to show that if we are given a pair of étale maps $\mathfrak{U} \to \mathfrak{Y} \leftarrow \mathfrak{V}$ where $\mathfrak{U}$ and $\mathfrak{V}$ are affine, then $\mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{V}$ is $(n-1)$-quasi-compact. If $\mathfrak{Y}_0$ and $\mathfrak{Y}_1$ are n-quasi-compact, then the spectral Deligne-Mumford stacks

$$\mathfrak{X}_0 \times_{\mathfrak{X}} (\mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{V}) \qquad \mathfrak{X}_1 \times_{\mathfrak{X}} (\mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{V})$$

are $(n-1)$-quasi-compact, so that the desired result follows from the inductive hypothesis. It therefore suffices to treat the case $n = 0$. Suppose that $\mathfrak{Y}$ can be described as the colimit of a diagram of open substacks $\{\mathfrak{Y}_\alpha\}_{\alpha \in P}$ indexed by a filtered poset P. Since $\mathfrak{Y}_0$ is quasi-compact, there exists an index $\alpha \in P$ such that $\mathfrak{Y}_\alpha \times_{\mathfrak{Y}} \mathfrak{Y}_0 \simeq \mathfrak{Y}_0$. Enlarging α if necessary, we may suppose that $\mathfrak{Y}_\alpha \times_{\mathfrak{Y}} \mathfrak{Y}_1 \simeq \mathfrak{Y}_1$. It then follows from (5) that the open immersion $\mathfrak{Y}_\alpha \hookrightarrow \mathfrak{Y}$ is an equivalence. □

Corollary 9.4. *Suppose we are given a pushout diagram of spectral Deligne-Mumford stacks*

$$\begin{array}{ccc} \mathfrak{X}_{01} & \xrightarrow{i} & \mathfrak{X}_0 \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathfrak{X}_1 & \longrightarrow & \mathfrak{X} \end{array}$$

where i and j are closed immersions. If $\mathfrak{X}_0$ and $\mathfrak{X}_1$ are separated spectral algebraic spaces, then $\mathfrak{X}$ is a separated spectral algebraic space.

Proof. We wish to show that the diagonal map $\delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is a closed immersion. Using Proposition 9.3, we may reduce to proving that each of the vertical maps appearing in the diagram

$$\begin{array}{ccccc} \mathfrak{X}_0 & \longleftarrow & \mathfrak{X}_{01} & \longrightarrow & \mathfrak{X}_1 \\ \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \delta_{01}} & & \downarrow{\scriptstyle \delta_1} \\ \mathfrak{X}_0 \times \mathfrak{X} & \longleftarrow & \mathfrak{X}_{01} \times \mathfrak{X} & \longrightarrow & \mathfrak{X}_1 \times \mathfrak{X} \end{array}$$

is a closed immersion. We will prove that δ_0 is a closed immersion; the proof in the other two cases are similar. We can factor δ_0 as a composition

$$\mathfrak{X}_0 \xrightarrow{\delta'} \mathfrak{X}_0 \times \mathfrak{X}_0 \xrightarrow{\delta''} \mathfrak{X}_0 \times \mathfrak{X}.$$

Here δ' is a closed immersion (by virtue of our assumption that $\mathfrak{X}_0$ is a separated spectral algebraic space), and δ'' is a pullback of the closed immersion $\mathfrak{X}_0 \to \mathfrak{X}$. □

10 Unramified Transformations

Suppose we are given a pullback diagram σ :

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

in the category of schemes (or, more generally, in the category of locally ringed spaces). If f is a closed immersion, then σ is also a pullback diagram in the larger category of ringed spaces. Our goal in this section is to place this observation into a larger context.

Definition 10.1. Let $\theta : \mathcal{T}' \to \mathcal{T}$ be a transformation of pregeometries, and $\Theta : \mathcal{T}\mathrm{op}(\mathcal{T}) \to \mathcal{T}\mathrm{op}(\mathcal{T}')$ the induced functor, given on objects by $(\mathcal{X}, \mathcal{O}) \mapsto (\mathcal{X}, \mathcal{O} \circ \theta)$. We will say that θ is *unramified* if the following conditions are satisfied:

(1) The pregeometries $\mathcal{T}$ and $\mathcal{T}'$ are unramified (see Definition 1.3).

(2) For every morphism $f : X \to Y$ in $\mathcal{T}$ and every object $Z \in \mathcal{T}$, the diagram

$$\begin{array}{ccc} \Theta \operatorname{Spec}^{\mathcal{T}}(X \times Z) & \longrightarrow & \Theta \operatorname{Spec}^{\mathcal{T}}(X) \\ \downarrow & & \downarrow \\ \Theta \operatorname{Spec}^{\mathcal{T}}(X \times Y \times Z) & \longrightarrow & \Theta \operatorname{Spec}^{\mathcal{T}}(X \times Y) \end{array}$$

is a pullback square in $\mathcal{T}\mathrm{op}(\mathcal{T}')$.

Remark 10.2. Let $\theta : \mathcal{T}' \to \mathcal{T}$ be a transformation of unramified pregeometries. For every object $U \in \mathcal{T}$, denote $\operatorname{Spec}^{\mathcal{T}}(U)$ by $(\mathcal{X}_U, \mathcal{O}_U)$. Suppose we are given a morphism $f : X \to Y$ in $\mathcal{T}$ and another object $Z \in \mathcal{T}$. Since $\mathcal{T}$ is unramified, the induced diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{X}_{X \times Z} & \xrightarrow{\alpha} & \mathcal{X}_X \\ \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} \\ \mathcal{X}_{X \times Y \times Z} & \xrightarrow{\alpha'} & \mathcal{X}_{X \times Y} \end{array}$$

is a pullback square in $\mathcal{T}\mathrm{op}$. Using Propositions T.4.3.1.5, T.4.3.1.9, and T.4.2.4.1, we see that condition (2) of Definition 10.1 is equivalent to the following:

(2′) For every geometric morphism of ∞-topoi $g^* : \mathcal{X}_{X \times Z} \to \mathcal{Y}$, the diagram

$$\begin{array}{ccc} g^* \alpha^* \beta^* (\mathcal{O}_{X \times Y} \circ \theta) & \longrightarrow & g^* \beta^* (\mathcal{O}_X \circ \theta) \\ \downarrow & & \downarrow \\ g^* \beta'^* (\mathcal{O}_{X \times Y \times Z} \circ \theta) & \longrightarrow & g^* (\mathcal{O}_{X \times Z} \circ \theta) \end{array}$$

is a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{Y})$.

Note that the horizontal maps in the above diagram admit sections, and are therefore effective epimorphisms. Since $\mathcal{T}'$ is unramified, Proposition 1.8 implies that it is sufficient to check the following weaker assertion:

(2″) The diagram

$$\begin{array}{ccc} \alpha^*\beta^*(\mathcal{O}_{X\times Y}\circ\theta) & \longrightarrow & \beta^*(\mathcal{O}_X\circ\theta) \\ \downarrow & & \downarrow \\ \beta'^*(\mathcal{O}_{X\times Y\times Z}\circ\theta) & \longrightarrow & (\mathcal{O}_{X\times Z}\circ\theta) \end{array}$$

is a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{X}_{X\times Z})$.

We can now formulate the main result of this section:

Proposition 10.3. *Let $\theta : \mathcal{T}' \to \mathcal{T}$ be an unramified transformation of pregeometries. Suppose we are given a pullback square σ :*

$$\begin{array}{ccc} (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) & \longrightarrow & (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \\ \downarrow & & \downarrow \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) & \longrightarrow & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

in $\mathcal{T}\mathrm{op}(\mathcal{T})$, such that the underlying geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$ induces an effective epimorphism $f^*\,\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}}$. Then the diagram σ' :

$$\begin{array}{ccc} (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}\circ\theta) & \longrightarrow & (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}\circ\theta) \\ \downarrow & & \downarrow \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}\circ\theta) & \longrightarrow & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}\circ\theta) \end{array}$$

is also a pullback square in $\mathcal{T}\mathrm{op}(\mathcal{T}')$.

We will give the proof of Proposition 10.3 at the end of this section. First, let us describe some examples.

Proposition 10.4. *Let $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}$, $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}$, and $\mathcal{T}^{\mathrm{Sp}}_{\acute{e}t}$ be the pregeometries of Remark VII.1.28, Definition VII.2.18, and Definition VII.8.26. Then the evident transformations*

$$\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}} \to \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}} \qquad \mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}} \to \mathcal{T}^{\mathrm{Sp}}_{\acute{e}t} \qquad \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}} \to \mathcal{T}^{\mathrm{Sp}}_{\acute{e}t}$$

are unramified.

Corollary 10.5. *Suppose we are given a pullback diagram σ :*

$$\begin{array}{ccc} \mathfrak{Y}' & \longrightarrow & \mathfrak{X}' \\ \downarrow & & \downarrow \\ \mathfrak{Y} & \xrightarrow{f} & \mathfrak{X} \end{array}$$

of spectral Deligne-Mumford stacks (spectral schemes), where f is a closed immersion. Then σ is a pullback diagram in the ∞-category RingTop *of spectrally ringed ∞-topoi.*

Proof. Combine Propositions 10.4 and 10.3. □

The proof of Proposition 10.4 will require a few facts about local and (strictly) Henselian $\mathbb{E}_\infty$-rings.

Lemma 10.6. *Let $\phi : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings which induces a surjection $\pi_0 A \to \pi_0 B$, and assume that B is nonzero. If A is local (strictly Henselian), then B is local (strictly Henselian). Moreover, if R is any local $\mathbb{E}_\infty$-ring, then a map $\psi : B \to R$ is local if and only if the composite map $\psi \circ \phi$ is local (in particular, the map ϕ is local).*

Proof. The assertion regarding strictly Henselian $\mathbb{E}_\infty$-rings follows from Corollary VIII.1.1.5. Suppose that A is local. To prove that B is local, it will suffice to show that for every element $x \in \pi_0 B$, either x or $1 - x$ is invertible in $\pi_0 B$. Since the map $\pi_0 A \to \pi_0 B$ is surjective, we can write x as the image of an element $\overline{x} \in \pi_0 A$. Since A is local, either $\overline{x}$ or $1 - \overline{x}$ is invertible in $\pi_0 A$, from which the desired result follows immediately.

