A NOTE ON GORENSTEIN INJECTIVE DIMENSION

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ABSTRACT. Suppose that M is a module over a commutative noetherian ring R. It is proved that the Gorenstein injective dimension of M, if finite, equals $\sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$.

INTRODUCTION

In 1976, Chouinard gave a general formula for injective dimension of a module, when it is finite (cf. [1]).

Theorem. Let M be an R-module of finite injective dimension. Then

$$\operatorname{inj.dim}_R M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Recall that for a module M over a local ring R, $\operatorname{width}_R M$ is defined as the $\inf\{\, i \mid \operatorname{Tor}_i^R(k, M) \neq 0\}$, where k is the residue field of R. This gives a general formula from which the so called Bass formula can be concluded, namely, the injective dimension of a finite module over a local ring is either infinite or equals the depth of the base ring.

Our theorem 1.2 extends the chouinard's formula for Gorenstein injective dimension.

Theorem. Let R be a noetherian ring and M an R-module of finite Gorenstein injective dimension. Then

$$\operatorname{Gid}_R M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

1. MAIN THEOREM

Definition 1.1. *An R-module G is said to be* Gorenstein injective *if and only if there exists an exact complex of injective R-modules,*

$$I = \cdots \to I_2 \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots$$

such that the complex $Hom_R(J, I)$ is exact for every injective R-module J and G is the kernel in degree 0 of I. The Gorenstein injective dimension *of an R-module M, $Gid_R(M)$, is defined to be the infemum of integers n such that there exists an exact sequence*

$$0 \to M \to G_0 \to G_{-1} \to \cdots \to G_{-n} \to 0$$

with all G_i's Gorenstein injective.

Theorem 1.2. *Let R be a commutative noetherian ring and M an R-module of finite Gorenstein injective dimension. Then*

$$Gid_R(M) = \sup\{depth\, R_{\mathfrak{p}} - width_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in Spec\,(R)\}.$$

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Proof. First assume that $\text{Gid}_R(M) = 0$. By definition, there is an exact sequence

$$E_\bullet : \cdots \to E_1 \to E_0 \to M \to 0$$

such that every E_i is injective. Set $K_i = \ker(E_{i-1} \to E_{i-2})$.
For any $\mathfrak{p} \in \text{Spec}(R)$ and every $R_\mathfrak{p}$-module T, we have $\text{Ext}^i_{R_\mathfrak{p}}(T, M_\mathfrak{p}) \cong \text{Ext}^{i+t}_{R_\mathfrak{p}}(T, (K_t)_\mathfrak{p})$ for any two positive integers i and t. Hence using [2, 5.3(c)] we get

$$\begin{aligned} 0 &= \sup\{ i \mid \text{Ext}^i_{R_\mathfrak{p}}(T, M_\mathfrak{p}) \neq 0, \text{ for some } R_\mathfrak{p}-\text{module } T \text{ with proj.dim}_{R_\mathfrak{p}} T < \infty\} \\ &\geq \text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} M_\mathfrak{p}. \end{aligned}$$

In addition, if $\mathfrak{p}$ is such that $\dim R/\mathfrak{p} = \dim_R M$ then, using [2, 5.3(c)] again, we get $\text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} M_\mathfrak{p} = 0$.
Therefore, $\sup\{\text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}(R)\} = 0$ for a Gorenstein injective module M.

Now assume that $n = \text{Gid}_R M > 0$. By [3, 2.16], there exists a short exact sequence

$$0 \to K \to L \to M \to 0,$$

where K is Gorenstein injective and $\text{inj.dim}_R L = \text{Gid}_R M = n$.
Thus

$$0 = \sup\{ \text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} K_\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}(R)\}.$$

On the other hand, by Chouinard's equality [**?**, 3.1], we have

$$\text{inj.dim}_R L = \sup\{\text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}(R)\}.$$

For any $\mathfrak{p} \in \text{Spec}(R)$, we denote $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Then, for any $\mathfrak{p} \in \text{Spec}(R)$, the exact sequence $0 \to K_\mathfrak{p} \to L_\mathfrak{p} \to M_\mathfrak{p} \to 0$ induces the long exact sequence

$$\cdots \to \text{Tor}^{R_\mathfrak{p}}_i(k(\mathfrak{p}), K_\mathfrak{p}) \to \text{Tor}^{R_\mathfrak{p}}_i(k(\mathfrak{p}), L_\mathfrak{p}) \to \text{Tor}^{R_\mathfrak{p}}_i(k(\mathfrak{p}), M_\mathfrak{p}) \to \text{Tor}^{R_\mathfrak{p}}_{i-1}(k(\mathfrak{p}), K_\mathfrak{p}) \to \cdots$$

This sequence gives rise to the following inequalities.

$$\begin{aligned} \text{width}_{R_\mathfrak{p}} L_\mathfrak{p} &\geq \min\{\text{width}_{R_\mathfrak{p}} M_\mathfrak{p}, \text{width}_{R_\mathfrak{p}} K_\mathfrak{p}\} \text{ and} \\ \text{width}_{R_\mathfrak{p}} M_\mathfrak{p} &\geq \min\{\text{width}_{R_\mathfrak{p}} L_\mathfrak{p}, \text{width}_{R_\mathfrak{p}} K_\mathfrak{p} + 1\} \end{aligned}$$

If $\mathfrak{p} \in \text{Spec}(R)$ is such that $\text{width}_{R_\mathfrak{p}} K_\mathfrak{p} > \text{width}_{R_\mathfrak{p}} M_\mathfrak{p}$ then, by the mentioned inequalities, $\text{width}_{R_\mathfrak{p}} M_\mathfrak{p} = \text{width}_{R_\mathfrak{p}} L_\mathfrak{p}$.
But for any $\mathfrak{p}$ with $\text{width}_{R_\mathfrak{p}} K_\mathfrak{p} \leq \text{width}_{R_\mathfrak{p}} M_\mathfrak{p}$, we also have $\text{width}_{R_\mathfrak{p}} K_\mathfrak{p} \leq \text{width}_{R_\mathfrak{p}} L_\mathfrak{p}$.
Thus

$$\begin{aligned} \text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} L_\mathfrak{p} &\leq \text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} K_\mathfrak{p} \leq 0 \text{ and} \\ \text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} M_\mathfrak{p} &\leq \text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} K_\mathfrak{p} \leq 0. \end{aligned}$$

Therefore we have

$$\begin{aligned} \text{Gid}_R(M) = \text{inj.dim}_R(L) &= \\ \sup\{\text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} L_\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}(R)\} &= \\ \sup\{\text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} L_\mathfrak{p} \mid \mathfrak{p} \text{ with width}_{R_\mathfrak{p}} M_\mathfrak{p} < \text{width}_{R_\mathfrak{p}} K_\mathfrak{p}\} &= \\ \sup\{\text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \text{ with width}_{R_\mathfrak{p}} M_\mathfrak{p} < \text{width}_{R_\mathfrak{p}} K_\mathfrak{p}\} &= \\ \sup\{\text{depth}\, R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}(R)\}&. \end{aligned}$$

□

Corollary 1.3. *Let M be an R-module and $\mathfrak{p} \subseteq \mathfrak{q}$ prime ideals of R. If $Gid_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ then*

$$Gid_{R_\mathfrak{p}} M_\mathfrak{p} \leq Gid_{R_\mathfrak{q}} M_\mathfrak{q}.$$

References

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