Now suppose we are given a map of $\mathbb{E}_\infty$-rings $\psi : B \to R$. We wish to show that $\psi \circ \phi$ is local if and only if ψ is local. Let $\mathfrak{m}_A$ denote the maximal ideal in $\pi_0 A$, and define $\mathfrak{m}_B$ and $\mathfrak{m}_R$ similarly. We wish to prove that $\psi^{-1}\mathfrak{m}_R = \mathfrak{m}_B$ if and only if $(\psi \circ \phi)^{-1}\mathfrak{m}_R = \mathfrak{m}_A$. To prove this, it suffices to show that $\mathfrak{m}_B$ is unique among those ideals $I \subseteq \pi_0 B$ such that $\phi^{-1} I = \mathfrak{m}_A$. This is clear, since ϕ^{-1} induces a bijection between the set of ideals in $\pi_0 B$ and the set of ideals in $\pi_0 A$ which contain the kernel of the map $\pi_0 A \to \pi_0 B$. ☐

Lemma 10.7. *Suppose we are given a pushout diagram of connective $\mathbb{E}_\infty$-rings*

$$\begin{array}{ccc} \overline{A} & \xrightarrow{\phi} & A \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} \\ \overline{B} & \xrightarrow{\phi'} & B \end{array}$$

where $\pi_0 \overline{A} \to \pi_0 A$ is surjective. Assume that $\overline{A}$, $\overline{B}$, and A are local (strictly Henselian), and that the maps ϕ and ψ are local. Then:

(1) *The $\mathbb{E}_\infty$-ring B is local (strictly Henselian).*

(2) *Let R be a local $\mathbb{E}_\infty$-ring. Then a map $\epsilon : B \to R$ is local if and only if the composite maps $\epsilon \circ \phi'$ and $\epsilon \circ \psi'$ are local. (In particular, the maps ϕ' and ψ' are local).*

Proof. Let $\kappa_{\overline{A}}$ denote the residue field of the local commutative ring $\pi_0 \overline{A}$, and define κ_A and $\kappa_{\overline{B}}$ similarly. There exists a map of $\mathbb{E}_\infty$-rings $B \to \kappa_A \otimes_{\kappa_{\overline{A}}} \kappa_{\overline{B}} \neq 0$, so that B is nonzero. Since ϕ induces a surjection $\pi_0 \overline{A} \to \pi_0 A$, the map ϕ' induces a surjection $\pi_0 \overline{B} \to \pi_0 B$. Assertion (1) now follows from Lemma 10.6. Moreover, Lemma 10.6 implies that a map $\epsilon : B \to R$ is local if and only if $\epsilon \circ \phi'$ is local. To complete the proof of (2), it will suffice to show that if this condition is satisfied, then $\epsilon \circ \psi'$ is also local. Using Lemma 10.6, we are reduced to checking that $\epsilon \circ \psi' \circ \phi \simeq \epsilon \circ \phi' \circ \psi$ is local. This is clear, since ψ is local by assumption. ☐

Proof of Proposition 10.4. We will prove that the transformation of pregeometries $\theta : \mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}} \to \mathcal{T}^{\mathrm{Sp}}_{\text{ét}}$ is unramified; the proof in the other two cases is similar. The geometric $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}$ is unramified because it is discrete (Example 1.4), and $\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}$ is unramified by Proposition 4.1. We will complete the proof by verifying criterion $(2'')$ of Remark 10.2. For this, it suffices to prove the following:

$(*)$ Let $\mathcal{X}$ be the underlying ∞-topos of an affine spectral Deligne-Mumford stack, let $\mathcal{C}$ denote the subcategory of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{X})$ whose objects are strictly Henselian sheaves of connective $\mathbb{E}_\infty$-rings on $\mathcal{X}$ and whose morphisms are local maps, and suppose we are given a pushout diagram σ :

$$\begin{array}{ccc} \overline{\mathcal{O}} & \xrightarrow{f} & \mathcal{O} \\ \downarrow & & \downarrow \\ \overline{\mathcal{O}}' & \longrightarrow & \mathcal{O}' \end{array}$$

in $\mathcal{C}$, where f induces an epimorphism $\pi_0 \overline{\mathcal{O}} \to \mathcal{O}$. Then σ is also a pushout diagram in $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{X})$.

To prove $(*)$, form a pushout diagram σ'

$$\begin{array}{ccc} \overline{\mathcal{O}} & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow \\ \overline{\mathcal{O}}' & \longrightarrow & \mathcal{A} \end{array}$$

in $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{X})$. To prove that σ and σ' are equivalent, it will suffice to show that $\mathcal{A}$ is strictly Henselian and that, for any object $\mathcal{A}' \in \mathcal{C}$, a map $\epsilon : \mathcal{A} \to \mathcal{A}'$ is local if and only if the induced maps $\mathcal{O} \to \mathcal{A}'$ and $\overline{\mathcal{O}}' \to \mathcal{A}$ are local. Since the ∞-topos $\mathcal{X}$ is 1-localic and its hypercompletion has enough points (Theorem VII.4.1), it will suffice to prove the analogous assertions after taking the stalk at any point of $\mathcal{X}$ (Remark VII.8.31). We complete the proof by invoking Lemma 10.7. □

We now turn to the proof of Proposition 10.3. We will need the following lemma:

Lemma 10.8. *Let $\theta : \mathcal{T}' \to \mathcal{T}$ be an unramified transformation of pregeometries. Let $\mathcal{Z}$ be an ∞-topos, and suppose we are given a pushout square σ :*

$$\begin{array}{ccc} \overline{\mathcal{O}} & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow \\ \overline{\mathcal{O}}_0 & \longrightarrow & \mathcal{O}_0 \end{array}$$

in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$, where the horizontal maps are effective epimorphisms. Let $\Theta : \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{Z})$ be given by composition with θ; then $\Theta(\sigma)$ is a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{Z})$.

Proof. Let $p : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z}) \to \mathcal{Z}$ be defined as in Notation 2.1. Since $\mathcal{T}$ is unramified, the proof of Proposition 1.8 shows that we can choose a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\triangleright} \times \Delta^1 \times \Delta^1 & \xrightarrow{\overline{Q}} & \underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z}) \\ \downarrow & & \downarrow p \\ \mathcal{C}^{\triangleright} & \xrightarrow{\overline{q}} & \mathcal{Z} \end{array}$$

with the following properties:

- (a) The map $\overline{q}$ carries the cone point v of $\mathcal{C}^{\triangleright}$ to a final object $\mathbf{1} \in \mathcal{Z}$.
- (b) The restriction $q = \overline{q}|\,\mathcal{C}$ is a Cartesian fibration, whose fibers are essentially small sifted ∞-categories.
- (c) The restriction $\overline{Q}|(\{v\} \times \Delta^1 \times \Delta^1)$ can be identified with the diagram σ in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z}^{/\mathbf{1}})$.
- (d) For each object $C \in \mathcal{C}$, the diagram $\overline{Q}|(\{C\} \times \Delta^1 \times \Delta^1)$ is a pushout square σ_C:

$$\begin{array}{ccc} \overline{\mathcal{O}}^C & \longrightarrow & \mathcal{O}^C \\ \downarrow & & \downarrow \\ \overline{\mathcal{O}}^C_0 & \longrightarrow & \mathcal{O}^C_0 \end{array}$$

  in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z}^{/q(C)})$, such that the restriction

$$\overline{\mathcal{O}}^C_0 \leftarrow \overline{\mathcal{O}}^C \to \mathcal{O}^C$$

  is an elementary diagram (see Definition 3.2).

(e) For each vertex $u \in \Delta^1 \times \Delta^1$, the restriction $\overline{Q}|\, \mathcal{C}^{\triangleright} \times \{u\}$ is a p-colimit diagram, which carries q-Cartesian morphisms in $\mathcal{C}$ to p-Cartesian morphisms in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$.

Let $p' : \underline{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{Z}) \to \mathcal{Z}$ be the forgetful functor. To complete the proof, it will suffice to show that $(\Theta\overline{Q})|(\{v\} \times \Delta^1 \times \Delta^1)$ is a p'-colimit diagram. Using (a), (b), (e), and Proposition 2.2, we conclude that $(\Theta\overline{Q})|\, \mathcal{C}^{\triangleright} \times \{u\}$ is a p'-colimit diagram for each $u \in \Delta^1 \times \Delta^1$. Using Lemma T.5.5.2.3, we are reduced to showing that for each $C \in \mathcal{C}$, the restriction $(\Theta\overline{Q})|(\{C\} \times \Delta^1 \times \Delta^1)$ is a p'-colimit diagram in $\underline{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{Z})$. This follows from (d) together with $(2')$ of Remark 10.2. $\square$

Proof of Proposition 10.3. It follows from Theorem 1.6 that the diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

is a pullback square in $\mathcal{T}\mathrm{op}$. Using Proposition T.4.3.1.5, we conclude that σ is a p-limit diagram, where $p : \mathcal{T}\mathrm{op}(\mathcal{T}) \to \mathcal{T}\mathrm{op}$ denotes the forgetful functor. Using Propositions T.4.3.1.10 and T.4.3.1.9, we conclude that for every geometric morphism of ∞-topoi $h^* : \mathcal{Y}' \to \mathcal{Z}$, the diagram τ :

$$\begin{array}{ccc} h^* g^* f^* \mathcal{O}_{\mathcal{X}} & \longrightarrow & h^* f'^* \mathcal{O}_{\mathcal{X}'} \\ \downarrow & & \downarrow \\ h^* g^* \mathcal{O}_{\mathcal{Y}} & \longrightarrow & h^* \mathcal{O}_{\mathcal{Y}'} \end{array}$$

is a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{Z})$. It follows from Lemma 10.8 that the image of τ is a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}'}(\mathcal{Z})$. Using Propositions T.4.3.1.10 and T.4.3.1.9 again, we conclude that σ' is a p'-limit diagram, where $p' : \mathcal{T}\mathrm{op}(\mathcal{T}') \to \mathcal{T}\mathrm{op}$. It now follows from Proposition T.4.3.1.5 that σ' is a pullback square in $\mathcal{T}\mathrm{op}(\mathcal{T}')$, as desired. $\square$

11 Derived Complex Analytic Geometry

Our goal for the remainder of this paper is to sketch the foundations for a theory of "derived" complex analytic spaces. In this section, we will set the stage by introducing a pregeometry $\mathcal{T}_{\mathrm{an}}$ which encodes the theory of complex analytic manifolds. Our main technical result, Lemma 11.10, asserts that there is an unramified transformation of pregeometries $\mathcal{T}_{\mathrm{disc}}(\mathbf{C}) \to \mathcal{T}_{\mathrm{an}}$. It follows that every $\mathcal{T}_{\mathrm{an}}$-structured ∞-topos is equipped with an underlying sheaf of $\mathbb{E}_{\infty}$-rings, which behave well under pullbacks along closed immersions (Proposition 11.12). This assertion has several applications which we will describe in §12.

Definition 11.1. We define a pregeometry $\mathcal{T}_{\mathrm{an}}$ as follows:

(1) The underlying ∞-category of $\mathcal{T}_{\mathrm{an}}$ is $\mathrm{N}(\mathcal{C})$, where $\mathcal{C}$ is the category of complex manifolds (which we assume to be Hausdorff and equipped with a countable neighborhood basis).

(2) A morphism $f : U \to X$ in $\mathcal{T}_{\mathrm{an}}$ is admissible if and only if it is locally biholomorphic.

(3) A collection of admissible morphisms $\{U_\alpha \to X\}$ in $\mathcal{T}_{\mathrm{an}}$ generates a covering sieve on X if and only if, for every point $x \in X$, some inverse image $U_\alpha \times_X \{x\}$ is nonempty.

Remark 11.2. Let $\mathcal{T}^0_{\mathrm{an}}$ be the pregeometry defined in the same way $\mathcal{T}_{\mathrm{an}}$, except that the class of admissible morphisms in $\mathcal{T}^0_{\mathrm{an}}$ is the class of *open immersions* of complex manifolds. Then the identity transformation $\mathcal{T}^0_{\mathrm{an}} \to \mathcal{T}_{\mathrm{an}}$ is a Morita equivalence; this follows immediately from Proposition V.3.2.5.

Remark 11.3. Let $\mathcal{T}^1_{\mathrm{an}}$ be the the full subcategory of $\mathcal{T}_{\mathrm{an}}$ whose objects are *Stein* manifolds: that is, complex manifolds X which can be realized as a closed submanifold of $\mathbf{C}^n$ for some n. Proposition V.3.2.8 implies that the inclusion $\mathcal{T}^1_{\mathrm{an}} \subseteq \mathcal{T}_{\mathrm{an}}$ is a Morita equivalence.

Proposition 11.4. *The pregeometry* $\mathcal{T}_{\mathrm{an}}$ *is compatible with* n*-truncations, for each* $n \geq 0$.

Proof. This follows immediately from Remark 11.2 and Proposition V.3.3.5. □

Remark 11.5. Let $X \in \mathcal{T}_{\mathrm{an}}$ be a complex manifold. The construction of spectra described in §V.2.2 shows that the underlying ∞-topos of $\mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(X)$ can be identified with $\mathcal{S}\mathrm{hv}(X)$.

Proposition 11.6. *The pregeometry* $\mathcal{T}_{\mathrm{an}}$ *is unramified.*

Proof. In view of Example 1.5, it suffices to show that the composite functor

$$\mathcal{T}_{\mathrm{an}} \stackrel{\mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}}{\longrightarrow} \mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}) \to \mathcal{T}\mathrm{op}$$

commutes with finite products. Since every complex manifold is locally compact, this follows from Remark 11.5 and Proposition T.7.3.1.11. □

We now explain how every $\mathcal{T}_{\mathrm{an}}$-structure on an ∞-topos $\mathcal{X}$ determines a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Construction 11.7. Let $\mathbf{C}$ denote the field of complex numbers. Let $\mathrm{Poly}_{\mathbf{C}}$ denote the category of polynomial algebras over $\mathbf{C}$: that is, the full subcategory of $\mathrm{CAlg}_{\mathbf{C}}$ spanned by those objects of the form $\mathbf{C}[x_1, \ldots, x_n]$ for $n \geq 0$. We let $\mathcal{T}_{\mathrm{disc}}(\mathbf{C})$ denote the ∞-category $\mathrm{N}(\mathrm{Poly}_{\mathbf{C}})^{op}$. If A is a polynomial algebra over $\mathbf{C}$, we let $\mathrm{Spec}\, A$ denote the corresponding object of $\mathcal{T}_{\mathrm{disc}}(\mathbf{C})$. Note that $\mathrm{Poly}_{\mathbf{C}}$ admits finite coproducts, so that $\mathcal{T}_{\mathrm{disc}}(\mathbf{C})$ admits finite products. We will regard $\mathcal{T}_{\mathrm{disc}}(\mathbf{C})$ as a discrete pregeometry: that is, the admissible morphisms in $\mathcal{T}_{\mathrm{disc}}(\mathbf{C})$ are precisely the equivalences and a collection of admissible morphisms $\{U_\alpha \to X\}$ in $\mathcal{T}_{\mathrm{disc}}(\mathbf{C})$ is a covering if and only if it is nonempty.

Let $\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}}$ denote the ∞-category of connective $\mathbb{E}_\infty$-algebras over $\mathbf{C}$. Combining Propositions V.4.1.9 and V.4.1.11, we see that for every ∞-topos $\mathcal{X}$ there is an equivalence of ∞-categories $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}(\mathbf{C})}(\mathcal{X}) = \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(\mathbf{C})}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}}}(\mathcal{X})$.

For every smooth $\mathbf{C}$-algebra A, the set $\mathrm{Hom}_{\mathbf{C}}(A, \mathbf{C})$ of $\mathbf{C}$-points of the algebraic variety $\mathrm{Spec}\, A$ has the structure of a complex analytic manifold $(\mathrm{Spec}\, A)^{\mathrm{an}}$. This construction determines a transformation of geometries $\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(\mathbf{C}) \to \mathcal{T}_{\mathrm{an}}$. Composing with the inclusion $\mathcal{T}_{\mathrm{disc}}(\mathbf{C}) \subseteq \mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(\mathbf{C})$ (see Remark VII.1.28), we get a transformation of pregeometries $\mathcal{T}_{\mathrm{disc}}(\mathbf{C}) \to \mathcal{T}_{\mathrm{an}}$. This transformation fits into a chain of transformations

$$\mathcal{T}_{\mathrm{disc}}(\mathbf{C}) \hookrightarrow \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(\mathcal{C}) \hookrightarrow \mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(\mathcal{C}) \to \mathcal{T}_{\mathrm{an}}$$

whose composition is a transformation of pregeometries $u : \mathcal{T}_{\mathrm{disc}}(\mathbf{C}) \to \mathcal{T}_{\mathrm{an}}$.

In particular, to every $\mathcal{T}_{\mathrm{an}}$-structured ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, we can associate an underlying sheaf $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$ of connective $\mathbb{E}_\infty$-algebras on $\mathcal{X}$. We have a canonical equivalence $\Omega^\infty \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}(\mathbf{C})$ of objects of $\mathcal{X}$. For each $n \geq 0$, we let $\pi_n \mathcal{O}_{\mathcal{X}}$ denote the nth homotopy group of $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$. Then $\pi_0 \mathcal{O}_{\mathcal{X}}$ is a commutative $\mathbf{C}$-algebra object of the underlying topos $\mathrm{h}(\tau_{\leq 0} \mathcal{X})$, and each $\pi_i \mathcal{O}_{\mathcal{X}}$ has the structure of a module over $\pi_0 \mathcal{O}_{\mathcal{X}}$.

Remark 11.8. Let $\mathcal{O}_{\mathcal{X}}$ be a $\mathcal{T}_{\mathrm{an}}$-structure on an ∞-topos $\mathcal{X}$. Since the transformation of pregeometries $u : \mathcal{T}_{\mathrm{disc}}(\mathbf{C}) \to \mathcal{T}_{\mathrm{an}}$ of Construction 11.7 factors through $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(\mathbf{C})$, we deduce that $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$ is a local sheaf of connective $\mathbf{C}$-algebras on $\mathcal{X}$, in the sense of Definition VII.2.5 (in fact, we can say more: since u factors through $\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(\mathcal{C})$, the sheaf $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$ is strictly Henselian). Moreover, for any map $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$, the associated map $f^* \mathcal{O}^{\mathrm{alg}}_{\mathcal{Y}} \to \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$ is a local map of sheaves of connective $\mathbf{C}$-algebras on $\mathcal{X}$.

Our next goal is to show that if $\mathcal{O}_{\mathcal{X}}$ is a local $\mathcal{T}_{\mathrm{an}}$-structure on an ∞-topos $\mathcal{X}$, then $\mathcal{O}_{\mathcal{X}}$ is to a large extent controlled by the sheaf of $\mathbf{C}$-algebras $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$. For example, we have the following:

Proposition 11.9. *Let $\mathcal{X}$ be an ∞-topos and let $f : \mathcal{O} \to \mathcal{O}'$ be a morphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{X})$. Then f is an equivalence if and only if the induced map $\mathcal{O}^{\mathrm{alg}} \to \mathcal{O}'^{\,\mathrm{alg}}$ is an equivalence.*

Proof. We will prove the "if" direction; the "only if" direction is obvious. Fix an object $M \in \mathcal{T}_{\mathrm{an}}$; we wish to show that the map $\mathcal{O}(M) \to \mathcal{O}'(M)$ is an equivalence in $\mathcal{X}$. Choose a covering $\{U_\alpha \to M\}$, where each U_α is biholomorphic to an open subset of $\mathbf{C}^n$ for some n. We have a pullback diagram

$$\begin{array}{ccc} \coprod \mathcal{O}(U_\alpha) & \longrightarrow & \coprod \mathcal{O}'(U_\alpha) \\ \downarrow & & \downarrow \\ \mathcal{O}(M) & \longrightarrow & \mathcal{O}'(M) \end{array}$$

where the vertical maps are effective epimorphisms. It will therefore suffice to show that the upper horizontal map is an equivalence: that is, we can replace M by U_α and thereby reduce to the case where there exists an open embedding $M \hookrightarrow \mathbf{C}^n$. Using the pullback diagram

$$\begin{array}{ccc} \mathcal{O}(M) & \longrightarrow & \mathcal{O}'(M) \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathbf{C}^n) & \longrightarrow & \mathcal{O}'(\mathbf{C}^n), \end{array}$$

we reduce to the case $M = \mathbf{C}^n$. Since $\mathcal{O}$ and $\mathcal{O}'$ commute with products, we can further assume that $n = 1$, in which case the result is obvious. □

Lemma 11.10. *The transformation $\mathcal{T}_{\mathrm{disc}}(\mathbf{C}) \to \mathcal{T}_{\mathrm{an}}$ of Construction 11.7 is unramified.*

Proof. The pregeometry $\mathcal{T}_{\mathrm{an}}$ is unramified by Proposition 11.6, and the pregeometry $\mathcal{T}_{\mathrm{disc}}(\mathbf{C})$ is unramified by Example 1.4. If M is a complex manifold, we denote $\mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(M)$ by $(\mathcal{S}\mathrm{hv}(M), \mathcal{O}_M)$. Given a map of topological spaces $M' \to M$, we let $\mathcal{O}^{\mathrm{alg}}_M | M'$ denote the image of $\mathcal{O}^{\mathrm{alg}}_M$ under the pullback functor $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S}\mathrm{hv}(M)) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{S}\mathrm{hv}(M'))$. According to Remark 10.2, it will suffice to show that for every map of complex manifolds $f : X \to Y$ and every other complex manifold Z (all assumed to be Hausdorff and second countable), the diagram σ :

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{alg}}_{X\times Y} | (X \times Z) & \longrightarrow & \mathcal{O}^{\mathrm{alg}}_{X} | (X \times Z) \\ \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{alg}}_{X\times Y\times Z} | (X \times Z) & \longrightarrow & \mathcal{O}^{\mathrm{alg}}_{X\times Z} \end{array}$$

is a pushout square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}(\mathbf{C})}(\mathcal{S}\mathrm{hv}(X \times Z)) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}}}(X \times Z)$. The space $X \times Z$ is a paracompact and has finite covering dimension, so that $\mathcal{S}\mathrm{hv}(X \times Z)$ has finite homotopy dimension (Theorem T.7.2.3.6) and therefore has enough points. It will therefore suffice to show that for every point $x \in X$ and every point $z \in Z$, the pullback of σ to the point $(x, z) \in X \times Z$ is a pushout diagram in $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}}}(\{(x, z)\}) \simeq \mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}}$. Let $y = f(x)$ be the image of x in the complex manifold Y.

If M is a complex manifold with a point p, let $\mathcal{O}^{\mathrm{alg}}_p \in \mathrm{CAlg}_{\mathbf{C}}$ denote the discrete $\mathbf{C}$-algebra of germs of holomorphic functions on M at the point p. Unwinding the definitions, we are required to show that the commutative diagram τ :

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{alg}}_{(x,y)} & \longrightarrow & \mathcal{O}^{\mathrm{alg}}_{x} \\ \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{alg}}_{(x,y,z)} & \longrightarrow & \mathcal{O}^{\mathrm{alg}}_{(x,z)} \end{array}$$

is a pushout square of connective $\mathbb{E}_\infty$-algebras over $\mathbf{C}$. This assertion is local on X, Y, and Z. We may therefore assume that $Y \simeq \mathbf{C}^d$ for some $d \geq 0$. We now proceed using induction on d. If $d = 0$ there is nothing to prove. If $d \geq 2$, then we can write $Y = Y' \times Y''$, so that $y = (y', y'')$ for some $y' \in Y'$ and $y'' \in Y''$. We then have a larger commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^{\mathrm{alg}}_{(x,y)} & \longrightarrow & \mathcal{O}^{\mathrm{alg}}_{(x,y')} & \longrightarrow & \mathcal{O}^{\mathrm{alg}}_{x} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}^{\mathrm{alg}}_{(x,y,z)} & \longrightarrow & \mathcal{O}^{\mathrm{alg}}_{(x,y',z)} & \longrightarrow & \mathcal{O}^{\mathrm{alg}}_{(x,z)} \end{array}$$

where the left and right squares are pushout diagrams by the inductive hypothesis, so that the outer square is a pushout diagram as well. We may therefore assume that $d = 1$, so that $f : X \to Y$ can be identified with a holomorphic function on X.

Using Proposition A.7.2.1.19, we are reduced to proving the following pair of assertions:

(a) The diagram τ induces an isomorphism

$$\mathrm{Tor}_0^{\mathcal{O}^{\mathrm{alg}}_{(x,y)}}(\mathcal{O}^{\mathrm{alg}}_{x}, \mathcal{O}^{\mathrm{alg}}_{(x,y,z)}) \to \mathcal{O}^{\mathrm{alg}}_{(x,z)}.$$

In other words, the diagram τ is a pushout square in the ordinary category of discrete $\mathbf{C}$-algebras.

(b) For $p > 0$, the groups $\mathrm{Tor}_p^{\mathcal{O}^{\mathrm{alg}}_{(x,y)}}(\mathcal{O}^{\mathrm{alg}}_{x}, \mathcal{O}^{\mathrm{alg}}_{(x,y,z)})$ vanish.

Let $\pi_0 : X \times Y \to X$ and $\pi_1 : X \times Y \to Y = \mathbf{C}$ be the projections onto the first and second factor, respectively. Let g be the holomorphic function $f \circ \pi_0 - \pi_1$ on $X \times Y$, so that multiplication by g determines a short exact sequence

$$0 \to \mathcal{O}^{\mathrm{alg}}_{(x,y)} \xrightarrow{g} \mathcal{O}^{\mathrm{alg}}_{(x,y)} \to \mathcal{O}^{\mathrm{alg}}_{x} \to 0.$$

Using this exact sequence to compute the relevant Tor-groups, we see that (a) and (b) are equivalent to the exactness of the sequence

$$0 \to \mathcal{O}^{\mathrm{alg}}_{(x,y,z)} \to \mathcal{O}^{\mathrm{alg}}_{(x,y,z)} \to \mathcal{O}^{\mathrm{alg}}_{(x,z)} \to 0,$$

which is evident. □

Lemma 11.11. *Let $\mathcal{X}$ be an ∞-topos and let $f : \mathcal{O} \to \mathcal{O}'$ be a morphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{X})$. The following conditions are equivalent:*

(1) *The map f is an effective epimorphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}}(\mathcal{X})$.*

(2) *The map $f^{\mathrm{alg}} : \mathcal{O}^{\mathrm{alg}} \to \mathcal{O}'^{\,\mathrm{alg}}$ is an effective epimorphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}(\mathbf{C})}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{C}}}(\mathcal{X})$.*

(3) *The induced map $\mathcal{O}(\mathbf{C}) \to \mathcal{O}'(\mathbf{C})$ is an effective epimorphism in $\mathcal{X}$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are immediate from the definitions. Suppose that (3) is satisfied; we will show that f is an effective epimorphism. Fix an object $M \in \mathcal{T}_{\mathrm{an}}$; we wish to show that the map $\mathcal{O}(M) \to \mathcal{O}'(M)$ is an effective epimorphism. Choose a covering $\{U_\alpha \to M\}$, where each U_α is biholomorphic to an open subset of $\mathbf{C}^n$ for some n. We have a commutative diagram

$$\begin{array}{ccc} \coprod \mathcal{O}(U_\alpha) & \longrightarrow & \coprod \mathcal{O}'(U_\alpha) \\ \downarrow & & \downarrow \\ \mathcal{O}(M) & \longrightarrow & \mathcal{O}'(M). \end{array}$$

Since $\mathcal{O}$ and $\mathcal{O}'$ are local, the vertical maps are effective epimorphisms. It will therefore suffice to show that the upper horizontal map is an effective epimorphism. In other words, we may replace M by some U_α and thereby assume the existence of an open embedding $M \hookrightarrow \mathbf{C}^n$. Consider the diagram

$$\begin{array}{ccc} \mathcal{O}(M) & \longrightarrow & \mathcal{O}'(M) \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathbf{C}^n) & \longrightarrow & \mathcal{O}'(\mathbf{C}^n). \end{array}$$

Since f is local, this diagram is a pullback square. We are therefore reduced to showing that the map $\mathcal{O}(\mathbf{C}^n) \to \mathcal{O}'(\mathbf{C}^n)$ is an effective epimorphism. Since $\mathcal{O}$ and $\mathcal{O}'$ commute with finite products, we can reduce to the case $n = 1$ in which case the desired result follows from (3). □

Combining Lemmas 11.11, 11.10, and Proposition 10.3, we obtain the following result:

Proposition 11.12. *Suppose we are given morphisms*

$$(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \xrightarrow{f} (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xleftarrow{g} (\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$$

in $\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$. *Assume that the induced map* $\theta : f^* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}} \to \mathcal{O}_{\mathcal{Y}}^{\mathrm{alg}}$ *is surjective, in the sense that that the fiber of* θ *is connective (as a sheaf of spectra on* $\mathcal{Y}$*). Then:*

(1) *There exists a pullback diagram*

$$\begin{array}{ccc} (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) & \xrightarrow{f'} & (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) & \xrightarrow{f} & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

in $\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$.

(2) *The underlying diagram of* ∞*-topoi*

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

is a pullback square.

(3) *The diagram*

$$\begin{array}{ccc} f'^* g^* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}} & \longrightarrow & f'^* \mathcal{O}_{\mathcal{Y}}^{\mathrm{alg}} \\ \downarrow & & \downarrow \\ g'^* \mathcal{O}_{\mathcal{Y}}^{\mathrm{alg}} & \longrightarrow & \mathcal{O}_{\mathcal{Y}'}^{\mathrm{alg}} \end{array}$$

is a pushout square in $\mathrm{Shv}_{\mathrm{CAlg}_{\mathbf{C}}^{\mathrm{cn}}}(\mathcal{Y}')$.

(4) *The map* $\theta' : f'^* \mathcal{O}_{\mathcal{X}'} \to \mathcal{O}_{\mathcal{Y}'}$ *is surjective (that is, the fiber of* θ *is a connective sheaf of spectra on* $\mathcal{Y}'$*).*

Remark 11.13. All of the results of this section carry over without essential change if we replace the pregeometry $\mathcal{T}_{\mathrm{an}}$ of Definition 11.1 with the pregeometry $\mathcal{T}_{\mathrm{Diff}}$ described in §V.4.4. The essential point is that if M is a smooth manifold and $f(x,t)$ is a smooth function on $M \times \mathbf{R}$ which vanishes on $M \times \{0\}$, then $f(x,t) = tg(x,t)$ for a *unique* smooth function $g(x,t) : M \times \mathbf{R} \to \mathbf{R}$. Note it is essential here that we work with infinitely differentiable functions: the analogue of Lemma 11.10 fails in the setting of C^r-manifolds for $r < \infty$.

12 Derived Complex Analytic Spaces

In §11, we introduced the pregeometry $\mathcal{T}_{\text{an}}$ underlying complex analytic geometry. In this section, we will use $\mathcal{T}_{\text{an}}$ to introduce the notion of a *derived complex analytic space*. The collection of derived complex analaytic spaces forms an ∞-category, which we will denote by $\mathcal{A}\text{n}^{\text{der}}_{\mathbf{C}}$. We will show that the ∞-category $\mathcal{A}\text{n}^{\text{der}}_{\mathbf{C}}$ contains the usual category of complex analytic spaces as a full subcategory (Theorem 12.8.

We begin with a brief review of the theory of complex analytic spaces (see [24] for more details).

Definition 12.1. An *complex analytic space* is a pair $(X, \mathcal{O}^{\text{alg}}_X)$, where X is a topological space and $\mathcal{O}^{\text{alg}}_X$ is a sheaf of (discrete) commutative **C**-algebras on X, such that for every point $x \in X$, there exists an open neighborhood $U \subseteq X$ containing x, an open subset $V \subseteq \mathbf{C}^n$, and a map of ringed spaces $(U, \mathcal{O}^{\text{alg}}_X | U) \rightarrow (V, \mathcal{O}^{\text{alg}}_V)$ (here $\mathcal{O}^{\text{alg}}_V$ denotes the sheaf of germs of holomorphic functions on V) satisfying the following conditions:

(a) The underlying map of topological spaces $f : U \rightarrow V$ is a homeomorphism of U onto a closed subset of V.

(b) The map $\mathcal{O}^{\text{alg}}_V \rightarrow f_*(\mathcal{O}^{\text{alg}}_X | U)$ is an epimorphism of sheaves, whose kernel is a coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}^{\text{alg}}_V$.

The collection of complex analytic spaces forms a category, which we will denote by $\mathcal{A}\text{n}_{\mathbf{C}}$ (it is a full subcategory of the topological spaces equipped with a sheaf of commutative **C**-algebras).

Remark 12.2. If $(X, \mathcal{O}^{\text{alg}}_X)$ is a complex analytic space, then X is (canonically) homeomorphic to the space of points of the locale of open subsets of X. In other words, the topological space X is sober: this follows from Lemma V.2.5.17 (since X is locally Hausdorff, and any Hausdorff topological space is sober).

We now introduce a derived version of the theory of complex analytic spaces.

Definition 12.3. Let $\mathcal{X}$ be an ∞-topos and $\mathcal{O}_{\mathcal{X}}$ a $\mathcal{T}_{\text{an}}$-structure on $\mathcal{X}$. We will say that the pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a *derived complex analytic space* if there exists an effective epimorphism $\coprod_i U_i \rightarrow \mathbf{1}$ in $\mathcal{X}$ (where $\mathbf{1}$ denotes the final object of $\mathcal{X}$) satisfying the following conditions, for each index i:

(a) The ∞-topos $\mathcal{X}_{/U_i}$ can be identified with the category of sheaves on a topological space X_i.

(b) The pair $(X_i, \pi_0(\mathcal{O}^{\text{alg}}_{\mathcal{X}} | U_i))$ is a complex analytic space (in the sense of Definition 12.1).

(c) For each $k \geq 0$, $\pi_k(\mathcal{O}^{\text{alg}}_{\mathcal{X}} | U_i)$ is a coherent sheaf of $\pi_0(\mathcal{O}^{\text{alg}}_{\mathcal{X}} | U_i)$-modules.

We let $\mathcal{A}\text{n}^{\text{der}}_{\mathbf{C}}$ denote the full subcategory of $\mathcal{T}\text{op}(\mathcal{T}_{\text{an}})$ spanned by the derived complex analytic spaces.

Remark 12.4. Let $\mathcal{O}$ be a $\mathcal{T}_{\text{an}}$-structure on an ∞-topos $\mathcal{X}$. Then $(\mathcal{X}, \mathcal{O})$ is a derived complex analytic space if and only if, for each $n \geq 0$, the truncation $(\mathcal{X}, \tau_{\leq n} \mathcal{O})$ is a derived complex analytic space.

Our main goal in this section is to show that Definition 12.3 generalizes Definition 12.1. Our first step is to make the relationship between these definitions more precise.

Construction 12.5. Let $(X, \mathcal{O}^{\text{alg}}_X)$ be a complex analytic space. We define a functor $\mathcal{O}_X : \mathcal{T}_{\text{an}} \rightarrow \mathcal{S}\text{hv}(X)$ by the formula

$$\mathcal{O}_X(M)(U) = \text{Hom}_{\mathcal{A}\text{n}_{\mathbf{C}}}((U, \mathcal{O}^{\text{alg}}_X | U), (M, \mathcal{O}^{\text{alg}}_M)),$$

where $\mathcal{O}^{\text{alg}}_M$ denote the sheaf of germs of holomorphic functions on the complex manifold $M \in \mathcal{T}_{\text{an}}$.

Lemma 12.6. *Let $(X, \mathcal{O}^{\text{alg}}_X)$ be a complex analytic space. Then $\mathcal{O}_X$ is a 0-truncated $\mathcal{T}_{\text{an}}$-structure on the ∞-topos $\mathcal{S}\text{hv}(X)$.*

Proof. The only nontrivial point is to verify that if $\{M_i \to M\}$ is an admissible cover of $M \in \mathcal{T}_{\mathrm{an}}$, then the induced map $\coprod_i \mathcal{O}_X(M_i) \to \mathcal{O}_X(M)$ is an effective epimorphism in $\mathcal{S}\mathrm{hv}(X)$. Unwinding the definitions, we must show that if $x \in X$ and $f : (U, \mathcal{O}_X^{\mathrm{alg}} | U) \to (M, \mathcal{O}_M^{\mathrm{alg}})$ is a map of complex analytic spaces defined in a neighborhood U of X, then (after shrinking U) we can assume that f factors through some map $(U, \mathcal{O}_X^{\mathrm{alg}} | U) \to (M_i, \mathcal{O}_{M_i}^{\mathrm{alg}})$. Choose any i such that $f(x) \in M$ is the image of some point $p \in M_i$. Since the map $q : M_i \to M$ is locally biholomorphic, we can choose an open sets $V \subseteq M_i$ containing p such that $q|V$ is an open embedding. Shrinking U, we can assume that $f(U) \subseteq q(V)$. Then f factors through $(V, \mathcal{O}_{M_i}^{\mathrm{alg}} | V)$. □

The construction $(X, \mathcal{O}_X^{\mathrm{alg}}) \mapsto (\mathcal{S}\mathrm{hv}(X), \mathcal{O}_X)$ determines a functor $\Phi : \mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}}) \to \mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$.

Remark 12.7. The notations of Constructions 12.5 and 11.7 are consistent: if $(X, \mathcal{O}_X^{\mathrm{alg}})$ is a complex analytic space, then the underlying sheaf of $\mathbf{C}$-algebras on $\mathcal{S}\mathrm{hv}(X)$ associated to $\mathcal{O}_X$ can be identified with the original structure sheaf $\mathcal{O}_X^{\mathrm{alg}}$. It follows that $(\mathcal{S}\mathrm{hv}(X), \mathcal{O}_X)$ is a derived complex analytic space.

Theorem 12.8. *The functor $\Phi : \mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}}) \to \mathcal{A}\mathrm{n}_{\mathbf{C}}^{\mathrm{der}}$ is a fully faithful embedding, whose essential image consists of those derived complex analytic spaces $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{X}$ is 0-localic and $\mathcal{O}_{\mathcal{X}}$ is 0-truncated.*

As a first step toward proving Theorem 12.8, we construct a partial inverse to Φ. Let ${\mathcal{A}\mathrm{n}_{\mathbf{C}}^{\mathrm{der}}}_{\leq 0}$ denote the full subcategory of $\mathcal{A}\mathrm{n}_{\mathbf{C}}^{\mathrm{der}}$ spanned by the derived complex analytic spaces $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ for which $\mathcal{X}$ is 0-localic. The collection of (-1)-truncated objects of $\mathcal{X}$ forms a locale, and Lemma V.2.5.21 implies that this locale has enough points. We therefore obtain an equivalence $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(X)$ for a sober topological space X, which is determined up to homeomorphism. Under this equivalence, we can identify $\mathcal{O}_{\mathcal{X}}$ with a $\mathcal{T}$-structure $\mathcal{O}_X$ on $\mathcal{S}\mathrm{hv}(X)$. It follows immediately from the definitions that $(X, \pi_0 \, \mathcal{O}_X^{\mathrm{alg}})$ is a complex analytic space in the sense of Definition 12.1. The construction $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (X, \pi_0 \, \mathcal{O}_X^{\mathrm{alg}})$ determines a functor

$$\Psi : {\mathcal{A}\mathrm{n}_{\mathbf{C}}^{\mathrm{der}}}_{\leq 0} \to \mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}}).$$

Lemma 12.9. *The composition $\Psi \circ \Phi$ is equivalent to the identity functor from $\mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}})$ to itself.*

Proof. Note that Φ takes values in the full subcategory ${\mathcal{A}\mathrm{n}_{\mathbf{C}}^{\mathrm{der}}}_{\leq 0} \subseteq \mathcal{A}\mathrm{n}_{\mathbf{C}}^{\mathrm{der}}$, so that $\Psi \circ \Phi$ is well-defined. The existence of a canonical equivalence of $\Psi \circ \Phi$ with the identity functor follows easily from Remark 12.2. □

To complete the proof of Theorem 12.8, it would suffice to show that $\Phi \circ \Psi$ is equivalent to the identity functor when restricted to the full subcategory of $\mathcal{A}\mathrm{n}_{\mathbf{C}}^{\mathrm{der}}$ spanned by the 0-truncated, 0-localic derived complex analytic spaces. It is not difficult to prove this directly. However, we will adopt another approach, which give a great deal more information about the ∞-category $\mathcal{A}\mathrm{n}_{\mathbf{C}}^{\mathrm{der}}$.

Proposition 12.10. *Suppose we are given maps of derived complex analytic spaces*

$$(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \xrightarrow{f} (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \leftarrow (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}),$$

where f is a closed immersion. Then:

(1) *There exists a pullback diagram σ :*

$$\begin{array}{ccc} (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) & \xrightarrow{f'} & (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \\ \downarrow & & \downarrow g \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) & \xrightarrow{f} & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

in the ∞-category $\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$.

(2) *The image of σ in $\mathcal{T}\mathrm{op}$ is a pullback diagram of ∞-topoi.*

(3) *The map f' is a closed immersion.*

(4) *The pair $(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$ is a derived complex analytic space.*

(5) *Assume that $\mathcal{X}'$, $\mathcal{X}$, and $\mathcal{Y}$ are 0-localic. Then $\mathcal{Y}'$ is 0-localic. Moreover, $\Psi(\sigma)$ is a pullback diagram in the category $\mathcal{A}\mathrm{n}_{\mathbf{C}}$ of complex analytic spaces.*

Lemma 12.11. *Let $f : (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a map of derived complex analytic spaces, where $\mathcal{X} = \mathcal{S}\mathrm{hv}(X)$ and $\mathcal{X}' = \mathcal{S}\mathrm{hv}(X')$ so that the pairs $(X, \pi_0 \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}})$ and $(X', \pi_0 \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}'})$ are complex analytic spaces. Suppose that $\mathcal{F}$ is a connective sheaf of $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$-modules on $\mathcal{X}$, and that each $\pi_k \mathcal{F}$ is a coherent sheaf of $\pi_0 \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$-modules. Then the tensor product $\mathcal{F}' = f^* \mathcal{F} \otimes_{f^* \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}} \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}'}$ is connective, and each $\pi_k \mathcal{F}'$ is a coherent sheaf of $\pi_0 \mathcal{O}_{\mathcal{X}'}$-modules.*

Proof. We first prove that for every $m \geq -1$ and every $x \in X$, there exists an open set $U \subseteq X$ containing X and a sequence of morphisms

$$0 = \mathcal{F}(-1) \to \mathcal{F}(0) \to \mathcal{F}(1) \to \cdots \to \mathcal{F}(m) \to \mathcal{F}\,|U$$

of $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}\,|U$-modules with the following properties:

(*a*) For $0 \leq i \leq m$, the fiber of the map $\mathcal{F}(i-1) \to \mathcal{F}(i)$ is equivalent to a direct sum of finitely many copies of $(\mathcal{O}_{\mathcal{X}}\,|U)[i]$.

(*b*) For $0 \leq i \leq m$, the fiber of the map $\mathcal{F}(i) \to \mathcal{F}\,|U$ is i-connective.

(*c*) For $-1 \leq i \leq m$, the homotopy groups $\pi_j \mathcal{F}(i)$ are coherent $(\pi_0 \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}})|U$-modules, which vanish for $j < 0$.

We proceed by induction on m. In case $m = -1$, we simply take $\mathcal{F}(-1) = 0$. Condition (*b*) holds in this case since $\mathcal{F}$ is assumed to be connective, and condition (*c*) is obvious. Assume therefore that the above conditions hold for some integer $m \geq 0$. Replacing X by U, we assume that we are given a sequence

$$0 = \mathcal{F}(0) \to \cdots \to \mathcal{F}(m) \to \mathcal{F}$$

satisfying the above conditions. Let $\mathcal{F}'$ be the fiber of the map $\mathcal{F}(m) \to \mathcal{F}$, so that $\mathcal{F}'$ is m-connective. We have an exact sequence

$$\pi_{m+1} \mathcal{F}(m) \to \pi_{m+1} \mathcal{F} \to \pi_m \mathcal{F}' \to \pi_m \mathcal{F}(m) \to \pi_m \mathcal{F}.$$

It follows that $\pi_m \mathcal{F}'$ is a coherent sheaf of $\pi_0 \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$-modules. In particular, there exists an open neighborhood U of X and a finite collection of sections which generate $\pi_m \mathcal{F}'$ as a module over $\pi_0 \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$. Shrinking X further, we obtain finitely many global sections which determine a map $(\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}})^n[m] \to \mathcal{F}'$. Let $\mathcal{F}(m+1)$ denote the cofiber of the composite map $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}})^n[m] \to \mathcal{F}' \to \mathcal{F}(m)$. It is obvious that $\mathcal{F}(m+1)$ satisfies condition (*a*). Let $\mathcal{F}''$ denote the fiber of the map $\mathcal{F}(m+1) \to \mathcal{F}$, so that we have a fiber sequence

$$(\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}})^n[m] \to \mathcal{F}' \to \mathcal{F}''.$$

The corresponding long exact sequence of homotopy groups shows that $\mathcal{F}''$ is $(m+1)$-connective, which proves (*b*). Finally, condition (*c*) follows immediately from the long exact sequence of homotopy groups

$$(\pi_{j-m} \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}})^n \to \pi_j \mathcal{F}(m) \to \pi_j \mathcal{F}(m+1) \to (\pi_{j-m-1} \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}})^n \to \pi_{j-1} \mathcal{F}(m).$$

The connectivity of $\mathcal{F}' = f^* \mathcal{F} \otimes_{f^* \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}} \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}'}$ is obvious. Fix $k \geq 0$; we wish to show that $\pi_k \mathcal{F}'$ is a coherent sheaf of $\pi_0 \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}'}$-modules. The assertion is local on X; we may therefore assume that there exists a sequence

$$0 = \mathcal{F}(-1) \to \mathcal{F}(0) \to \cdots \to \mathcal{F}(k+1) \to \mathcal{F}$$

satisfying conditions (a), (b), and (c). In particular, the fiber $\mathcal{F}'$ of the map $\mathcal{F}(k+1) \to \mathcal{F}$ is $(k+1)$-connective. It follows that the induced map

$$f^* \mathcal{F}(k+1) \otimes_{f^* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}} \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}} \to f^* \mathcal{F} \otimes_{f^* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}} \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}}$$

also has a $(k+1)$-connective fiber, and therefore induces an isomorphism

$$\pi_k(f^* \mathcal{F}(k+1) \otimes_{f^* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}} \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}}) \to \pi_k(f^* \mathcal{F} \otimes_{f^* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}} \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}}).$$

We will complete the proof by showing that for $-1 \leq i \leq k+1$, $\pi_k(f^* \mathcal{F}(i) \otimes_{f^* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}} \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}})$ is a coherent sheaf of $\pi_0 \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}}$-modules. The proof proceeds by induction on i, the case $i = -1$ being obvious. To handle the inductive step, it suffices to observe that the fiber sequence

$$(\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}})^n[i] \to \mathcal{F}(i) \to \mathcal{F}(i+1)$$

induces a long exact sequence

$$\cdots \to (\pi_{k-i} \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}})^n \to \pi_k(f^* \mathcal{F}(i) \otimes_{f^* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}} \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}}) \to \pi_k(f^* \mathcal{F}(i+1) \otimes_{f^* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}} \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}}) \to (\pi_{k-i-1} \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}})^n \to \cdots .$$

□

Proof of Proposition 12.10. Assertions (1), (2), and (3) follow from Proposition 11.12. We will prove (4). The assertion is local on $\mathcal{Y}'$; we may therefore assume that $\mathcal{X}' = \mathcal{S}\mathrm{hv}(X')$, $\mathcal{Y} = \mathcal{S}\mathrm{hv}(Y)$, and $\mathcal{X} = \mathcal{S}\mathrm{hv}(X)$ for some topological spaces X', Y, and X (which are the underlying topological spaces of complex analytic spaces). Using the map f, we will identify Y with a closed subset of X, whose inverse image in X' is a closed subset $Y' \subseteq X'$ such that $\mathcal{S}\mathrm{hv}(Y) \simeq \mathcal{Y}'$ (see Corollary T.7.3.2.10). Note that f induces a closed immersion $(Y, \pi_0 \mathcal{O}_Y^{\mathrm{alg}}) \to (X, \pi_0 \mathcal{O}_X^{\mathrm{alg}})$. It follows that for each $k \geq 0$, the pushforward $f_* \pi_k \mathcal{O}_Y^{\mathrm{alg}}$ is a coherent sheaf of $\pi_0 \mathcal{O}_X^{\mathrm{alg}}$-modules on X. Using Lemma 12.11 and Proposition 11.12, we conclude that for each $k \geq 0$, the pushforward $f'_* \pi_k \mathcal{O}_{\mathcal{Y}'}^{\mathrm{alg}}$ is a coherent sheaf of $\pi_0 \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}}$-modules. Taking $k = 0$, it will follow that $(Y', \pi_0 \mathcal{O}_{\mathcal{Y}'}^{\mathrm{alg}})$ is a complex analytic space and that f' induces a closed immersion of complex analytic spaces $(Y', \pi_0 \mathcal{O}_{\mathcal{Y}'}^{\mathrm{alg}}) \to (X', \pi_0 \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}})$. Then each $\pi_k \mathcal{O}_{\mathcal{Y}'}^{\mathrm{alg}}$ is a coherent $\pi_0 \mathcal{O}_{\mathcal{Y}'}^{\mathrm{alg}}$-module, since its pushforward is coherent as a $\pi_0 \mathcal{O}_{\mathcal{X}'}^{\mathrm{alg}}$-algebra.

The first assertion of (5) follows immediately from (3). The assertion that $\Psi(\sigma)$ is a pullback diagram in $\mathcal{A}\mathrm{n}_{\mathbf{C}}$ follows from the description of $\pi_0 \mathcal{O}_{\mathcal{Y}}^{\mathrm{alg}}$ supplied by Proposition 11.12. □

We next use Proposition 12.10 to establish the existence of fiber products in the ∞-category $\mathcal{A}\mathrm{n}_{\mathbf{C}}^{\mathrm{der}}$ of derived complex analytic spaces.

Proposition 12.12. *Suppose we are given maps of derived complex analytic spaces*

$$(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \leftarrow (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}).$$

Then:

(1) *There exists a pullback diagram σ :*

$$\begin{array}{ccc} (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) & \longrightarrow & (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \\ \downarrow & & \downarrow{\scriptstyle g} \\ (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) & \longrightarrow & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \end{array}$$

in the ∞-category $\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$.

(2) *The pair* $(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$ *is a derived complex analytic space.*

(3) *The image of* σ *in* $\mathcal{T}\mathrm{op}$ *is a pullback diagram of* ∞*-topoi.*

The proof of Proposition 12.12 will require some preliminaries.

Lemma 12.13. *Let* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *be a derived complex analytic space, and let* $\mathbf{1}$ *denote the final object of* $\mathcal{X}$*. Then there exists an effective epimorphism* $\coprod U_i \to \mathbf{1}$ *in* $\mathcal{X}$ *and a collection of closed immersions* $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(\mathbf{C}^{n_i})$*, where each* $M_i \in \mathcal{T}_{\mathrm{an}}$ *is a complex manifold.*

Proof. Without loss of generality we may assume that $\mathcal{X} = \mathcal{S}\mathrm{hv}(X)$, where $(X, \pi_0 \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}})$ is a complex analytic space. The assertion is local on X. We may therefore assume that there exists an open subset $U \subseteq \mathbf{C}^n$ and a closed immersion of complex analytic spaces $(X, \pi_0 \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}) \to (U, \mathcal{O}_U^{\mathrm{alg}})$. The composite map $(X, \pi_0 \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}) \to (\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}^{\mathrm{alg}})$ is determined by a sequence of n global sections of the sheaf $\pi_0 \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$. Shrinking X and U, we may assume that these global sections are given by $f_1, \ldots, f_n \in \mathcal{O}_{\mathcal{X}}(\mathbf{C})(X)$. These sections determine a morphism of derived complex analytic spaces $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}} \mathbf{C}^n$. The morphism f determines a continuous map $X \to \mathbf{C}^n$ which (by construction) factors through U, so that f is given by a composition a composition

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{f'} \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(U) \to \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(\mathbf{C}^n)$$

(see Remark V.2.3.4). It follows from Lemma 11.11 that f' is a closed immersion. □

Lemma 12.14. *Let* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *and* $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ *be derived complex analytic spaces. Then:*

(1) *There exists a product* $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \simeq (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \times (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ *in the* ∞*-category* ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})^{op}$*.*

(2) *The pair* $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ *is a derived complex analytic space.*

(3) *The underlying* ∞*-topos of* $\mathcal{Z}$ *is a product of* $\mathcal{X}$ *and* $\mathcal{Y}$ *in* $\mathcal{T}\mathrm{op}$*.*

(4) *Suppose that* $\mathcal{X}$ *and* $\mathcal{Y}$ *are* 0*-localic, so that* $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(X)$ *and* $\mathcal{Y} = \mathcal{S}\mathrm{hv}(Y)$ *for some topological spaces* X *and* Y*. Then* $\mathcal{Z} \simeq \mathcal{S}\mathrm{hv}(X \times Y)$*; in particular,* $\mathcal{Z}$ *is* 0*-localic.*

(5) *The functor* $\Psi : \mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}\,\leq 0} \to \mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}})$ *preserves finite products.*

Proof. Assertions (1), (2), and (3) are of a local nature; we may therefore assume that there exist closed immersions $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(M)$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(M')$. Using Proposition 12.10, we can reduce to the case $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(M)$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \simeq \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(M')$. Then $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) = \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(M \times M')$; this proves the first three assertions. Assertion (4) follows from (3) and Proposition T.7.3.1.11. To prove (5), we can again work locally and use Proposition 12.10 to reduce to the case where $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ lie in the essential image of $\mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}$. It then suffices to observe that if M and M' are complex manifolds, then the complex analytic space $(M \times M', \mathcal{O}^{\mathrm{alg}}_{M \times M'})$ is a product of $(M, \mathcal{O}^{\mathrm{alg}}_M)$ with $(M', \mathcal{O}^{\mathrm{alg}}_{M'})$ in the category $\mathcal{A}\mathrm{n}_{\mathbf{C}}$. □

Lemma 12.15. *Let* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *be a derived complex analytic space such that* $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(X)$*, where* X *is a Hausdorff topological space. Then the diagonal map* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \times (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *is a closed immersion of* 0*-localic derived complex analytic spaces.*

Proof. Using Lemma 12.14 we can identify the underlying ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \times (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ with $\mathcal{S}\mathrm{hv}(X \times X)$. Since X is Hausdorff, the diagonal map $\delta_* : \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}\mathrm{hv}(X \times X)$ is a closed immersion of ∞-topoi. Let $\mathcal{O}_{\mathcal{X} \times \mathcal{X}}$ be the structure sheaf of the product $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \times (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. The map $\delta^* \mathcal{O}_{\mathcal{X} \times \mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is obviously an effective epimorphism, since it admits a section. □

Proof of Proposition 12.12. Working locally on $\mathcal{X}$, we may assume that $\mathcal{X} = \mathcal{S}\mathrm{hv}(X)$ for some Hausdorff topological space X. Using Lemma 12.14, we deduce the existence of a product $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) = (\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \times (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in $\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$. Similarly, we have a product $(\mathcal{S}\mathrm{hv}(X \times X), \mathcal{O}_{X \times X}) = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \times (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Lemma 12.15 shows that the diagonal map $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (X \times X, \mathcal{O}_{X\times X})$ is a closed immersion. We now apply Proposition 12.10 to the produce a pullback square

$$\begin{array}{ccc} (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) & \longrightarrow & (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \\ \downarrow & & \downarrow \\ (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) & \longrightarrow & (\mathcal{S}\mathrm{hv}(X \times X), \mathcal{O}_{X\times X}). \end{array}$$

It is easy to see that $(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$ has the desired properties. □

Corollary 12.16. *The ∞-category $\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}$ of derived complex analytic spaces admits finite limits. The full subcategory $\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}{}_{\leq 0}$ of 0-localic derived complex analytic spaces is closed under finite limits in $\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}$. The forgetful functor $\Psi : \mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}{}_{\leq 0} \to \mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}})$ preserves finite limits.*

Proof. Since $\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}$ contains a final object (given applying the functor $\mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}$ to a single point) and admits finite limits (Proposition 12.12, it admits all finite limits by Corollary T.4.4.2.4. It follows from Proposition 12.12 that the forgetful functor $\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}} \to \mathcal{T}\mathrm{op}$ preserves finite limits (use Corollary T.4.4.2.5). Since the collection of 0-localic ∞-topoi is stable under small limits in $\mathcal{T}\mathrm{op}$, we conclude that $\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}{}_{\leq 0}$ is stable under finite limits in $\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}$. It remains to prove that Ψ preserves finite limits. Since Ψ obviously preserves final objects, it will suffice to show that Ψ preserves pullback squares. This follows as in the proof of Proposition 12.12, since Ψ preserves finite products (Lemma 12.14) and pullbacks along closed immersions (Proposition 12.10). □

Example 12.17. Let V be a finite dimensional complex vector space. We will regard V as a complex manifold, so that $\mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(V) \in \mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}$ is defined and equipped with a canonical base point (given by the origin in V). By iterated application of Proposition 12.12, we deduce that there exists an n-fold loop space $\Omega^n \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(V) \in \mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}$. Unwinding the definitions, we can characterize $\Omega^n \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(V)$ by the following universal property: for every object $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$, there is a canonical homotopy equivalence

$$\mathrm{Map}_{\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})}((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), \Omega^n \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(V)) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{C}}}(V^{\vee}[n], \Gamma(\mathcal{X}, \mathcal{O}^{\mathrm{alg}}_{\mathcal{X}})),$$

where $V^{\vee}$ denote the dual space of V.

Lemma 12.18. *Let $n \geq 1$ and let V be a finite dimensional complex vector space. Then $\Omega^n \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(V)$ is equivalent to $(\mathcal{S}\mathrm{hv}(*), \mathcal{O})$, and the canonical map $V^{\vee}[n] \to \mathcal{O}^{\mathrm{alg}}$ induces an equivalence $\mathrm{Sym}^*(V[n]) \simeq \mathcal{O}^{\mathrm{alg}}$. in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}_{\mathbf{C}}}(*) \simeq \mathrm{CAlg}_{\mathbf{C}}$.*

Proof. We proceed by induction on n. Assume first that $n = 1$. We begin by treating the case $V = \mathbf{C}$. We have a pullback diagram of derived complex analytic spaces

$$\begin{array}{ccc} \Omega \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(\mathbf{C}) & \longrightarrow & \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(*) \\ \downarrow & & \downarrow \\ \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(*) & \xrightarrow{f} & \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(\mathbf{C}), \end{array}$$

where the lower horizontal map is closed immersion. It follows from Proposition 11.12 that $\Omega \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}(\mathbf{C})} \simeq (\mathcal{S}\mathrm{hv}(*), \mathcal{O})$, where $\mathcal{O}^{\mathrm{alg}} \simeq \mathbf{C} \times_{f^* \mathcal{O}^{\mathrm{alg}}_{\mathbf{C}}} \mathbf{C}$. Here $f^* \mathcal{O}^{\mathrm{alg}}_{\mathbf{C}}$ can be identified with the (discrete) commutative ring R of germs of holomorphic functions at the origin. Let z denote the germ of the identity map from $\mathbf{C}$ to itself, regarded as an object of R; then z determines a map of polynomial rings

$$\mathrm{Sym}^*(\mathbf{C}) \simeq \mathbf{C}[t] \xrightarrow{f} R$$

given by $t \mapsto z$. We wish to prove that f induces an equivalence of $\mathbb{E}_\infty$-rings

$$\mathbf{C} \otimes_{\mathbf{C}[t]} \mathbf{C} \to \mathbf{C} \otimes_R \mathbf{C}.$$

In other words, we wish to prove that the map $\operatorname{Tor}_i^{\mathbf{C}[t]}(\mathbf{C}, \mathbf{C}) \to \operatorname{Tor}_i^R(\mathbf{C}, \mathbf{C})$ is an isomorphism for $i \geq 0$. To prove this, we note that we have a map of projective resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{C}[t] & \xrightarrow{t} & \mathbf{C}[t] & \longrightarrow & \mathbf{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R & \xrightarrow{z} & R & \longrightarrow & \mathbf{C} & \longrightarrow & 0. \end{array}$$

We now continue to assume that $n = 1$, but allow the vector space V to be arbitrary. Choose a basis $\{\lambda_1, \ldots, \lambda_m\}$ for $V^\vee$, inducing an isomorphism of complex manifolds $V \to \prod_{1 \leq i \leq m} \mathbf{C}$. Then $\Omega \operatorname{Spec}^{\mathcal{T}_{\mathrm{an}}}(V)$ can be identified with the product $\prod_{1 \leq i \leq m} (\Omega \operatorname{Spec}^{\mathcal{T}}_{\mathrm{an}}(\mathbf{C}))$. The argument above shows that the projection $\Omega \operatorname{Spec}^{\mathcal{T}}_{\mathrm{an}}(\mathbf{C}) \to \operatorname{Spec}^{\mathcal{T}}_{\mathrm{an}}(*)$ is a closed immersion. By iteratively applying Proposition 11.12, we deduce that $\prod_{1 \leq i \leq m} (\Omega \operatorname{Spec}^{\mathcal{T}}_{\mathrm{an}}(\mathcal{C})) \simeq (\mathcal{S}\mathrm{hv}(*), \mathcal{O}')$, where $\mathcal{O}'^{\mathrm{alg}}$ is a tensor product of m copies of the algebra $\mathcal{O}$ appearing above. It follows that the canonical map $\theta : \operatorname{Sym}^* V^\vee[1] \to \mathcal{O}'^{\mathrm{alg}}$ is a tensor product of m copies of the equivalence $\operatorname{Sym}^*(\mathbf{C}[1]) \to \mathcal{O}^{\mathrm{alg}}$ appearing above, and is therefore an equivalence.

If $n > 1$, we have a pullback diagram of derived complex analytic spaces

$$\begin{array}{ccc} \Omega^n \operatorname{Spec}^{\mathcal{T}_{\mathrm{an}}}(V) & \longrightarrow & \operatorname{Spec}^{\mathcal{T}_{\mathrm{an}}}(*) \\ \downarrow & & \downarrow \\ \operatorname{Spec}^{\mathcal{T}_{\mathrm{an}}}(*) & \xrightarrow{f} & \Omega^{n-1} \operatorname{Spec}^{\mathcal{T}_{\mathrm{an}}}(V). \end{array}$$

Using the inductive hypothesis, we deduce again that the lower horizontal map is a closed immersion; the desired result now follows immediately by combining the inductive hypothesis with Proposition 11.12. □

Lemma 12.19. *Let $(\mathcal{S}\mathrm{hv}(X), \mathcal{O}_X)$ be a 0-localic derived complex analytic space and let $V \simeq \mathbf{C}^n$ be a finite dimensional complex vector space. Fix $n \geq 0$ and a map $V^\vee[n] \to \Gamma(X; \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}})$ which is adjoint to a map of $\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$-modules $\phi : (V^\vee \otimes_{\mathbf{C}} \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}})[n] \to \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \Omega^n \operatorname{Spec}^{\mathcal{T}_{\mathrm{an}}}(V)$ be the corresponding map of derived complex analytic spaces, and form a pullback diagram*

$$\begin{array}{ccc} (\mathcal{S}\mathrm{hv}(X'), \mathcal{O}_{X'}) & \xrightarrow{g} & (X, \mathcal{O}_X) \\ \downarrow & & \downarrow f \\ \operatorname{Spec}^{\mathcal{T}_{\mathrm{an}}}(*) & \longrightarrow & \Omega^n \operatorname{Spec}^{\mathcal{T}_{\mathrm{an}}}(V). \end{array}$$

Then:

(1) *If $n = 0$, then g induces an identification of $(X', \pi_0 \mathcal{O}_{X'}^{\mathrm{alg}})$ with the closed complex analytic subspace of $(X, \pi_0 \mathcal{O}_X^{\mathrm{alg}})$ defined by the coherent ideal sheaf* $\operatorname{im}(\phi)$.

(2) *If $n > 0$, then g induces a homeomorphism $X' \simeq X$ and a map $\alpha : \operatorname{cofib}(\phi) \to \mathcal{O}_{X'}^{\mathrm{alg}}$ whose cofiber is $(2n+2)$-connective.*

Proof. Assertion (1) follows immediately from Proposition 11.12. Suppose that $n > 0$, and let $R = \operatorname{Sym}^* V[n]$. Using Proposition 11.12 and Lemma 12.18, we can identify X' with X and $\mathcal{O}_{X'}^{\mathrm{alg}}$ with $\mathbf{C} \otimes_R \mathcal{O}_X^{\mathrm{alg}}$. Let M denote the cofiber of the canonical map $\beta : V[n] \otimes_{\mathbf{C}} R \to R$. We observe that β induces an isomorphism $\pi_i(V[n] \otimes_{\mathbf{C}} R) \to \pi_i R$ for $0 < i < 2n$, and an surjection when $i = 2n$. It follows that $\pi_i M \simeq 0$ for

$0 < i \leq 2n$, so that the cofiber of the map $M \to \mathbf{C}$ is $(2n+2)$-connective. We conclude that the induced map

$$\mathrm{cofib}(\phi) \simeq M \otimes_R \mathcal{O}_X^{\mathrm{alg}} \to \mathbf{C} \otimes_R \mathcal{O}_X^{\mathrm{alg}} \simeq \mathcal{O}_{X'}^{\mathrm{alg}}$$

has $(2n+2)$-connective cofiber. □

In what follows, let us choose a geometric envelope $\theta : \mathcal{T}_{\mathrm{an}} \to \mathcal{G}_{\mathrm{an}}$ for the pregeometry $\mathcal{T}_{\mathrm{an}}$. If $K \in \mathcal{G}_{\mathrm{an}}$, we use the functor θ to identify $\mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K)$ with an object in $\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$, which we will denote by $(\mathcal{X}_K, \mathcal{O}_K)$.

Remark 12.20. Let V be a finite dimensional complex vector space, and regard V as a (pointed) object of $\mathcal{T}_{\mathrm{an}}$. Since $\mathcal{G}_{\mathrm{an}}$ admits finite limits, there exists an n-fold loop space $\Omega^n \theta(V) \in \mathcal{G}_{\mathrm{an}}$. We note that $\mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(\Omega^n \theta(V))$ can be identified with the derived complex analytic space $\Omega^n \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(V)$ of Example 12.17.

Proposition 12.21. *For each object $K \in \mathcal{G}_{\mathrm{an}}$, the spectrum $\mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K) \in \mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$ is a derived complex analytic space.*

Proof. Let $\mathcal{C} \subseteq \mathcal{G}_{\mathrm{an}}$ be the full subcategory spanned by those objects $K \in \mathcal{G}_{\mathrm{an}}$ such that $\mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K)$ is a derived complex analytic space. It follows from Proposition 12.12 that $\mathcal{C}$ is closed under finite limits. Since $\mathcal{C}$ is also closed under retracts and contains the essential image of $\theta : \mathcal{T}_{\mathrm{an}} \to \mathcal{G}_{\mathrm{an}}$, it follows that $\mathcal{C} = \mathcal{G}_{\mathrm{an}}$. □

Corollary 12.22. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})^{op}$ be a $\mathcal{G}_{\mathrm{an}}$-scheme which is locally of finite presentation. Then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a derived complex analytic space.*

The converse of Corollary 12.22 is false. However, we have the following slightly weaker result:

Proposition 12.23. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a $\mathcal{T}_{\mathrm{an}}$-structured ∞-topos. The following conditions are equivalent:*

(1) *The pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a derived complex analytic space.*

(2) *For each $n \geq 0$, there exists an effective epimorphism $\coprod U_i \to \mathbf{1}$ in $\mathcal{X}$ (where $\mathbf{1}$ denotes the final object of $\mathcal{X}$) a collection of objects $K_i \in \mathcal{G}_{\mathrm{an}}$, and a collection of maps $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}} | U_i) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(X_i)$ which induce equivalences $f_i^* : \mathcal{X}_{K_i} \simeq \mathcal{X}_{/U_i}$ and $f_i^* \tau_{\leq n} \mathcal{O}_{K_i} \to \mathcal{O}_{\mathcal{X}} | U_i$.*

Proof. Suppose first that condition (2) is satisfied. We claim that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a derived complex analytic space. In view of Remark 12.4, it suffices to show that $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ is a derived complex analytic space for each $n \geq 0$. This assertion is local on $\mathcal{X}$; using (2), we can assume that there exists $K \in \mathcal{G}_{\mathrm{an}}$ such that $\mathcal{X} = \mathcal{X}_K$ and $\tau_{\leq n} \mathcal{O}_{\mathcal{X}} = \tau_{\leq n} \mathcal{O}_K$. The desired result now follows from Remark 12.4 and Proposition 12.21.

Now suppose that $\mathcal{X}$ is a derived complex analytic space. In view of Proposition 11.9, it will suffice to prove the following:

(2′) For each $n \geq 0$, there exists an effective epimorphism $\coprod U_i \to \mathbf{1}$ in $\mathcal{X}$ a collection of objects $K_i \in \mathcal{G}_{\mathrm{an}}$, and a collection of maps $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}} | U_i) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(X_i)$ which induce equivalences $f_i^* : \mathcal{X}_{K_i} \simeq \mathcal{X}_{/U_i}$ and maps $\alpha_i : f_i^* \mathcal{O}_{K_i}^{\mathrm{alg}} \to \mathcal{O}_{\mathcal{X}} | U_i$ with $(n+1)$-connective fibers.

The proof of (2′) proceeds by induction on n. We begin with the case $n = 0$. The assertion is local on $\mathcal{X}$; we may therefore assume that there exists a closed immersion $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}} M$ for some complex manifold $M \in \mathcal{T}_{\mathrm{an}}$ (Lemma 12.13). In particular, we can identify $\mathcal{X}$ with $\mathcal{S}\mathrm{hv}(M_0)$ where M_0 is a closed subset of M. Let $i : M_0 \to M$ be the inclusion and let $\mathcal{I}$ denote the fiber of the map $\mathcal{O}_M^{\mathrm{alg}} \to i_* \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$. Then $\pi_0 \mathcal{I}$ is a coherent sheaf of $\mathcal{O}_M^{\mathrm{alg}}$-modules. Passing to an open cover of M, we may assume that there exists a finite dimensional complex vector space V and a map $u : V^\vee \otimes_{\mathbf{C}} \mathcal{O}_M^{\mathrm{alg}} \to \mathcal{I}$ such that the composite map

$$V^\vee \otimes_{\mathbf{C}} \mathcal{O}_M^{\mathrm{alg}} \to \mathcal{I} \to \pi_0 \mathcal{I}$$

is an epimorphism. Similarly, we can assume that there is a finite dimensional vector space W and a map $v : W^\vee \otimes_{\mathbf{C}} \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}})[1] \to \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$ which induces an epimorphism

$$W^\vee \otimes_{\mathbf{C}} \pi_0 \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}} \to \pi_1 \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$$

Let $K_0, K_1 \in \mathcal{G}_{\mathrm{an}}$ be the fiber products $\theta(M) \times_{\theta(V)} \theta(*)$ and $\Omega\theta(W)$, respectively. Then u and v determine a map

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K_0) \times \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K_1) \simeq \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K_0 \times K_1)$$

which is easily checked to have the desired properties.

Now assume that $n > 0$. We will assume that assertion ($2'$) holds for the integer $n-1$. Working locally on $\mathcal{X}$, we may assume that there exists a map $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K)$ which induces an equivalence $f^* : \mathcal{X}_K \to \mathcal{X}$ and a map $\alpha : f^* \mathcal{O}_K^{\mathrm{alg}} \to \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$ such that the fiber $\mathrm{fib}(\alpha)$ is n-connective. In particular, f induces an isomorphism of sheaves of discrete commutative rings $\pi_0 \mathcal{O}_K^{\mathrm{alg}} \to \pi_0 \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$. The long exact sequence

$$f^*\pi_{n+1} \mathcal{O}_K^{\mathrm{alg}} \to \pi_{n+1} \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}} \to \pi_n \mathrm{fib}(\alpha) \to f^*\pi_n \mathcal{O}_K^{\mathrm{alg}} \to \pi_n \mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$$

shows that $\pi_n \mathrm{fib}(\alpha)$ is a coherent $f^*\pi_0 \mathcal{O}_K^{\mathrm{alg}}$-module. Passing to an admissible cover of K (and the corresponding cover of $\mathcal{X}$), we may assume that there exists a finite dimensional complex vector space V and a map $u : V^\vee \otimes_{\mathbf{C}} f^* \mathcal{O}_K^{\mathrm{alg}})[n] \to \mathrm{fib}(\alpha)$ which induces an epimorphism of (discrete) sheaves $V^\vee \otimes_{\mathbf{C}} f^*\pi_0 \mathcal{O}_K^{\mathrm{alg}} \to \pi_n \mathrm{fib}(\alpha)$. The composite map

$$V^\vee \otimes_{\mathbf{C}} \mathcal{O}_K^{\mathrm{alg}}[n] \overset{f_*(u)}{\to} f_* \mathrm{fib}(\alpha) \to \mathcal{O}_K^{\mathrm{alg}}$$

determines a morphism $\mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(\Omega^n\theta(V))$. Passing to a further cover of K, we may assume that this map is the image of a morphism $K \to \Omega^n\theta(V)$ in $\mathcal{G}_{\mathrm{an}}$. We have a diagram

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) & \xrightarrow{f} & \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K) \\ \downarrow & & \downarrow \\ \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(*) & \longrightarrow & \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(\Omega^n\theta(V)) \end{array}$$

in $\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}})$ which commutes up to canonical homotopy, thereby inducing a map $f' : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K \times_{\Omega^n\theta(V)} *)$. Using Lemma 12.19 we easily deduce that f' satisfies the requirements of ($2'$). $\square$

Proposition 12.24. *The canonical equivalence* $u : \mathrm{id}_{\mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}})} \to \Psi \circ \Phi$ *is the unit of an adjunction between the functors* $\Phi : \mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}}) \to \mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}\,\leq 0}$ *and* $\Psi : \mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}\,\leq 0} \to \mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}})$.

Proof. Let $(X, \mathcal{O}_X^{\mathrm{alg}})$ be a complex analytic space and let $(\mathcal{S}\mathrm{hv}(Y), \mathcal{O}_Y)$ be a 0-localic derived complex analytic space. We wish to show that composition with u induces a homotopy equivalence

$$\phi : \mathrm{Map}_{\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}}(\Phi(X, \mathcal{O}_X^{\mathrm{alg}}), (\mathcal{S}\mathrm{hv}(Y), \mathcal{O}_Y)) \to \mathrm{Hom}_{\mathcal{A}\mathrm{n}_{\mathbf{C}}}((X, \mathcal{O}_X^{\mathrm{alg}}), (Y, \pi_0 \mathcal{O}_Y^{\mathrm{alg}})).$$

In other words, we wish to show that for every map $f : (X, \mathcal{O}_X^{\mathrm{alg}}) \to (Y, \pi_0 \mathcal{O}_Y^{\mathrm{alg}})$ of complex analytic spaces, the inverse image $\phi^{-1}\{f\} \subseteq \mathrm{Map}_{\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}}(\Phi(X, \mathcal{O}_X^{\mathrm{alg}}), (\mathcal{S}\mathrm{hv}(Y), \mathcal{O}_Y))$ is contractible. The problem is local on X and Y. We may therefore use Proposition 12.23 to choose a map $g : (\mathcal{S}\mathrm{hv}(Y), \mathcal{O}_Y) \to \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K)$ which induces equivalences $g^* \mathcal{X}_K \simeq \mathcal{S}\mathrm{hv}(Y)$ and $\tau_{\leq 0} g^* \mathcal{O}_K \to \tau_{\leq 0} \mathcal{O}_Y$. Since $\mathcal{O}_X^{\mathrm{alg}}$ is 0-truncated, we can replace $\mathcal{O}_Y$ by $g^* \mathcal{O}_K$ and thereby reduce to the case where $(\mathcal{S}\mathrm{hv}(Y), \mathcal{O}_Y) = \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K)$ for some $K \in \mathcal{G}_{\mathrm{an}}$.

Let $\mathcal{C} \subseteq \mathcal{G}_{\mathrm{an}}$ be the full subcategory of $\mathcal{G}_{\mathrm{an}}$ spanned by those objects K for which the map $\phi_K : \mathrm{Map}_{\mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}}(\Phi(X, \mathcal{O}_X^{\mathrm{alg}}), \mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K)) \to \mathrm{Hom}_{\mathcal{A}\mathrm{n}_{\mathbf{C}}}((X, \mathcal{O}_X^{\mathrm{alg}}), \Psi \,\mathrm{Spec}^{\mathcal{G}_{\mathrm{an}}}(K))$ is a homotopy equivalence. Using Corollary 12.16, we see that $\mathcal{C}$ is closed under finite limits in $\mathcal{G}_{\mathrm{an}}$; it is obviously closed under retracts as well. Since $\mathcal{G}_{\mathrm{an}}$ is generated under retracts and finite limits by the essential image of the functor $\theta : \mathcal{T}_{\mathrm{an}} \to \mathcal{G}_{\mathrm{an}}$, it will suffice to show that $\mathcal{C}$ contains the essential image of θ. In other words, we are reduced to proving that ϕ is an equivalence when $(\mathcal{S}\mathrm{hv}(Y), \mathcal{O}_Y) = \mathrm{Spec}^{\mathcal{T}_{\mathrm{an}}}(M)$ for some complex manifold $M \in \mathcal{T}_{\mathrm{an}}$. The result in this case follows immediately from the definition of $\Phi(X, \mathcal{O}_X^{\mathrm{alg}})$. $\square$

Proof of Theorem 12.8. It follows from Proposition 12.24 and Lemma 12.9 that the functor $\Phi : \mathrm{N}(\mathcal{A}\mathrm{n}_{\mathbf{C}}) \to \mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}{}_{\leq 0}$ is fully faithful, and it is obvious that the essential image of Φ consists of 0-truncated objects $(\mathcal{S}\mathrm{hv}(X), \mathcal{O}_X) \in \mathcal{A}\mathrm{n}^{\mathrm{der}}_{\mathbf{C}}{}_{\leq 0}$. Conversely, suppose that $(\mathcal{S}\mathrm{hv}(X), \mathcal{O}_X)$ is a 0-truncated, 0-localic derived complex analytic space. We wish to show that the counit map $\Phi(X, \pi_0 \, \mathcal{O}^{\mathrm{alg}}_X) \to (\mathcal{S}\mathrm{hv}(X), \mathcal{O}_{\mathcal{X}})$ is an equivalence of derived complex analytic spaces. Using Proposition 11.9, we are reduced to showing that the truncation map $\mathcal{O}^{\mathrm{alg}}_X \to \pi_0 \, \mathcal{O}^{\mathrm{alg}}_X$ is an equivalence of sheaves of $\mathbb{E}_\infty$-rings on X, which follows from our assumption that $\mathcal{O}_X$ is 0-truncated. □

Remark 12.25. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a derived complex analytic space. In view of Theorem V.2.3.13, there exists an étale map $(\mathcal{X}, \tau_{\leq 0} \, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, where $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a 0-truncated, 1-localic derived complex analytic space. If $\mathcal{Y}$ is 0-localic, then we can identify it with an ordinary complex analytic space (Theorem 12.8). In general, we can view $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ as a ringed topos which is locally equivalent to the underlying topos of a complex analytic space; in other words, we can think of $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ as a *complex analytic orbifold.*

Remark 12.26. The transformation of pregeometries $\mathcal{T}_{\text{ét}}(\mathbf{C}) \to \mathcal{T}_{\mathrm{an}}$ of Construction 11.7 determines a relative spectrum functor $\mathrm{Sch}(\mathcal{T}_{\text{ét}}(\mathbf{C})) \to \mathrm{Sch}(\mathcal{T}_{\mathrm{an}})$. This functor carries derived Deligne-Mumford stacks which are locally almost of finite presentation over $\mathbf{C}$ to derived complex analytic spaces (generally not 0-localic; see Remark 12.25); we will refer to this as the *analytification* functor.

Warning 12.27. The category of complex analytic spaces can be identified with a full subcategory the $\mathrm{RingSpace}_{\mathbf{C}}$ of ringed spaces over $\mathbf{C}$. The derived analogue of this statement is *false*. The structure sheaf $\mathcal{O}_{\mathcal{X}}$ of a derived complex analytic space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ has more structure than that of a $\mathrm{CAlg}_{\mathbf{C}}$-valued sheaf on $\mathcal{X}$. Roughly speaking, this structure allows us to compose sections of $\mathcal{O}_{\mathcal{X}}$ with arbitrary complex analytic functions ϕ defined on open sets $U \subseteq \mathbf{C}^n$; this structure descends to the algebraic structure sheaf $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}} \,|\, \mathcal{T}_{\mathrm{disc}}(\mathbf{C})$ only if ϕ is a polynomial function. The fact that the algebraic structure sheaf $\mathcal{O}^{\mathrm{alg}}_{\mathcal{X}}$ determines $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ in the 0-truncated case is somewhat remarkable, and depends on strong finiteness properties enjoyed by the class of Stein algebras (which do not hold in the derived setting).

Remark 12.28. With some additional effort, the ideas presented in this section can be carried over to the setting of rigid analytic geometry. We will return to this subject in a future work.

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